MATH 304: Mathematical Journeys IV

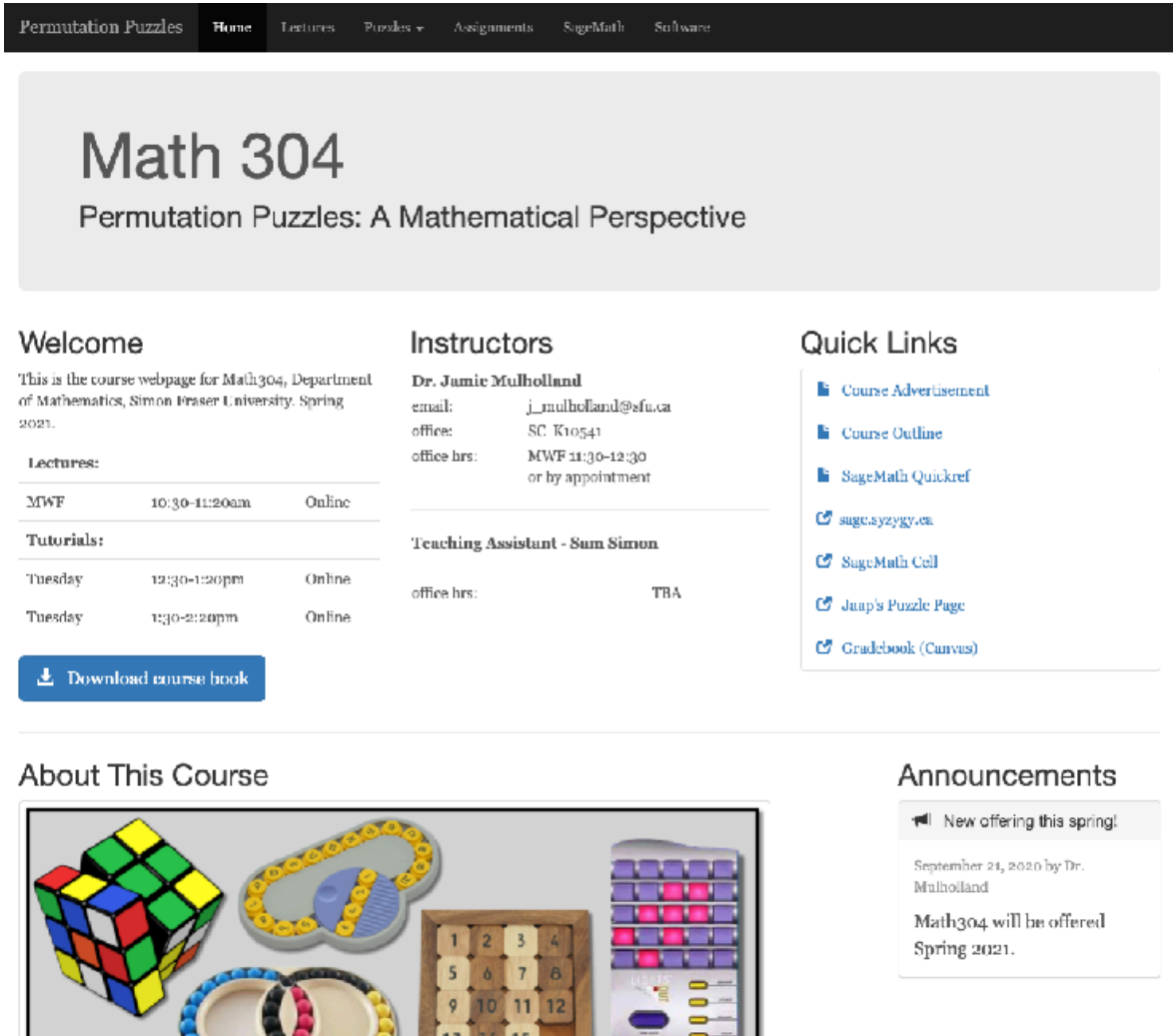
Permutation Puzzles: A Mathematical Perspective

Instructor: Jamie Mulholland

TA: Sam Simon

Course Webpage: https://www.sfu.ca/~jtmulhol/permutationpuzzles

Note: page redirects to http://www.sfu.ca/~jtmulhol/math302/



Grading Scheme:

Assignments	30%
Midterm 1	25%
Midterm 2	25%
Final Project	20%

You can check your grades on Canvas.

