Representing Positions/Moves by Permutations :

Assume the puzzle pieces and their home positions have been labeled by numbers in $[n]$, where n is the number of moving pieces.

Definition 5.1.1 — Puzzle Position → Permutation. For a given position (scrambling) of the puzzle, the **permutation corresponding to this position** is $\alpha : S_n \to S_n$ where

$$\alpha(i) = j \quad \text{if the piece labelled } i \text{ is in the position labelled } j.$$

Ex : Write down the permutation corresponding to each position.

(a)

1	2	3	4	5
5	4	1	2	3

piece moved to box

(b)

1	2	3	4	5	6	7
7	4	6	1	3	2	5

(c)

1: 6	2: 9	3: 10	4: 13
5: 12	6: 4	7: 5	8: 3
9: 7	10: 8	11: 1	12: 14
13: 11	14:	15: 15	16: 2

Ex : Draw the puzzle position corresponding to the permutation

$\alpha = (1\ 5\ 3\ 2)(4\ 8)$

1	2	3	4	5	6	7	8

Definition 5.1.2 — Puzzle Move → Permutation. For a given move sequence applied to the puzzle, the **permutation corresponding to this move sequence** is $\beta : S_n \to S_n$ where

$$\beta(i) = j \quad \text{if the piece in position labelled } i \text{ moved to position labelled } j.$$

Ex: Write down the permutation corresponding to the move sequence.

(a)

1	2	3	4	5
5	3	1	4	2

$\xrightarrow{\alpha}$

1	2	3	4	5
5	3	2	4	1

piece in box   moved to box $\Big\} \Rightarrow \alpha =$

(b)

1	2	3	4	5	6
4	1	3	6	2	5

$\xrightarrow{\beta}$

1	2	3	4	5	6
6	5	4	3	2	1

(c) The move sequence corresponding to α is applied to the puzzle shown left. Determine the resulting position of the pieces.

1	2	3	4	5
4	5	1	3	2

$\xrightarrow{\alpha = (1\ 4\ 5)}$

1	2	3	4	5

Activity:

Try the following (fill out the table below):

1. For each arrangement of the tiles of the Swap puzzle write down the corresponding "position" permutation α

$$\alpha$$

2. Solve the puzzle using legal moves (swaps) write down the permutations corresponding to the moves

$$\beta_1, \beta_2, \cdots \beta_k$$

Observations: Is there a relationship between α and $\beta_1\beta_2\cdots\beta_k$?

α	$\beta_1, \beta_2, \ldots, \beta_k$	$\alpha\beta_1\beta_2\cdots\beta_k$

Conclusion/observation:

Activity 2: Now try the following:

1. Randomly place the tiles on the swap board, write down the permutation corresponding to the position, α.

2. Using legal moves rearrange the tiles into another configuration γ. Write down the permutations corresponding to moves.

$$\beta_1, \beta_2, \dots, \beta_k$$

Observation: What is the relationship between $\alpha, \beta_1, \beta_2, \dots, \beta_k$ and γ?

α	$\beta_1, \beta_2, \dots, \beta_k$	$\alpha\beta_1\beta_2 \cdots \beta_k$	γ

Conclusion/Observation:

Definition 5.1.1 — Puzzle Position $\to$ Permutation. For a given position (scrambling) of the puzzle, the **permutation corresponding to this position** is $\alpha : S_n \to S_n$ where

$$\alpha(i) = j \quad \text{if the piece labelled } i \text{ is in the position labelled } j.$$

Definition 5.1.2 — Puzzle Move $\to$ Permutation. For a given move sequence applied to the puzzle, the **permutation corresponding to this move sequence** is $\beta : S_n \to S_n$ where

$$\beta(i) = j \quad \text{if the piece in position labelled } i \text{ moved to position labelled } j.$$

Theorem 5.1.1 — Multiplying Moves. Let α be the permutation corresponding to the current position of the puzzle, and $\beta_1, \beta_2, \ldots \beta_k$ be a move sequence applied to the puzzle which results in a final position γ. Then

$$\alpha\beta_1\beta_2\cdots\beta_k = \gamma.$$

Consequences of Theorem:

① if $\alpha = \varepsilon$, then the product of moves results in the permutation corresponding to puzzle position.

