# Chapter 13 - Commutators

**Definition 13.1.1** If $g$, $h$ are two elements of a group $G$, then we call the element

$$[g,h] = ghg^{-1}h^{-1}$$

the **commutator** of $g$ and $h$.

Note: If $g$ and $h$ commute then

$[g,h] =$

$[g,h]$ provides a measure for how much $g$ and $h$ fail to commute.

If $\alpha, \beta$ are permutations, and $\alpha, \beta$ fail to commute by "just a little bit" then $[\alpha,\beta]$ will be "close" to $\varepsilon$ i.e. it will only permute a few numbers.

Example: In $S_3$ let $\alpha = (1\ 3)$, $\beta = (1\ 2\ 3)$

## Creating Puzzle Moves with commutators:

We will concentrate on permutations in $S_n$.

Definitions: For $\alpha \in S_n$, define the <u>fixed set of</u> $\alpha$ by

$$\text{fix}(\alpha) = \{ m \in [n] \mid \alpha(m) = m \}$$

(This is just the set of numbers that would appear as 1-cycles in the disjoint cycle form of $\alpha$.)

The <u>moved set</u> of $\alpha$ is the complement of $\text{fix}(\alpha)$:

$$\text{mov}(\alpha) = \overline{\text{fix}(\alpha)} = \{ m \in [n] \mid \alpha(m) \neq m \}$$

(This is the set of all numbers which appear in cycles of length $\geq 2$ in the disjoint cycle form of $\alpha$.)

For $A \subset [n]$, and $\alpha \in S_n$ we define

$$\alpha A = \{ \alpha(a) \mid a \in A \}$$

called the <u>image of $A$ under $\alpha$</u>.

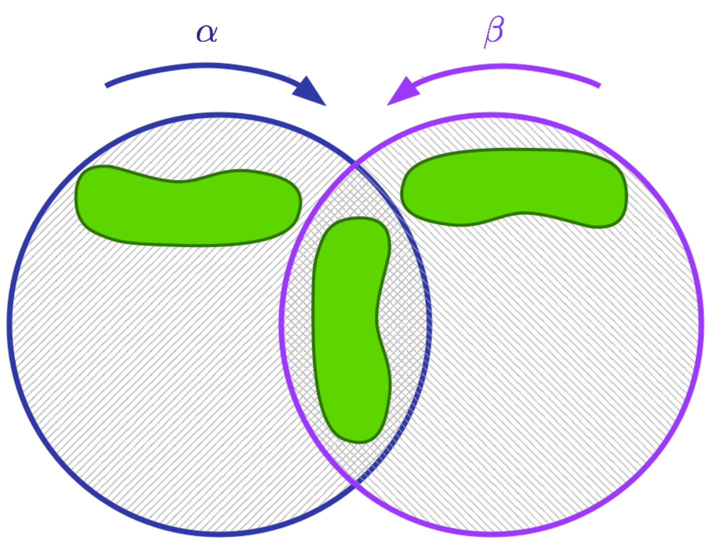
Note: $|\alpha A| = |A|$ since $\alpha$ is injective.

Example: Let $\alpha = (1\ 5\ 3\ 9)(4\ 11) \in S_{13}$

## <u>When is $[\alpha, \beta]$ close to the identity:</u>

We'll look at conditions for which $\mathrm{mov}([\alpha,\beta])$ is small.

First, consider the diagram:




$$\mathrm{mov}([\alpha,\beta]) \subset \mathrm{mov}(\alpha,\beta) \cup \alpha^{-1}\mathrm{mov}(\alpha,\beta) \cup \beta^{-1}\mathrm{mov}(\alpha,\beta)$$

If $m \in [n]$ is moved by $[\alpha,\beta]$, i.e. $m \in mov([\alpha,\beta])$ then both:

(a) $m \in mov(\alpha)$ or $\beta(m) \in mov(\alpha)$; and
(b) $m \in mov(\beta)$ or $\alpha(m) \in mov(\beta)$.

In other words,

$$(*) \qquad mov([\alpha,\beta]) = \left(mov(\beta) \cup \alpha^{-1} mov(\beta)\right) \cap \left(mov(\alpha) \cup \beta^{-1} mov(\alpha)\right)$$

Proof of (a), (b) :

(a)

(b) Proof similar to part (a). $\square$

Another way to write $(*)$ is

$$\boxed{mov([\alpha,\beta]) \subset \underbrace{mov(\alpha,\beta)}_{\substack{\text{pieces moved}\\ \text{by both } \alpha \\ \text{and } \beta \\ \text{(common intersection)}}} \cup \underbrace{\alpha^{-1} mov(\alpha,\beta)}_{\substack{\text{pieces moved}\\ \text{by } \alpha \text{ to common}\\ \text{intersection}}} \cup \underbrace{\beta^{-1} mov(\alpha,\beta)}_{\substack{\text{pieces moved}\\ \text{by } \beta \text{ to}\\ \text{common intersection.}}}.} \tag{13.2}$$