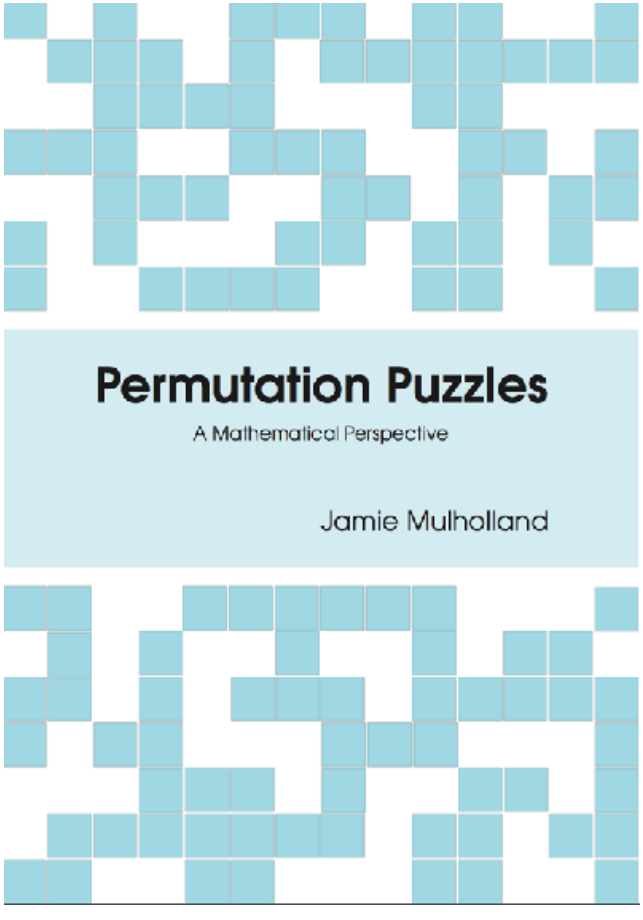
Important Dates:

Midterm 1	Fri. Feb. 26
Midterm 2	Fri. Mar. 26
Final Project	TBA

Weekly Assignments due Fridays.

Textbook:

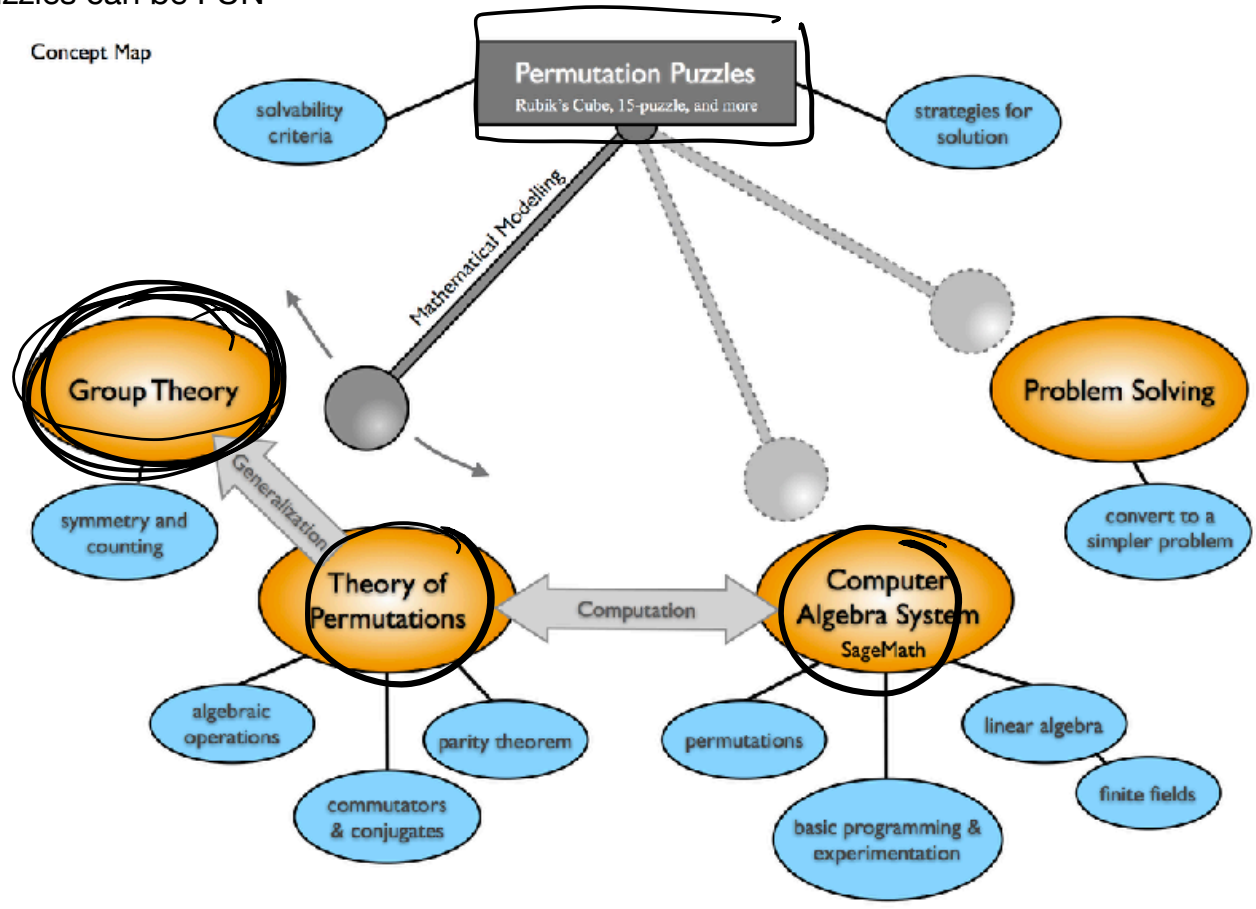
Available for download from course webpage.



