The Alternating Group : (a.k.a the group of even permutations)

Permutations come in one of two types : even or odd

$$A_n = \{\alpha \in S_n : \alpha \text{ is even}\}$$
$$O_n = \{\alpha \in S_n : \alpha \text{ is odd}\}$$

$$S_n = A_n \cup O_n \quad , \quad \text{and} \quad A_n \cap O_n = \emptyset .$$

Properties of A_n

1. $\varepsilon \in A_n$
2. A_n is closed under composition :
$$\alpha, \beta \in A_n \Rightarrow \alpha\beta \in A_n$$
3. A_n is closed under taking inverses :
$$\alpha \in A_n \Rightarrow \alpha^{-1} \in A_n$$

Notice $\varepsilon \notin O_n$ and O_n is not closed under composition. In fact, if $\alpha, \beta \in O_n$ then $\alpha\beta \in A_n$.

A_n is called the Alternating Group of degree n.

Theorem 8.2.1 — Cardinality of A_n. $|A_n| = |O_n| = \dfrac{n!}{2}$, for $n \geq 2$.

Proof :

Example: List the elements of A_2, A_3, A_4.

Example: How many elements of order 5 are there in A_8?

Products of 3-cycles:

We know every element of S_n, for $n \geq 2$, can be expressed as a product of 2-cycles. We say

S_n is generated by 2-cycles.

Theorem 8.3.1 Every permutation in A_n, for $n \geq 3$, can be expressed as a product of 3 cycles.

Example: For $\alpha \in A_9$ write it as a product of 3-cycles:

$$\alpha = (1\ 3\ 7)(2\ 8\ 5\ 4)(6\ 9)$$

Swap Variation :

Variation : Legal move is to pick any 3 boxes and cycle their contents either to the left or to the right.

Observation : A permutation is obtainable from the solved state, through legal moves, if and only if it is expressible as a product of 3-cycles.

Corollary 8.4.1 — Solvability of Swap Variation. The Swap puzzle, where the legal moves consist of 3-cycles on any three boxes, is solvable if and only if the starting position is an even permutation.

Example : Determe the solvability of each puzzle in this variation of swap.

(a)

1	2	3	4	5	6	7	8
8	7	6	5	4	3	2	1

(b)

1	2	3	4	5	6	7	8
8	1	2	3	4	5	6	7

(c)

1	2	3	4	5	6	7	8
7	8	1	2	3	4	5	6

(d)

1	2	3	4	5	6	7	8
1	2	3	4	5	6	8	7