This says :

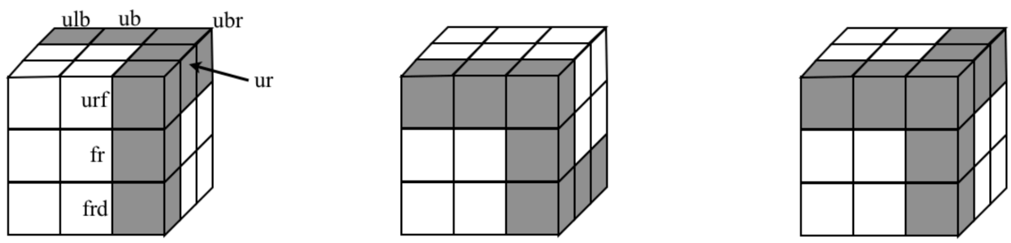
If $\alpha, \beta$ are puzzle moves then $[\alpha,\beta]$ only affects pieces that are in, or moved to, locations that are moved by both $\alpha$ and $\beta$.

To create a move which only affects a few pieces choose $\alpha$ and $\beta$ to have very little overlap.

# Creating Moves on Rubik's Cube :

## Puzzle moves $[x,y]$ :

$x = U$, $y = R$ $\qquad$ $x = F$, $y = R$ $\qquad$ $x = F$, $y = R^{-1}$




(a) Possible cubies moved by $URU^{-1}R^{-1}$.

(b) **Z-commutator**: Shading indicates locations changed by $FRF^{-1}R^{-1}$

(c) **Y-commutator**: Shading indicates locations changed by $FR^{-1}F^{-1}R$

Figure 13.1: Y- and Z- commutators

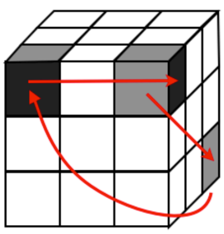


Figure 13.2: cubies moved by $[LD^2L^{-1}, U]$.

Consider

$x = LD^2L^{-1}$ (swaps rdb and lfu)

$y = U$

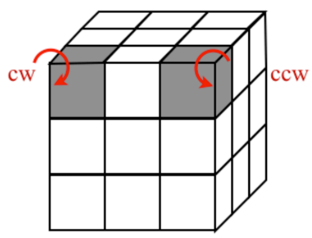
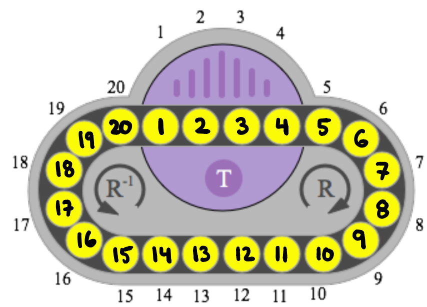



Figure 13.3: cubies moved by $[LD^2L^{-1}F^{-1}D^2F, U]$.

$x = LD^2L^{-1}F^{-1}D^2F$ (twists ufl )

$y = U$

# Oval Track Puzzle :

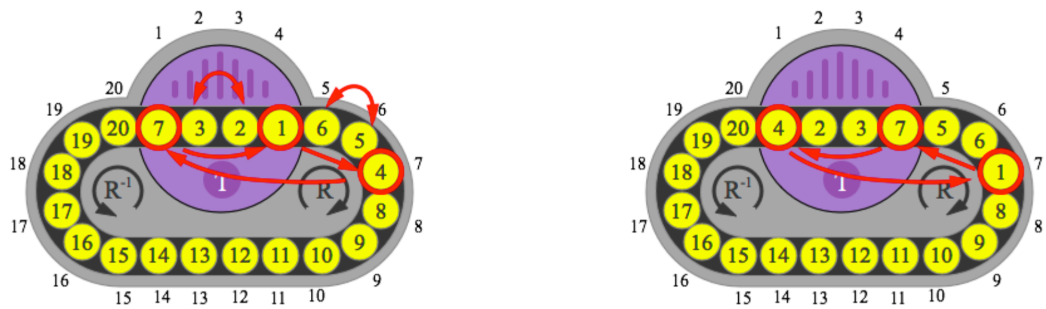



$\text{mov}(T) = \{1, 2, 3, 4\}$

$\text{mov}(R) = \{1, 2, \ldots, 20\}$

$[R^{-3}, T] = (1\,4\,7)(2\,3)(5\,6)$

$[R^{-3}, T]^2 = (1\,7\,4)$




(a) $R^{-3}TR^{3}T^{-1} = (1\,4\,7)(2\,3)(5\,6)$

(b) $(R^{-3}TR^{3}T^{-1})^2 = (1\,7\,4)$

Figure 13.8: Basic commutators on the Oval Track puzzle