② if $\gamma = \varepsilon$, then $\alpha = \beta_k^{-1}\beta_{k-1}^{-1}\cdots\beta_1^{-1}$.

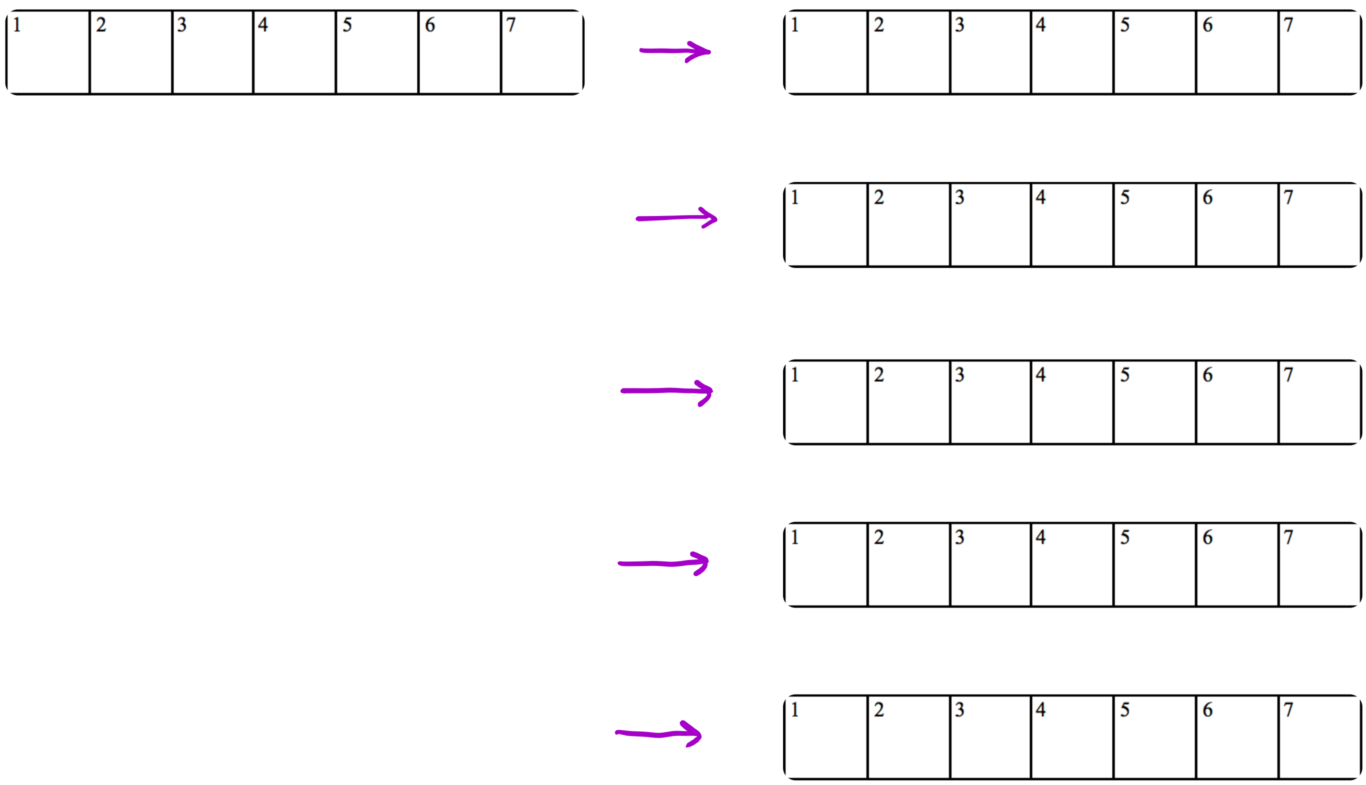
Therefore, solving α is equivalent to factoring α into puzzle moves.

