# Permutations : The Parity Theorem

## Individual Activity:

Start with the initial configuration

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 4 | 8 | 1 | 7 | 6 | 5 | 3 | 2 |

Count how many "swaps" it takes to solve the puzzle.

( Try solving it in different ways:

1. solve in numerical order (1 first, then 2, then 3 ...)
2. solve in reverse numerical order
3. solve in random order.
4. use only swaps involving box 1
5. use our "quick method" for writing a permutation as a product of 2-cycles.

⋮

whatever other method you can think of. )

Observation:

**Theorem 7.1.1 — The Parity Theorem.** If a permutation $\alpha$ can be expressed as a product of an even number of 2-cycles, then every decomposition of $\alpha$ into 2-cycles must have an even number. On the other hand, if $\alpha$ can be expressed as a product of an odd number of 2-cycles, then every decomposition of $\alpha$ into 2-cycles must have an odd number. In symbols, if

$$\alpha = \tau_1\tau_2\cdots\tau_r = \sigma_1\sigma_2\cdots\sigma_s$$

where the $\tau_i$'s and $\sigma_i$'s are 2-cycles, then $r$ and $s$ are both even or both odd.

Even permutation : one that can be expressed as a product of an EVEN number of 2-cycles.

Odd permutation : one that can be expressed as a product of an ODD number of 2-cycles.

Sign of a permutation :

$$\operatorname{sgn}(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is an even permutation} \\ -1 & \text{if } \alpha \text{ is an odd permutation} \end{cases}$$

Examples : Determine the parity of each of the following.

(a) $\varepsilon$

(b) $(1\ 2)$

(c) $(1\ 5\ 4)$

(d) $(1\ 7\ 3\ 5\ 6\ 8)$

(e) $(1\ 4\ 7)(2\ 6\ 3\ 10)(5\ 9)$

Parity of a cycle :

An $m$-cycle $(a_1 a_2 \cdots a_m)$ is even if $m-1$ is even and odd if $m-1$ is odd. Since

$$(a_1\ a_2\ \cdots\ a_m) = \underbrace{(a_1 a_2)(a_1 a_3)\cdots(a_1\ a_m)}_{m-1 \text{ transpositions}}$$

## Variation of Swap:

Legal move: Pick any 3 boxes and cycle their contents either to the left or right.

position is solvable $\iff$

$\iff$

Example: Consider the solvability of the following configuration.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 7 | 4 | 3 | 1 | 6 | 8 | 2 | 5 |

position is $\alpha =$

$=$

$\alpha$ is , therefore puzzle is with 3-cycles.

# Proof of Parity Theorem :

**Proposition 7.3.1** Any expression for the identity permutation $\varepsilon$ as a product of transpositions uses an even number of them. That is, if

$$\varepsilon = \tau_1 \tau_2 \cdots \tau_m$$

where the $\tau_i$'s are transpositions, then $m$ is an even integer.

Prop 1 $\Rightarrow$ Parity Theorem :

**Proposition 7.3.2** If there is an expression $\tau_1 \tau_2 \cdots \tau_m$ for the identity permutation $\varepsilon$ that uses $m$ transpositions, then there is an expression for $\varepsilon$ that uses $m-2$ transpositions.

Prop 2 $\Rightarrow$ Prop 1 ($\Rightarrow$ Parity Theorem)

## Proof of Proposition 2 :

Example: Consider the product

$$\varepsilon = (1\,3)(2\,5)(1\,4)(2\,3)(4\,5)(3\,5)(1\,4)(1\,2)$$

We will try to write it using two fewer transpositions. Consider the number 1, which appears in the rightmost transposition. Note

$$(1\,4)(1\,2) =$$

Proof of Prop 2: Let $\varepsilon = \tau_1\tau_2\cdots\tau_m$ and let $a$ be a number in the right-most transposition: $\tau_m = (a\,b)$. Then $\tau_{m-1}\tau_m$ can be expressed in one of the following ways:

1. $(a\,b)(a\,b) = \varepsilon$
2. $(a\,c)(a\,b) = (a\,b)(b\,c)$
3. $(c\,d)(a\,b) = (a\,b)(c\,d)$
4. $(b\,c)(a\,b) = (a\,c)(c\,b)$

In case 1 we omit $\tau_{m-1}\tau_m$ and get an expression for $\varepsilon$ using $m-2$ transpositions. In the other 3 cases we can push the right-most occurence of $a$ to the $m-1$ transposition. We can continue to push the right most occurence of $a$ to the left until eventually two consecutive transpositions cancel or the only occurrence of of $a$ appears in the first transposition. The latter would not fix $a$, hence could not happen. Therefore two transpositions must cancel. $\square$