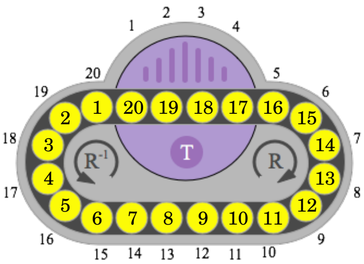
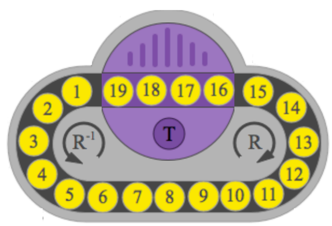
1. In the end-game phase of the Oval Track puzzle show that any 4-cycle can always be reduced to a 2-cycle by using a 3-cycle. (Note: this is stated in the solution flow chart from the notes, here you are asked to prove this always works.)

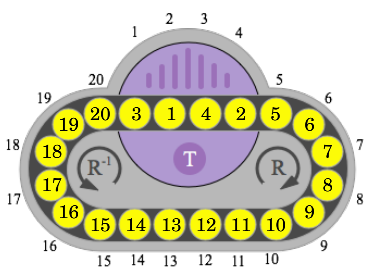
2. (a) For the oval track puzzle with 20-disks is the following configuration of disks solvable? Explain.



(b) For the oval track puzzle with 19-disks is the following configuration of disks solvable? Explain.



3. Consider the configuration of the oval track puzzle shown below. To solve the puzzle we just need to use two fundamental moves: $\sigma_2 = (1\ 3)$ and $\sigma_3 = (1\ 7\ 4)$, and their conjugates. Provide an outline of the steps involved in solving this configuration, indicate which move (a 2-cycle or a 3-cycle) you are using at each step, and draw the resulting configuration. You do not need to find the sequence β to conjugate σ_2 or σ_3, just provide an outline of the solution steps.



4. **Permutations: decompositions into 2-cycles of the form $(1\ m)$:**

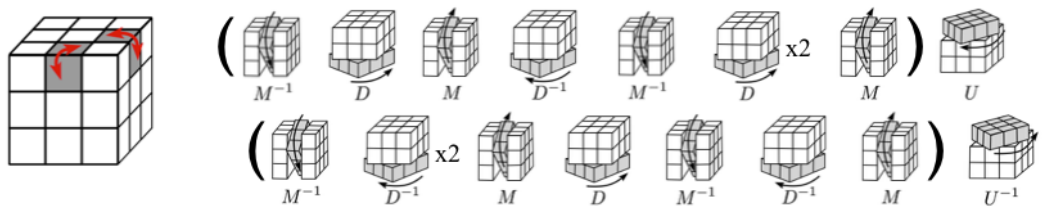
   We know that every permutation in S_n can be expressed as a product of 2-cycles. Show the stronger result that every permutation in S_n can be expressed as a product of 2-cycles of the form $(1\ m)$, where $2 \leq m \leq n$.

   (This is equivalent to showing that every permutation is obtainable on the Swap puzzle where the only legal move is to swap the contents of any box with box 1. You may use the Swap puzzle to investigate this statement, but the argument you present should be described in terms of permutations.)

5. **Permutations: decompositions into 4-cycles:**

   Can every permutation in S_n, for $n \geq 4$, be written as a product of 4-cycles? Justify your answer.

6. (a) Consider the following move sequence for flipping adjacent edges. What is the purpose of the initial part of the sequence consisting of $M^{-1}DMD^{-1}M^{-1}D^2M$?



(b) Using the above sequence, or a slight modification of it, come up with three different ways to flip two opposite edges (the uf and the ub edges).

7. Suppose $G = \{e, a, b, c, d, f\}$ is a group with Cayley table

	e	a	b	c	d	f
e	e					
a		e				
b		f				
c				e	a	
d		c	a			
f		b	c	a	e	

Fill in the blank entries.

8. **All about commutators.**
   (a) Let $\alpha, \beta \in S_n$. Show that the commutator $[\alpha, \beta]$ is an even permutation.
   (b) For g, h in a group G, show that $[g, h]^{-1} = [h, g]$.
   (c) For g, h, k in a group G, show that $[g, h]^k = [g^k, h^k]$.

9. Let G be a group of order 34. What are the possible orders of the elements of G?

10. Let $G = \{e, a, b, c\}$ be a group, where e is the identity.
(a) Assume G has an element of order 4, say a. Then there must be a second element of order 4, say c. Write out the Cayley table for G.
(b) Assume G does not have an element of order 4, then every (non-identity) element has order 2. If $ab = c$ write out the Cayley table for G.

11. List all subgroups of $\mathbb{Z}_{28}$ and the generators for each subgroup.

12. List all the elements of $U(8)$ and write out its Cayley table.

13. List all the elements of $U(21)$. What is the order of 4? What is the order of 5?

14. List all the elements of $U(16)$. What is the order of 9? What is the order of 15?

15. Show that for $n \geq 3$ the group $U(2^n)$ is not cyclic?
Hint: Can you find two elements of order 2? Further hint: have another look at the previous question.

16. $U(49)$ is a cyclic group with 42 elements. If b is a generator, what are the other generators?

17. Consider the regular 7-gon shown in the picture. Let r denote a clockwise rotation through $\frac{360}{7}$ degrees. The elements of D_7 are

$$D_7 = \{1, r, r^2, r^3, r^4, r^5, r^6, f_1, f_2, f_3, f_4, f_5, f_6, f_7\}$$

where f_i denotes a reflection across a line as shown the figure. Determine the element of D_7 corresponding to $f_1 r f_6$.