Chapter 12 - Puzzle Groups

A **permutation puzzle** is a one person game (solitaire) with a finite set $T = \{1, 2, \ldots, n\}$ of puzzle pieces satisfying the following four properties:

1. For some $n > 1$ depending only on the puzzle's construction, each move of the puzzle corresponds to a unique permutation of the numbers in T,
2. If the permutation of T in (1) corresponds to more than one puzzle move then the two positions reached by those two respective moves must be indistinguishable,
3. Each move, say M, must be "invertible" in the sense that there must exist another move, say M^{-1}, which restores the puzzle to the position it was at before M was performed, In this sense, we must be able to "undo" moves.
4. If M_1 is a move corresponding to a permutation τ_1 of T and if M_2 is a move corresponding to a permutation τ_2 of T then $M_1 \cdot M_2$ (the move M_1 followed by the move M_2) is either
   - ~~not a legal move, or~~
   - corresponds to the permutation $\tau_1 \tau_2$.

won't consider this here

Let Puz be a permutation puzzle (without gaps)
i.e. Rubik's cube, Oval track, Hungarain rings, etc

Let M(Puz) be the set of all inequivalent move-sequences.

(two moves are considered equivalent if the two positions reached by these moves are indistinguishable.)

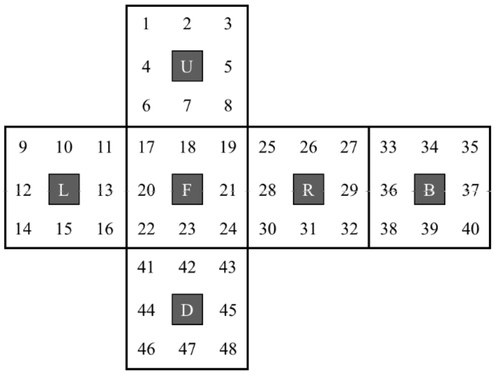
Theorem: M(Puz) is a group under move composition. It is called the <u>puzzle group</u>.

Rubik's Cube :

Let RC_3 denote the Rubik's cube group

$$RC_3 = \langle R, L, U, D, F, B \rangle$$

We can view RC_3 as a subgroup of S_{48} :



$R = (25\ 27\ 32\ 30)(26\ 29\ 31\ 28)(3\ 38\ 43\ 19)(5\ 36\ 45\ 21)(8\ 33\ 48\ 24)$
$L = (9\ 11\ 16\ 14)(10\ 13\ 15\ 12)(1\ 17\ 41\ 40)(4\ 20\ 44\ 37)(6\ 22\ 46\ 35)$
$U = (1\ 3\ 8\ 6)(2\ 5\ 7\ 4)(9\ 33\ 25\ 17)(10\ 34\ 26\ 18)(11\ 35\ 27\ 19)$
$D = (41\ 43\ 48\ 46)(42\ 45\ 47\ 44)(14\ 22\ 30\ 38)(15\ 23\ 31\ 39)(16\ 24\ 32\ 40)$
$F = (17\ 19\ 24\ 22)(18\ 21\ 23\ 20)(6\ 25\ 43\ 16)(7\ 28\ 42\ 13)(8\ 30\ 41\ 11)$
$B = (33\ 35\ 40\ 38)(34\ 37\ 39\ 36)(3\ 9\ 46\ 32)(2\ 12\ 47\ 29)(1\ 14\ 48\ 27)$

In [1]:
```
S48=SymmetricGroup(48)
R=S48("(25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)")
L=S48("(9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)")
U=S48("(1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)")
D=S48("(41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)")
F=S48("(17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)")
B=S48("(33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)")
RC3=S48.subgroup([R,L,U,D,F,B])    # define Rubik's cube group to be RC3
```

Determine order of the :

In [2]:
```
RC3.order()
```
Out[2]:
```
43252003274489856000
```
In [3]:
```
factor(RC3.order())
```
Out[3]:
```
2^27 * 3^14 * 5^3 * 7^2 * 11
```

$$|RC_3| = 2^{27} 3^{14} 5^3 7^2 \cdot 11 \approx 4.3 \times 10^{19}$$

Possible to flip an edge ?

In [4]:
```
S48("(7,18)") in RC3
```
Out[4]:
```
False
```

Possible to flip two edges ?