Why Study Puzzles?

- Puzzles are the common persons experience with abstraction, and math is all about abstraction
- Provides a wonderful hands-on approach to *group theory* (an important branch of mathematics)
- They help to illustrate and reinforce the fundamental technique of problem solving:

  if you don't know how to solve a problem, find a simpler one you do know how to solve.
- Puzzles can be FUN



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Broadly speaking, group theory is the study of **symmetry**.

We apply the label symmetry to anything which stays invariant under some transformation.

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Physics

- **Conservation laws** of physics are related to the symmetry of physical laws under various transformations.
- Modern particle physics would not exist without group theory; in fact, group theory predicted the existence of many elementary particles before they were found experimentally.

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Chemistry

- The structure and behaviour of molecules and crystals depends on their different symmetries.

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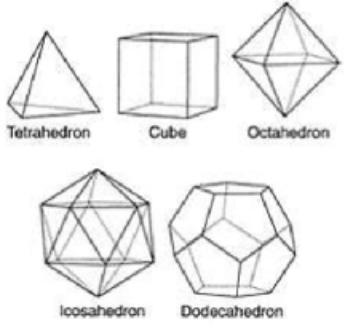
Mathematics

- Group theory is very closely linked to symmetry in **geometry**.
- Classical problems in **algebra** have been resolved with group theory.
  (no general solution to a degree 5 polynomial equation, unlike the degree 2 case where we have the quadratic formula)



Day to day life

- The mathematics of **public-key cryptography** uses group theory.
  - this is what is used when doing online purchases
- **Identification numbers** are all around us:
  - ISBN number for a book
  - VIN (Vehicle Identification Number) for your car
  - Bar code on a UPS package
  - Credit card numbers

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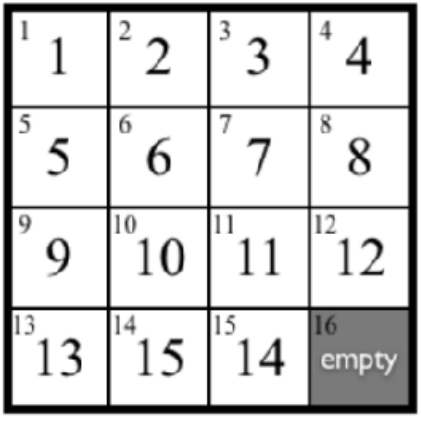
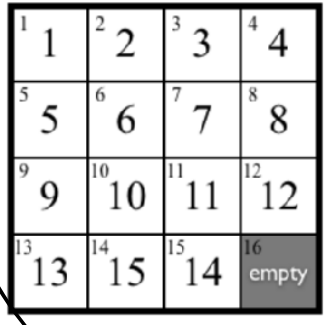
For Fun

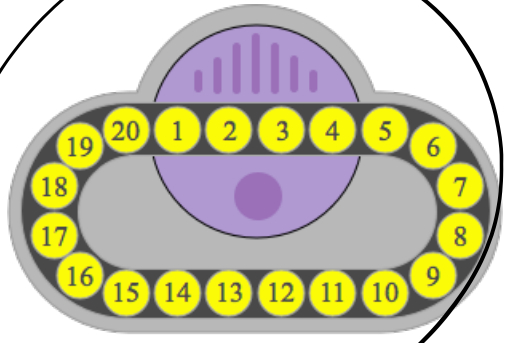
- With group theory we can completely understand **Rubik's Cube**, and other puzzles**.**
- The 15-puzzle drove the world crazy in the 1880's and group theory provided the cure.
- Science fiction shows (Futurama, Stargate SG1) feature problems requiring group theory.



