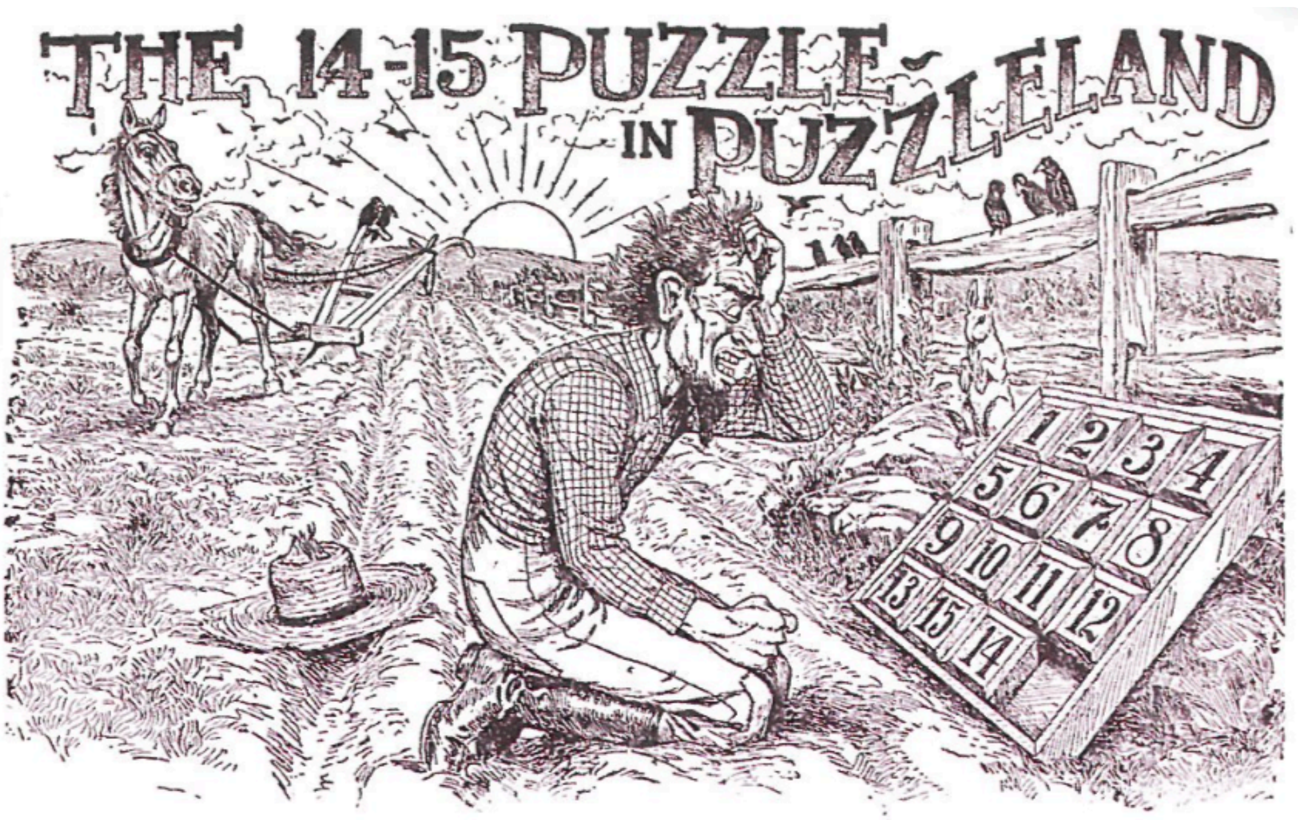
Gem Puzzle
Manufactured by Matthias Rice
December 1879 (Boston)

To solve the puzzle, you must first dump the fifteen pieces out, then randomly place them back in the box and slide them around until they are in "regular order".

When people proceeded to try to put the blocks in numerical order, something mysterious happened. They became fascinated with the puzzle, and could not put it down or stop trying to solve it.

The 15 Puzzle, Jerry Slocum



The Craze of 1880

In less than twenty-four hours every house in Beacon street contained at least one;

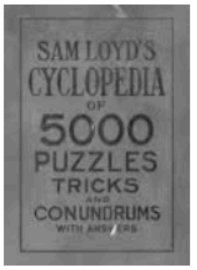
In less than a week the puzzles were to be bought at every store and any stall, and in less than a month it controlled the hour in banks and barrooms, legislation halls and gambling houses, hotels and horsecars, everywhere in short.

March 17, 1880, Boston Daily Advertiser

Salesmen left samples against the wishes of shop owners, who did not believe it would sell. But it did, better than any puzzle before.

To the minds of the Bostonians there was something more than a mere puzzle in the game: it was a mathematical study, and its solution a science.