Proof:

Permutations as a product of 2-cycles:

Ex: Write the permutation $\alpha = (1\,7\,4\,5)(2\,6\,3)$ as a product of 2-cycles.

Consider the corresponding arrangement of the swap puzzle, then solve it.



Ex: Write the 3-cycle $(1\,2\,3)$ as a product of 2-cycles.

Permutation as a product of 3-cycles:

Ex: Express the permutation $\alpha = (1\,4\,3\,2\,5)$ as a product of 3-cycles.

Consider the corresponding configuration of swap, and solve the puzzle using 3-cycles.

1	2	3	4	5

$\longrightarrow$

1	2	3	4	5

$\longrightarrow$

1	2	3	4	5

15 Puzzle:

Consider the following sequence of moves:

1	2	3	4
	15	9	8
14	6	5	12
11	7	10	13

$\xrightarrow{\tau_1}$

1	2	3	4
14	15	9	8
	6	5	12
11	7	10	13

$\xrightarrow{\tau_2}$

1	2	3	4
14	15	9	8
6		5	12
11	7	10	13

Puzzle:

moves:

position:

At home: Explore the representations with your own 15-puzzle by doing the following

1. randomly arrange pieces on the board, write out the permutation α.
2. perform a sequence of moves, write down the permutations $\tau_1, \tau_2, \ldots, \tau_k$
3. write down permutation for final arrangement: β

Verify $\alpha \tau_1 \tau_2 \cdots \tau_k = \beta$ and write the move sequences as a single permutation.

$$\sigma = \tau_1 \tau_2 \cdots \tau_k$$

Ex: Express, in cycle form, the permutation describing the move sequence.

^{1} 6	^{2} 1	^{3}
^{4} 3	^{5} 8	^{6} 7
^{7} 2	^{8} 5	^{9} 4

$$\xrightarrow{\alpha}$$

^{1} 1	^{2} 2	^{3} 3
^{4}	^{5} 5	^{6} 4
^{7} 7	^{8} 6	^{9} 8

Ex: The move sequence corresponding to α is applied to the puzzle shown on the left. Determine the resulting position of the pieces.

$$\alpha = (1\ 4\ 3\ 2\ 6\ 5)$$

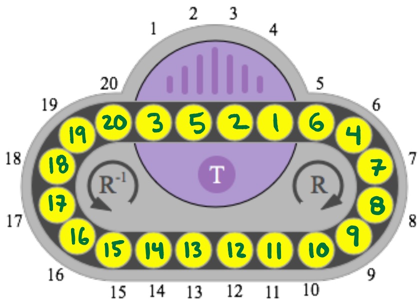
^{1} 4	^{2} 6	^{3} 2
^{4} 3	^{5} 1	^{6} 5
^{7}	^{8} 8	^{9} 7

$$\xrightarrow{\alpha}$$

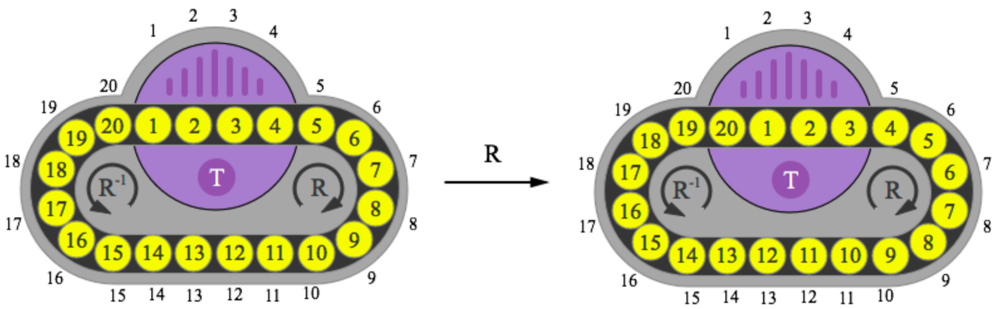
^{1}	^{2}	^{3}
^{4}	^{5}	^{6}
^{7}	^{8}	^{9}