The 15-Puzzle

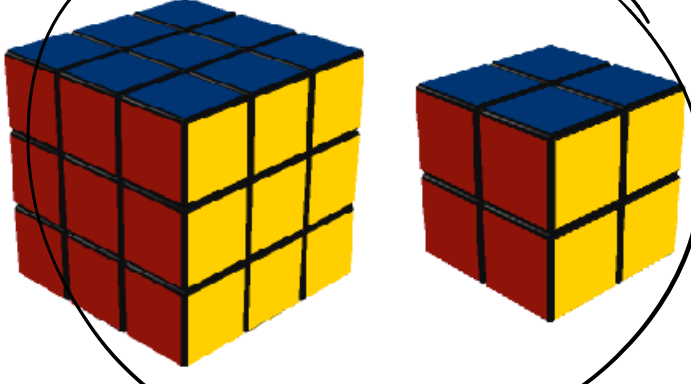


Oval Track (TopSpin)

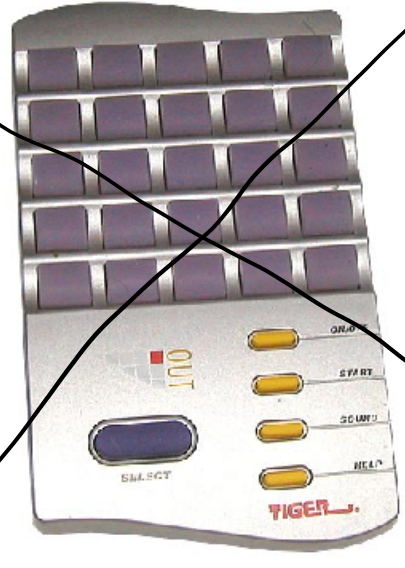


Hungarian Rings

Rubik's Cube

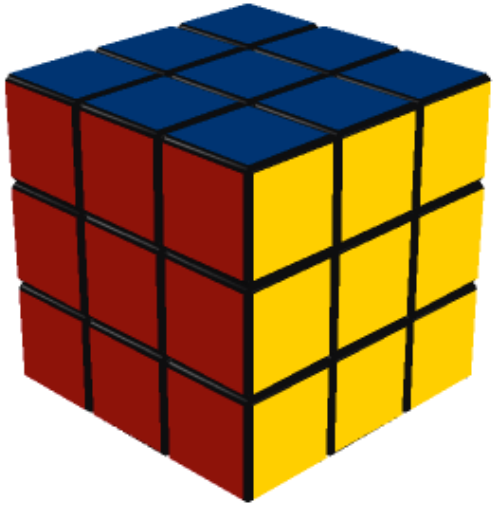


Lights Out



Jaap's Puzzle Page:
https://www.jaapsch.net/puzzles/

Rubik's Cube



Ideal Toy Company stated on the package of the original Rubik cube that there were more than three billion possible states the cube could attain. It's analogous to McDonald's proudly announcing that they've sold more than 120 hamburgers.
(J. A. Paulos, Innumeracy)

There are actually

$$8! \cdot 12! \cdot 3^7 \cdot 2^{10} = 43{,}252{,}003{,}274{,}489{,}856{,}000$$

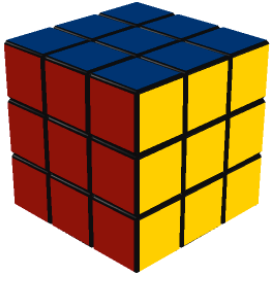
configurations of the cube.

Some questions we will answer in this course:

Are all configurations of the puzzle pieces solvable?

If not, can we determine…
- … a criteria for which puzzles are solvable?
- … the number of solvable configurations?
- … the probability that a random configuration of the pieces is solvable? $\frac{1}{12}$ ← ??

Can we find a strategy for solving the puzzle?



Why do we know there exist configurations of Rubik's Cube that no cube has ever been twisted into?