Feb 27, 1880 , The New York Sun



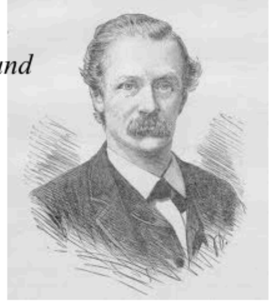
People became infatuated with the puzzle and ludicrous tales are told of shopkeepers who neglected to open their stores; of a distinguished clergyman who stood under a street lamp all through a wintery nigh trying to recall the way he had performed the feat.

The mysterious feature of the puzzle is that no one seems to be able to recall the sequence of moves whereby they feel sure they succeeded in solving the puzzle.

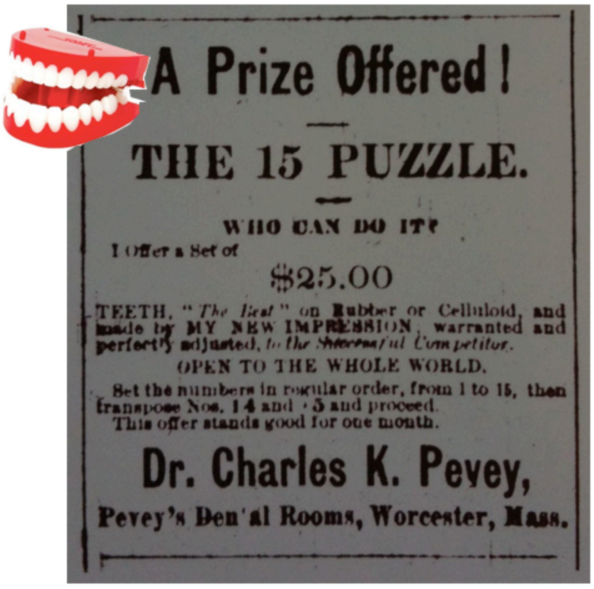
Pilots are said to have wrecked their ships, engineers rush their trains past stations and businesses generally became demoralized.

A famous Baltimore editor tells us how he went for his noon lunch and was discovered by his frantic staff long past midnight pushing little pieces of pie around on a plate!

Farmers are known to have deserted their plows.



Sam Loyd



A Prize Offered!

THE 15 PUZZLE.

WHO CAN DO IT?

I Offer a Set of

$25.00

TEETH, "*The Best*" on Rubber or Celluloid, and made by MY NEW IMPRESSION, warranted and perfectly adjusted, *to the Successful Competitor.*

OPEN TO THE WHOLE WORLD.

Set the numbers in regular order, from 1 to 15, then transpose Nos. 14 and 15 and proceed.

This offer stands good for one month.

Dr. Charles K. Pevey,

Pevey's Dental Rooms, Worcester, Mass.

Worcester Evening Gazette

January 24, 1880
Set of teeth, "the best".

January 29, 1880
teeth + $100

February 13, 1880
teeth + $100 + $1000

Solvability Criteria for the 15-puzzle:

Theorem 9.1.1 — Solvability Criteria for 15-Puzzle - Part 1. A permutation α of the 15-puzzle which fixes 16, is solvable if and only if it is even: i.e. $\alpha \in A_{15}$.

Example: Determine the solvability of the following configurations.

(a)

1: 13	2: 4	3: 7	4: 14
5: 5	6: 8	7: 1	8: 3
9: 2	10: 9	11: 6	12: 15
13: 12	14: 10	15: 11	16:

(b)

1: 6	2: 11	3: 15	4: 5
5: 13	6: 14	7: 1	8: 8
9: 9	10: 10	11: 4	12: 12
13: 3	14: 7	15: 2	16:

(c)

1: 5	2: 1	3: 3	4: 4
5: 2	6: 6	7: 7	8: 8
9: 9	10: 10	11: 11	12: 12
13: 13	14: 14	15: 15	16:

The number of solvable positions, where the empty space is in box 16, is

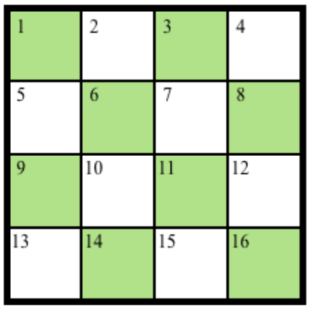
$$|A_n| = \frac{15!}{2!} = 653{,}837{,}184{,}000$$

Solvability in the case where the empty space is not in box 16 :

Example :

box 1: 1	box 2: 2	box 3: 3	box 4: 4
box 5: 5	box 6: 6	box 7: 7	box 8: 8
box 9: 9	box 10:	box 11: 10	box 12: 11
box 13: 13	box 14: 14	box 15: 15	box 16: 12

Parity of a Box :



Colour boxes in a checkerboard style.

Define :

shaded box <u>even</u>
white box <u>odd</u>

Reason for definition : If empty space is in a white box then it takes an <u>ODD</u> number of moves (swaps) to get it to box 16.
Similarly, it is an even number of moves from a shaded box to box 16.

> **Theorem 9.1.2 — Solvability Criteria for 15-Puzzle - Part 2.** A permutation of the 15-puzzle is solvable if and only if the parity of the permutation is the same as the parity of the location of the empty space.