In [5]:
```
S48("(7,18)(5,26)") in RC3
```
Out[5]:
```
True
```

Rubik's 2×2×2 Cube :

RC_2 = [group of moves of the Pocket Cube]

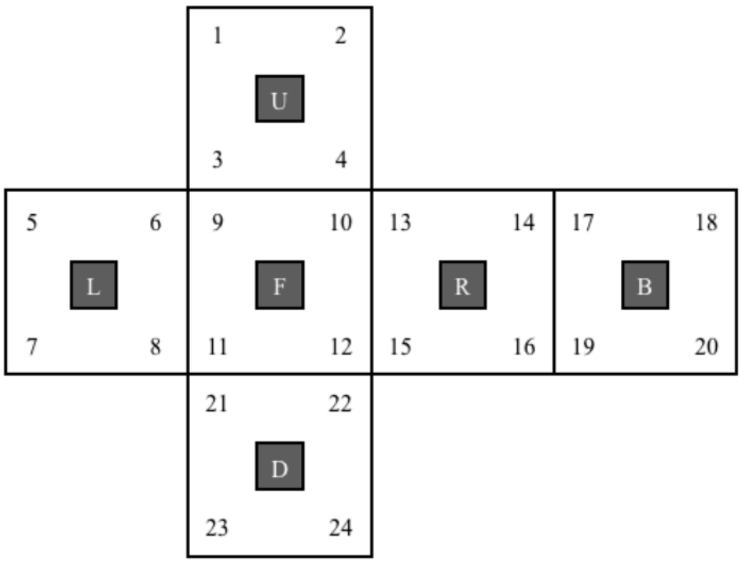
$= \langle R, L, U, D, F, B \rangle$

Note: $RL^{-1} = 1$ for pocket cube. Similarly, $UD^{-1} = FB^{-1} = 1$.

$$RC_2 = \langle R, D, F \rangle$$

These are the moves that keep the ubl corner fixed.

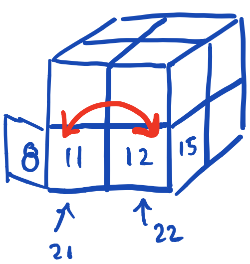
Viewing RC_2 as a subgroup of S_{24} :



$R = (13\ 14\ 16\ 15)(10\ 2\ 19\ 22)(12\ 4\ 17\ 24)$
$L = (5\ 6\ 8\ 7)(3\ 11\ 23\ 18)(1\ 9\ 21\ 20)$
$U = (1\ 2\ 4\ 3)(9\ 5\ 17\ 13)(10\ 6\ 18\ 14)$
$D = (21\ 22\ 24\ 23)(11\ 15\ 19\ 7)(12\ 16\ 20\ 8)$
$F = (9\ 10\ 12\ 11)(3\ 13\ 22\ 8)(4\ 15\ 21\ 6)$
$B = (17\ 18\ 20\ 19)(1\ 7\ 24\ 14)(2\ 5\ 23\ 16)$

$$|RC_2| = 2^4 \cdot 3^8 \cdot 5 \cdot 7$$
$$= 3{,}674{,}160$$

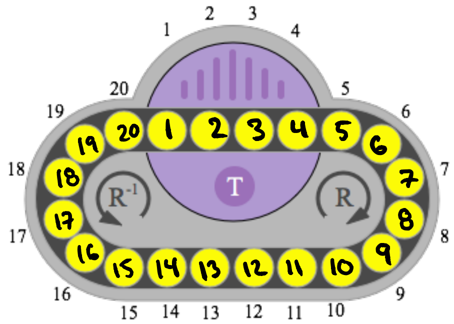
Corner swaps?



$(8\ 12)(11\ 15)(21\ 22) \in RC_2$

$(8\ 12\ 11\ 15\ 21\ 22) \notin RC_2$

Oval Track :



$OT = $ [group of moves on the oval track puzzle]

$$OT = \langle R, T \rangle \leq S_{20}$$

$$R = (1\ 2\ 3\ \dots\ 19\ \ 20)$$
$$T = (1\ 4)(2\ 3)$$

Since $|OT| = 20!$ then

$$OT = S_{20}.$$