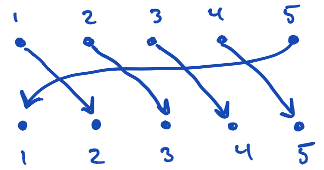
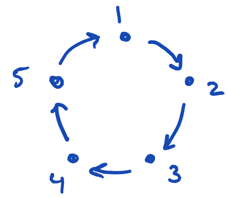
Cycle Notation:

Consider the 5-cycle $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$.

arrow diagram



cycle arrow form



cycle form: Leaving out the arrows in cycle arrow form, we can write the 5-cycle as

$$\alpha = (1\ 2\ 3\ 4\ 5)$$

This represents the fact that each number maps to the one on the right, and the last one maps back to the start.

non-unique: cycle form isn't unique, you can begin a cycle from any number, all that matters is that the number to the right is the image of the number on the left. So all these are equivalent expressions of α.

Example: Write $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 6 & 8 & 5 & 1 & 4 & 2 \end{pmatrix}$ in cycle form.

Cycle form is a compact way to represent a permutation. It still contains all the information, and it reveals more structure about the permutation than array form.

Convention: Leave off 1-cycles in cycle form, so any number not present in cycle form is assumed to map to itself.

Example: (a) Express the permutation in cycle form:

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 9 & 5 & 2 & 3 & 7 & 10 & 8 & 1 & 6 \end{pmatrix}$$

$\gamma =$

(b) For the permutation $\alpha = (1\ 5\ 3\ 7\ 2)(4\ 6\ 9)$ determine:

(i) $\alpha(3) =$

(ii) $\alpha(9) =$

(iii) $\alpha^2(1) =$

(iv) $\alpha(8) =$

Products of Permutations revisited:

Example: Find the product $\alpha\beta$ of $\alpha = (1357)$, $\beta = (14)(253)$

Properties of cycle form:

① Every permutation can be expressed as a product of disjoint cycles.

② Disjoint cycles commute (shirts and socks analogy)

Ex: $\alpha = (1\,3\,4)$, $\beta = (2\,5)$

Ex: For α, β above, determine $\beta\alpha\beta$.

If α, β commute, then

$$(\alpha\beta)^m = \alpha^m \beta^m \quad \text{for any } m$$

Inverse of a permutation revisited:

Ex: Find the inverse of $\alpha = (1\,4\,7\,3\,2)$.

① To find the inverse of a cycle, write the cycle backwards!

② If α is expressed as a product of disjoint cycles

$$\alpha = \sigma_1 \sigma_2 \cdots \sigma_k$$

then

$$\alpha^{-1} =$$

Taking ① & ② together:

To get from the cycle form of α to the cycle form of α^{-1} just write the representation for α down in the reverse order (both order of the cycles, and the numbers in the cycles).

Ex: Find the inverse of $\alpha = (2\,5\,4\,9)(3\,7)(6\,10\,8)$

Order of a permutation revisited:

Order of a permutation $\alpha \in S_n$ is the smallest number m for which $\alpha^m = \varepsilon$. We denote this number by $\text{ord}(\alpha)$.

We'll see how cycle form can be used to "eyeball" the order.

Ex: Determine the order of $\beta = (1\ 3\ 2\ 5)$

In general, the order of an m-cycle $(a_1\ a_2 \cdots a_m)$ is m.

Ex: Determine the order of $\alpha = (1\ 3)(2\ 4\ 5)$

Ex: What is the order of $\beta = (2\ 4\ 5)(3\ 1\ 7)(6\ 9\ 10\ 11)$?

What is the cycle structure of

β^3 ?

β^4 ?

Ex: If α has order 7, what is α^{35} ?
What about α^{106} ?

Theorem 4.4.1 — Order of a Permutation. The order of a permutation written in disjoint cycle form is the least common multiple of the lengths of the cycles.

Ex: The move sequence RU of Rubik's cube corresponds to the permutation consisting of a 15-cycle, a 3-cycle, and two 7-cycles

$$RU = \sigma_1 \sigma_2 \sigma_3 \sigma_4$$

15 cycle   3 cycle   7 cycles

What is the order?

Ex: Let $\alpha = (2\ 4\ 3)(1\ 5)$. If α^k is a 3-cycle what can you say about k?

Theorem 3.8.2 Let α be a permutation. If $\alpha^m = \varepsilon$ then $\text{ord}(\alpha)$ divides m.

Proof: