Math 304, Spring 2021
Midterm 1

1. **Permutations: products, inverses and orders:**
Consider the following permutations in S_9, given in cycle form.

$$\alpha = (1\ 9\ 7\ 4)(2\ 5)(3\ 8\ 6), \quad \beta = (1\ 3\ 4\ 2), \quad \gamma = (1\ 5)(3\ 9)(4\ 7).$$

Compute each of the following. Provide justification where appropriate.

[2] (a) $\alpha\beta = (1974)(25)(386)(1342)$

$= (19725)(3864)$

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[2] (b) $\alpha^{-1} = (1479)(25)(368)$

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[2] (c) $\mathrm{ord}(\alpha) = \mathrm{lcm}(4,2,3) = 12$

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[2] (d) $\mathrm{ord}(\gamma) = 2$

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[2] (e) Write α in **array** form.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 5 & 8 & 1 & 2 & 3 & 4 & 6 & 7 \end{pmatrix}$$

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[2] (f) Express α as a product of 2-cycles.

$\alpha = (1974)(25)(386) = (19)(17)(14)(25)(38)(36)$

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[2] (g) Express α as a product of 3-cycles.

$\alpha = (1974)(25)(386) = (197)\underbrace{(14)(25)}_{=(14)(12)(12)(25)}(386)$

$= (197)(142)(152)(386)$

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2. **Permutations:**

[2] (a) The diagram shows the configuration of the swap board before and after move sequence α was applied. Write the move sequence α in disjoint cycle form.

1	2	3	4	5	6	7	8
5	8	1	2	3	7	4	6

$\xrightarrow{\alpha}$

1	2	3	4	5	6	7	8
7	4	2	8	3	6	1	5

$\alpha = (1\ 8\ 6)(2\ 4\ 3\ 7)$

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[2] (b) Write down a solution to the 3-cycle swap puzzle with the initial configuration shown. Your answer should list the 3-cycles in order that they need to be applied to solve the puzzle.

1	2	3	4	5	6	7	8
1	7	3	2	5	4	8	6

$\beta = (2\ 4\ 6\ 8\ 7) = (2\ 4\ 6)(2\ 8\ 7)$

Solution: $(2\ 7\ 8)$, $(2\ 6\ 4)$

1 | 7 | 3 | 2 | 5 | 4 | 8 | 6 $\xrightarrow{(2\,7\,8)}$ 1 | 6 | 3 | 2 | 5 | 4 | 7 | 8 $\xrightarrow{(2\,6\,4)}$ 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8

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[2] (c) Consider the permutation $\alpha = (2\ 5\ 6\ 3)$. Give an example of a permutation β in S_4 that does not commute with α. For full credit you must verify your example doesn't commute with α.

$\beta = (1\ 2)$

$\alpha\beta = (1\ 2)(2\ 5\ 6\ 3) = (1\ 5\ 6\ 3\ 2)$

$\beta\alpha = (2\ 5\ 6\ 3)(1\ 2) = (1\ 2\ 5\ 6\ 3)$ $\neq$ α, β do not commute

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[2] (d) Give examples of permutations α and β such that $\text{ord}(\alpha) = 2$, $\text{ord}(\beta) = 3$ and $\text{ord}(\alpha\beta) = 4$.

$\alpha = (1\ 2)$, $\beta = (1\ 3\ 4)$

$\alpha\beta = (1\ 2)(1\ 3\ 4) = (1\ 2\ 3\ 4) \implies \text{ord}(\alpha\beta) = 4$.

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[2] (e) How many elements of order 4 are in A_6? Justify your answer.

Element of order 4 in A_6 has form: (2-cycle)(4-cycle)

There are,

$$\binom{6}{2}1! \cdot \binom{4}{4} \cdot 3! = \frac{6!}{\cancel{2!}\,4!}\cancel{3!}^{\,3} = \frac{6!}{4 \cdot 2} = 6 \cdot 5 \cdot 3 = \boxed{90}$$

Choose #'s for 2-cycle

make 2-cycle

Choose #'s for 4-cycle

make 4-cycle

3. **Permutations: Proofs involving parity**

[4] (a) Prove that the product of two even permutations is an even permutation.

If $\alpha, \beta \in S_n$ are both even, write

$$\alpha = \tau_1 \cdots \tau_m ,$$
$$\beta = \sigma_1 \cdots \sigma_k ,$$

where τ_i, σ_j are 2-cycles, and m, k are even.

Then

$$\alpha\beta = \tau_1 \cdots \tau_m \sigma_1 \cdots \sigma_k$$

is a product of $m+k$ (even number) of 2-cycles.

Therefore,

$\alpha\beta$ is even.

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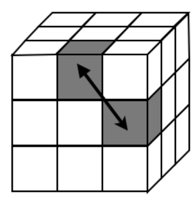
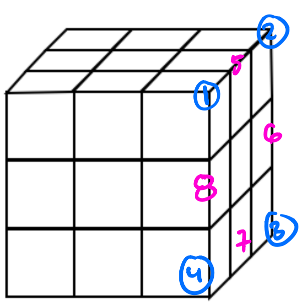
[4] (b) Show that it is impossible to find a move sequence that **swaps exactly two edge cubies** of the Rubik's cube, while leaving every other cubie in its home location (that is, with the exception of the two cubies that were swapped, all other cubies are in their correct location).

This question is asking you to show that a move like the one shown in the diagram on the right is impossible to do.



(Draw a cube like the one below, use it to help explain your argument.)



Label each cubie (8 corners, 12 edges).

A basic move on Rubik's cube is a quarter twist of a face. As a permutation of cubies this has the form

$$\underset{\text{(edges)}}{(4\text{-cycle})}\underset{\text{(corners)}}{(4\text{-cycle})}$$

which is even. Therefore only even permutations of cubies are possible.

edge swap is odd $\Rightarrow$ not solvable.

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4. **15-puzzle:**

[4] (a) Show that the following arrangement of the 15-puzzle is unsolvable. To receive full credit your argument must be clearly written, referencing any main results developed in this course.

1	2	3	4
8	12	9	11
6		5	7
13	14	15	10

position: (5 11 8)(6 9 7 12)(10 16)

$\boxed{\text{even}}$ permutation

empty space: in box 10 which is <u>3</u> moves from box 16, so it is in an $\boxed{\text{odd}}$ parity box.

∴ not solvable.

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[4] (b) The following arrangement of the 15-puzzle is solvable. Write down a sequence of moves that solves the puzzle.

Either write the moves using transpositions, or use the first letter of the words *(u)p, (d)own, (l)eft, (r)ight*, to indicate the direction the next tile is to be moved.

You may use your tiles and game board to play around with the puzzle. To grade this question I will simply follow your directions for solution, if it works you get full points, if it doesn't work you loose points. So make your steps for solution clear.

1	2	3	4
5	6	7	8
9	**12**	**10**	**11**
13	14	15	

(d r r u l)(d l u r)(r d l l u)

get 10,11,12 together

cycle 10,11,12

undo set-up move