Math 304, Spring 2021
Midterm 2

1. **Computations in various groups...**

[2] (a) In the group $\mathbb{Z}_{18}$ determine the order of 10. (That is, determine ord(10).)

$10,\ 10+10=2,\ 3\cdot 10=12,\ 4\cdot 10=4,\ 5\cdot 10=14,\ 6\cdot 10=6,\ 8\cdot 10=8,\ 9+10=0 \Rightarrow \text{ord}(10)=9.$

alternative 1: The smallest value of $k$ such that $k\cdot 10 \equiv 0 \pmod{18}$ is 9. $\therefore$ $\text{ord}(10)=9$.

alternative 2: $\gcd(5,18)=1 \Rightarrow$ 5 has order 18 $\Rightarrow$ $2\cdot 5=10$ has order 9. $\therefore$ $\text{ord}(10)=9$.

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[3] (b) List all the elements $a \in \mathbb{Z}_{15}$ for which $\mathbb{Z}_{15} = \langle a \rangle$.

generators of $\mathbb{Z}_{15}$ are the numbers relatively prime to 15:

1, 2, 4, 7, 8, 11, 13, 14

(there should be $\varphi(15) = 2\cdot 4 = 8$ numbers listed)

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[3] (c) Consider the permutation $\alpha = (1\ 2)(3\ 4\ 5\ 6)(7\ 8\ 9\ 10)$. Which of the following permutations $\beta_1$ and $\beta_2$ is conjugate to $\alpha$ in $S_{10}$? For the $\beta_i$ that is conjugate to $\alpha$ give an example of a $\gamma \in S_{10}$ such that $\beta_i = \gamma^{-1}\alpha\gamma$.

$$\beta_1 = (1\ 2\ 3\ 4)(5\ 6)(7\ 8)(9\ 10), \qquad \beta_2 = \overset{(3\,4\,5\,6)(1\,2)(7\,8\,9\,10)}{(1\ 2\ 3\ 4)(5\ 6)(7\ 8\ 9\ 10)}$$

$\beta_2$ has same cycle structure as $\alpha$, so $\alpha$ and $\beta_2$ are conjugate.

$\gamma = (1\ 5\ 3)(2\ 6\ 4)$ (other options involve ways to match up entries in
$(3\ 4\ 5\ 6)(1\ 2)$
& $(1\ 2\ 3\ 4)(5\ 6)$ )

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[2] (d) Suppose $H$ is a cyclic group and 18 divides $|H|$. If $b$ is one element of order 18, list the other elements of order 18 in terms of $b$.

$b^k$ s.t. $1 \le k \le 18$ & $\gcd(k, 18) = 1$

$\Rightarrow b^1, b^5, b^7, b^{11}, b^{13}, b^{17}$ (should be $\varphi(18) = 6$ in total)

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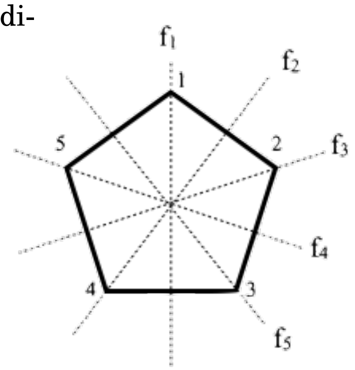
[2] (e) Consider the pentagon shown. Let $r$ denote a rotation in the clockwise direction through an angle of 72 degrees. The elements of $D_5$ are

$$D_5 = \{1, r, r^2, r^3, r^4, f_1, f_2, f_3, f_4, f_5\}.$$

Determine the element of $D_5$ corresponding to $\underbrace{r^2 f_3 r\ f_4}_{\text{rotation, since two reflections}}$.



where does it take 1:

$$1 \xrightarrow{r^2} 3 \xrightarrow{f_3} 1 \xrightarrow{r} 2 \xrightarrow{f_4} 3$$

$$\therefore \quad r^2 f_3 r^3 f_4 = \boxed{r^2}$$

alternate: $\underbrace{(12345)(12345)}_{r^2}\underbrace{(13)(54)}_{f_3}\underbrace{(12345)}_{r}\underbrace{(14)(23)}_{f_4} = (1\,3\,5\,2\,4) = \boxed{r^2}$