Consequence : There are $\frac{16!}{2}$ solvable configurations, out of $16!$.

2,615,348,736,000

1950's : Manufactures still can't count the number of solvable configurations correctly!

Instructions say number of configurations is :
2,615,348,736,000
but the correct number is

$$\frac{16!}{2} = 8(15!) = 10,461,394,944,000$$

(i.e. 4 times larger)

Example: Determine the solvability of the following configurations.

(a)

9	5		2
3	6	1	8
7	13	4	12
14	10	15	11

(b)

1	2	3	4
5	6	7	8
14	12	9	
13	11	10	15

Proof of Thm 9.1.1 : α solvable $\iff$ α is even

1) α solvable $\Rightarrow$ α even

If α is solvable then

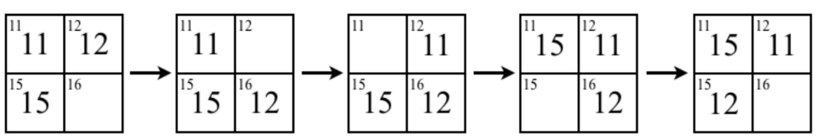
$$\alpha = \tau_k \tau_{k-1} \dots \tau_2 \tau_1$$

for some transpositions τ_i. Since the empty space must start in box 16 and return to box 16, it must haved moved an even number of times. Therefore k is even, and so α is an even permutation.

2) α even $\Rightarrow$ α solvable

Since every even permutation is a product of 3-cycles it suffices to show every 3-cycle can be obtained through puzzle moves.

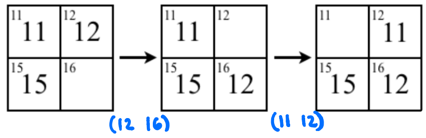
First note we can obtain the 3-cycle $\sigma = (11\ 12\ 15)$:



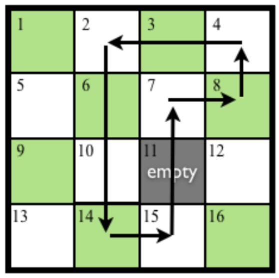
Using σ we can produce any 3-cycle of the form $(11\ 12\ i)$ for $i \neq 11, 12$

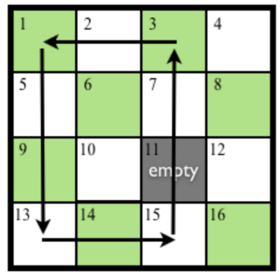
1) Pick any box i $(i \neq 11, 12)$

2) α is the move sequence that hides tiles from boxes 11, 12 into boxes 12, 16.



3) β is the move sequence that moves the tile in box i to box 15, using one of the following paths:





(Don't worry that this move seem to mix up othe tiles, those will get fixed as we proceed.)

4) apply α^{-1} (i.e. move tiles 11 and 12 back)

5) apply the 3-cycle $\sigma = (11\ 12\ 15)$

6) reverse steps 4 through 2.

The whole move sequence (steps 2–6) will be:

$$\alpha\beta\alpha^{-1}\sigma\alpha\beta^{-1}\alpha^{-1} = (\alpha\beta^{-1}\alpha^{-1}(11)\ \ \alpha\beta^{-1}\alpha^{-1}(12)\ \ \alpha\beta^{-1}\alpha^{-1}(15))$$

$$= (11\ 12\ i)$$

Finally, we can produce any 3-cycle $(j\ k\ i)$. To see how, first observe

$$(11\ 12\ k)(11\ 12\ j) = (11\ j)(12\ k)$$

and so

$$(11\ 12\ k)(11\ 12\ j)(11\ 12\ i)(11\ 12\ k)(11\ 12\ j)$$

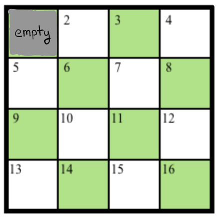
$$= (11\ j)(12\ k)(11\ 12\ i)(11\ j)(12\ k)$$

$$= (i\ j\ k)$$

Hence, we can produce any 3-cycle on the 15-puzzle. $\square$

Proof of Thm 9.1.2 : α soluable $\Leftrightarrow$ parity of α equals parity of box containing empty space

Reduce the problem:



Let α be the permutation representing the puzzles position.
Move the empty space to box 16 by moves $\tau_1, \dots, \tau_k$.
Let α^* be the resulting configuration

$$\alpha^* = \alpha \tau_1 \tau_2 \cdots \tau_k$$

α is soluble $\Leftrightarrow$ α^* is soluble
$\Leftrightarrow$ α^* is even (Thm 9.1.1)
$\Leftrightarrow$ parity of α equals parity of k
$\Leftrightarrow$ parity of α equals parity of box that contained the empty space

□

Strategy for solving the 15 puzzle :

We now know how to decide if a configuration is soluable or not. But we still don't have a strategy for actually solving it.

Activity : Randomly arange the tiles on the 15 puzzle board. Try to solve the puzzle.