Why is it impossible to flip only one edge sub-cube of Rubik's Cube?

What is the maximum number of moves required to solve Rubik's Cube? 20

How can we use computational tools to help us understand and solve this puzzles?

Note:

We will not be interested in “speedsolving” Rubik’s cube.

Our attention is on *understanding* the cube.

Any solution strategies we use for solving the cube we will discover. We won’t be focussing on any of the algorithms which can be found on the web.

SageMath: a free open-source mathematics software package

We will use SageMath in two ways:

(1) through a SageMath server: **https://sage.syzygy.ca**
- requires a GitHub account (which is free)
- cloud storage for all your files

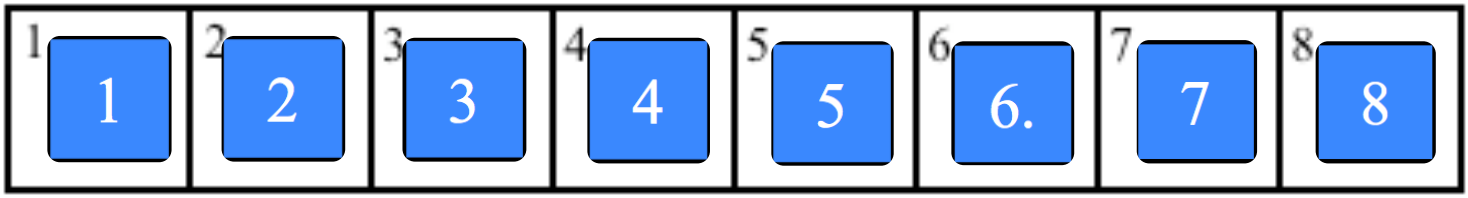
(2) SageMath cell server (for quick calculations):

```
1 # enter code here
```

Evaluate

Appendix A in the course text has all the information needed to get up and running with SageMath.

The Swap Puzzle



Rules (legal moves) : Pick any two boxes and swap the contents

Goal : Start with the files randomly arranged in the boxes, then try to put them in their proper order using legal moves.

Here are some initial configurations to try :

Puzzle 1 :

1	2	3	4	5	6	7	8
8	7	6	5	4	3	2	1

Puzzle 2 :

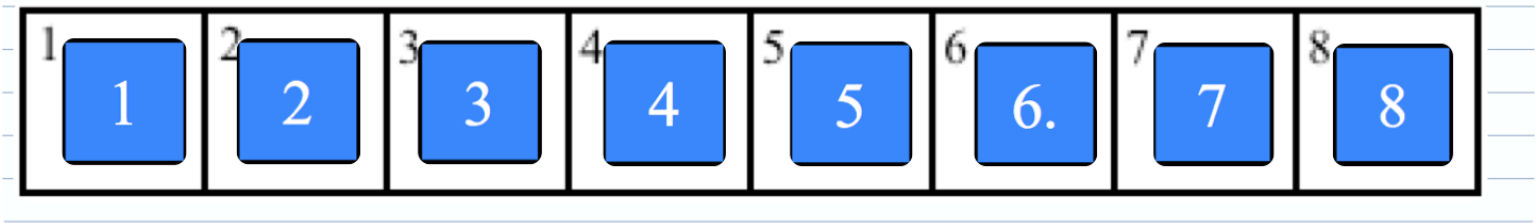
1	2	3	4	5	6	7	8
8	1	2	3	4	5	6	7

Puzzle 3 :

1	2	3	4	5	6	7	8
7	8	1	2	3	4	5	6

Puzzle 4 :

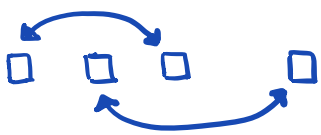
1	2	3	4	5	6	7	8
1	2	3	4	5	6	8	7



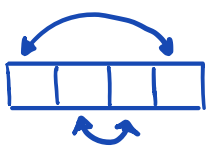
Variation 2:
Legal Moves - Swap the contents of any box with box 1.

Variation 3:
Legal Moves - Pick any 3 boxes, shift the contents either left or right one box (we call this move a 3-cycle)

Variation 4:
Legal Moves - Pick any 4 boxes, split the contents into pairs. Swap the pairs.



Variation 5:
Legal Moves - Pick any 4 *consecutive boxes* (assume end wrap around), and swap the inner two tiles, and then the outer two tiles.



Variation 6:
Legal Moves - Pick any 4 boxes, shift the contents either left or right one box (we call this a 4-cycle).