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2. **A Group of Permutations:**

Let $G$ be a subgroup of $S_6$ with 12 elements. Ten elements of $G$ are listed here:

$$G = \{\varepsilon, \quad (1\,3)(4\,6), \quad (1\,5)(2\,4), \quad (1\,2)(3\,6)(4\,5), \quad (1\,4)(2\,3)(5\,6), \quad (1\,4)(2\,5)(3\,6),$$
$$(1\,6)(2\,5)(3\,4), \quad (1\,3\,5)(2\,4\,6), \quad (1\,5\,3)(2\,6\,4), \quad (1\,2\,3\,4\,5\,6), \quad \underline{(1\,6\,5\,4\,3\,2)}, \quad \underline{(2\,6)(3\,5)}\}$$

*(Note: the following questions can be done in any order. If you get stuck on one, move on to the next.)*

[4] (a) Find the two elements of $G$ that are missing.

$$(123456)^{-1} = (1\,6\,5\,4\,3\,2)$$

Conjugate: $(123456)^{-1}\,\underbrace{(1\,5)(2\,4)}\,(1\,2\,3\,4\,5\,6) = (2\,6)(3\,5)$

[2] (b) Is $G$ a cyclic group? Justify your answer.

No, there is no element of order 12 (if it had one, there would need to be $\varphi(12) = 4$ of them in total)

alternate: If it was cyclic there would be only one element of order 2. But $G$ has at least two: $(13)(46)$, $(15)(24)$ $\therefore$ not cyclic.

[3] (c) Find a subgroup of $G$ of order 3.

$$\{\varepsilon, (1\,3\,5)(2\,4\,6), (1\,5\,3)(2\,6\,4)\}$$

[2] (d) Does $G$ contain a subgroup of order 7? If so, find one. It not, explain why not.

No, $7 \nmid 12$. By Lagrange's theorem, $G$ cannot have a subgroup of order 7

3. **Groups, Cayley Table, Commutators and Conjugates...**

[3] (a) Complete the statement for the definition of a **group**.

*A **group** consists of a nonempty set G together with* a function $* : G \times G \to G$

*that satisfies the following three properties:*

1. (associative) $a*(b*c) = (a*b)*c$ for all $a,b,c \in G$
2. (identity) there is an $e \in G$ s.t. $a*e = e*a = a$ for all $a \in G$
3. (inverse) for $a \in G$ there exists $b \in G$ s.t. $a*b = b*a = e$.

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[3] (b) Suppose $G = \{e, a, b, c, d, f\}$ is a group with the following multiplication (Cayley) table. Fill in the blank entries. You do not need to justify you answers.

|  | $e$ | $a$ | $b$ | $c$ | $d$ | $f$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ | $d$ | $f$ |
| $a$ | $a$ | $e$ | d | $f$ | b | c |
| $b$ | $b$ | $f$ | c | $e$ | $a$ | $d$ |
| $c$ | $c$ | d | e | b | $f$ | $a$ |
| $d$ | $d$ | c | f | a | e | $b$ |
| $f$ | $f$ | b | a | $d$ | c | $e$ |

[4] (c) Prove the following formula which shows that a commutator of a product is a product of commutators.

$$[\alpha, (\beta\gamma)] = [\alpha, \beta][\beta\alpha\beta^{-1}, \beta\gamma\beta^{-1}]$$

$$[\alpha, (\beta\gamma)] = \alpha\beta\gamma\alpha^{-1}(\beta\gamma)^{-1}$$

$$= \alpha\beta\gamma\alpha^{-1}\gamma^{-1}\beta^{-1} \qquad (*)$$

On the other hand,

$$[\alpha, \beta][\beta\alpha\beta^{-1}, \beta\gamma\beta^{-1}] = \alpha\beta\cancel{\alpha^{-1}}\cancel{\beta^{-1}}\ \cancel{\beta}\cancel{\alpha}\cancel{\beta^{-1}}\ \cancel{\beta}\gamma\cancel{\beta^{-1}}\ \cancel{\beta}\alpha^{-1}\cancel{\beta^{-1}}\ \cancel{\beta}\gamma^{-1}\beta^{-1}$$

$$= \alpha\beta\gamma\alpha^{-1}\gamma^{-1}\beta^{-1}$$

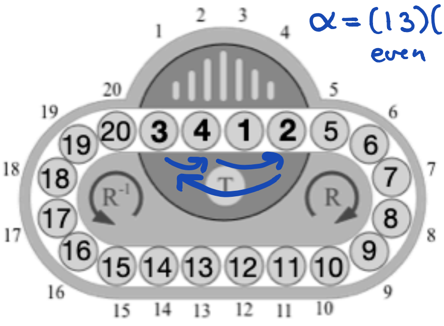
$$= [\alpha, \beta\gamma] \qquad \text{by } (*)$$