Oval Track Puzzle :

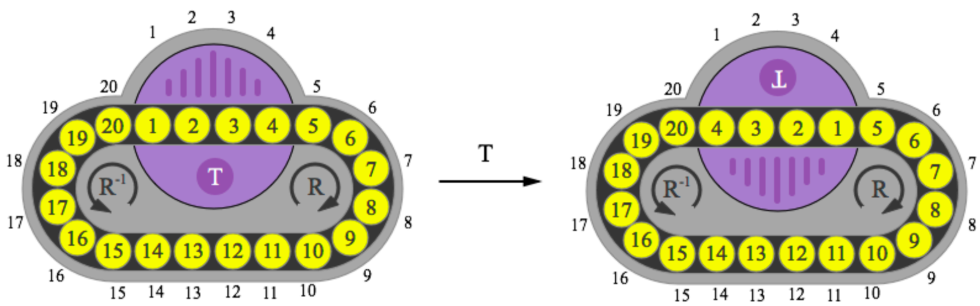
Exc : Express the puzzle position as a permutation in cycle form.



Permutations for the basic moves :



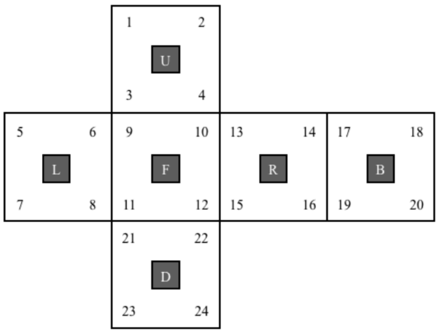
$$R = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20)$$



$$T = (1\ 4)(2\ 3)$$

Rubik's Cube:

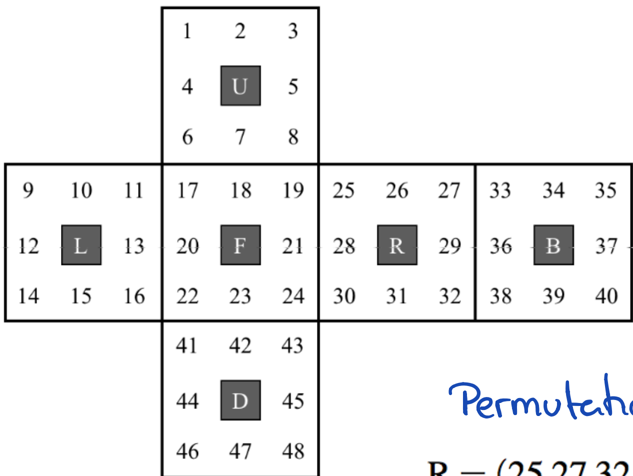
2×2×2 Cube : Labelling all facets



Permutations for cube moves:

R = (13 14 16 15)(10 2 19 22)(12 4 17 24)
L = (5 6 8 7)(3 11 23 18)(1 9 21 20)
U = (1 2 4 3)(9 5 17 13)(10 6 18 14)
D = (21 22 24 23)(11 15 19 7)(12 16 20 8)
F = (9 10 12 11)(3 13 22 8)(4 15 21 6)
B = (17 18 20 19)(1 7 24 14)(2 5 23 16)

3×3×3 Cube: Labelling of facets



Permutations for cube moves:

R = (25 27 32 30)(26 29 31 28)(3 38 43 19)(5 36 45 21)(8 33 48 24)
L = (9 11 16 14)(10 13 15 12)(1 17 41 40)(4 20 44 37)(6 22 46 35)
U = (1 3 8 6)(2 5 7 4)(9 33 25 17)(10 34 26 18)(11 35 27 19)
D = (41 43 48 46)(42 45 47 44)(14 22 30 38)(15 23 31 39)(16 24 32 40)
F = (17 19 24 22)(18 21 23 20)(6 25 43 16)(7 28 42 13)(8 30 41 11)
B = (33 35 40 38)(34 37 39 36)(3 9 46 32)(2 12 47 29)(1 14 48 27)