4. **Oval Track Puzzle and Variation:**

[4] (a) **Mapping out your solution strategy:** Consider the following configuration of the oval track puzzle. To solve the puzzle we only need to use two fundamental moves: $\sigma_3 = (1\ 7\ 4)$ and $\sigma_2 = (1\ 3)$, and their conjugates. Provide an outline of the steps involved in solving this configuration: indicate which move (a 2-cycle or a 3-cycle) you are using at each step, identify the disks you are permuting (i.e. give the permutation), and draw the resulting configuration. **Do not use a** 2**-cycle unless it is necessary**. You **do not** need to find the sequence $\beta$ to conjugate $\sigma_2$ or $\sigma_3$, just provide an outline of the solution steps. (You may not need all the images below, just write "solved" under the image in which you've finished your solution.)

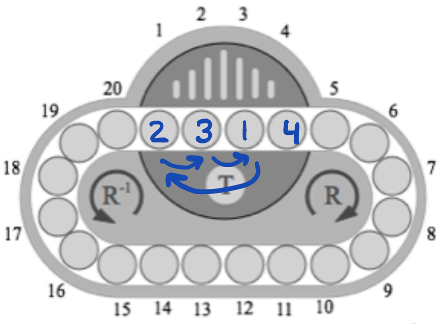
$\alpha = (13)(24) = (132)(142)$

even



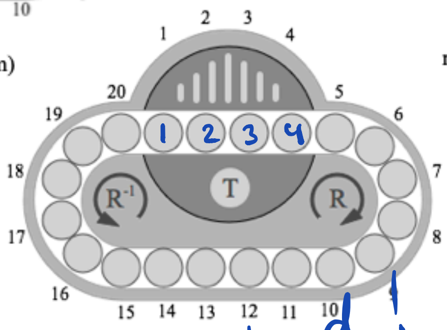
move: (permutation in cycle form)

(1 2 4)



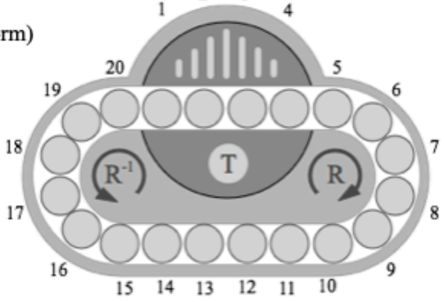
move: (permutation in cycle form)

(1 2 3)



Solved!

move: (permutation in cycle form)



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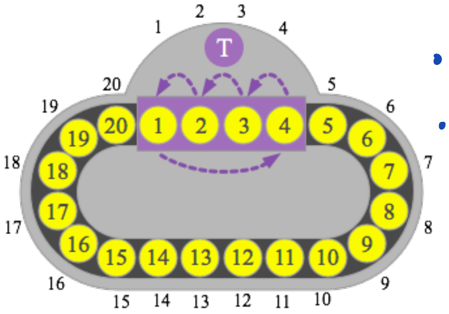
[3] (b) Consider the variation of the Oval Track puzzle on 20 disks where the turntable move $T$ corresponds to the permutation $T = (1\ 4\ 3\ 2)$. You are given the fact that any 3-cycle is possible to do on this puzzle. Using this fact, show that it is possible to do a 2-cycle.



- Any 3-cycle possible ⇒ any even permutation is possible.
- R is an odd move

} ⇒ all of $S_n$ is possible

∴ 2-cycles are possible

alternate: $T = (1432) = (143)(12)$

⇒ It is possible to do

$(134)\,T = (134)(143)(12) = (12)$

∴ a 2-cycle is possible.