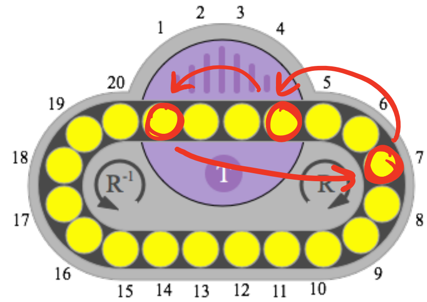
Chapter 15 - Oval Track



Creating 3-cycles :

Fundamental 3-cycle : $\gamma = [R^{-3}, T]^2 = (1\ 7\ 4)$

Lets create the 3-cycle $(1\ 2\ 3)$:

① Move 1,2,3 to 1,4,7 in any way, call this β^{-1} (note: "1 chases 2")

Ex: β^{-1} =

② Apply $\gamma^{-1} = (1\ 4\ 7)$

③ Apply β

Result : $\beta^{-1}\gamma\beta = (\beta(1)\ \beta(4)\ \beta(7)) = (1\ 2\ 3)$.

Create 3-cycle $(a\ b\ c)$:

Step ① Move tiles from a,b,c to 1,4,7 in any way. Call this β^{-1}.
(note " a chases b")
* there is enough flexibility in the puzzle to do this

Step ② Apply $\gamma = (1\ 4\ 7)$ or $\gamma^{-1} = (1\ 7\ 4)$ depending on where a,b,c are and recalling a chases b.

Step ③ Apply β.

Result : $\beta^{-1}\gamma\beta$ or $\beta^{-1}\gamma^{-1}\beta$ is $(a\ b\ c)$.

Theorem 15.1.1 — Solvability Criteria for Oval Track puzzle. For the Oval Track puzzle with 20 disks and $T = (1\ 4)(2\ 3)$, every permutation $\alpha \in S_{20}$ is solvable. In other words, $OT = S_{20}$.

Proof:

Creating a 2-cycle on the Oval Track:

Since $OT = S_{20}$ we know it is possible to create a 2-cycle. Lets try to find one.

Note TR^{-1} is a product of a 17-cycle and a 2-cycle:

$$TR^{-1} = (17\text{-cycle})(2\text{-cycle})$$

Therefore,

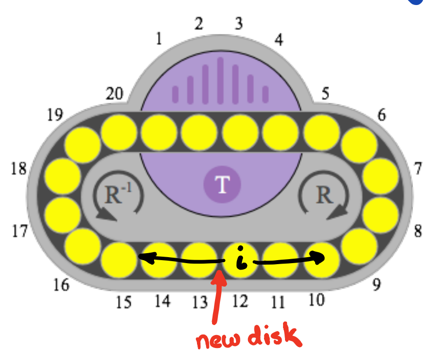
$$(TR^{-1})^{17} = (1\ 3)$$

This takes 34 moves to produce! Can we do better?

Theorem: If α is a 2-cycle $(a\,b)$ corresponding to a sequence of moves, then during the move sequence every piece (except possibly a,b) would have to be flipped in the turntable at least once.

In other words, a move sequence to produce α has to involve every piece on the track.

Proof: If α can be performed by not putting every piece $(\neq a,b)$ in the turntable, thus there is a piece, say the i^{th} disk that never gets put in the turntable. This disk just rocks back and forth, and eventually gets returned to its home location.



Add a new disk to the puzzle next to i, then α would just rock this new disk back and forth before sending it home. In other words α would produce a 2-cycle on the new 21-disk version of the puzzle.

This is impossible since on this 21-disk version both R and T are even, hence only even permutations are possible.

Therefore, if it is possible to produce a 2-cycle on the 20-disk puzzle all 18 pieces which get returned home would have been flipped in the turntable. $\square$

$(TR^{-1})^{17}$ does precisely this.

Oval Track – Strategy for solution

① Put disks 20 through 5 in numerical order.

② The permutation α of the final 4 disks is odd or even.
This is the endgame phase.

[Keep in mind the fundamental cycles:
$\sigma_3 = [R^{-3}, T]^2 = (1\ 7\ 4)$ & $\sigma_2 = (TR^{-1})^{17} = (1\ 3)$.]

(a) α is even:

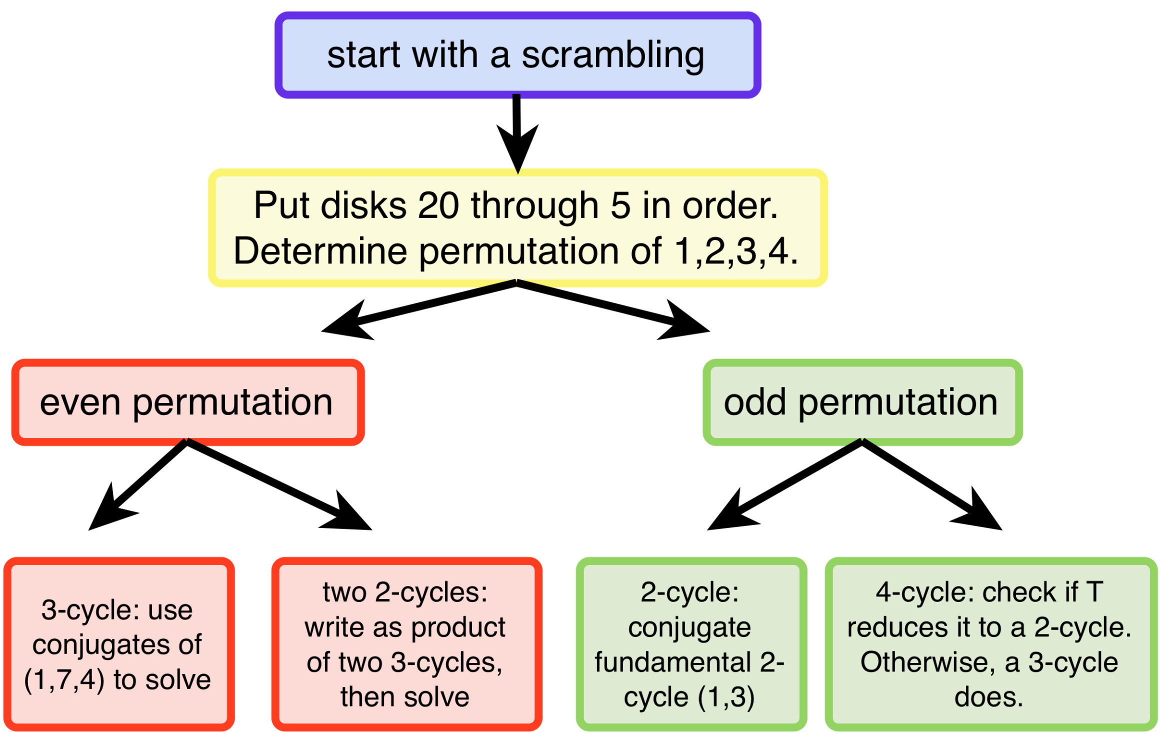
(i) α is a 3-cycle → use a conjugate of σ_3 or σ_3^{-1}

(ii) $\alpha = (\_\ \_)(\_\ \_)$ → if $\alpha = (1\,4)(2\,3)$ then apply T
else write it as two 3-cycles and
use conjugates of σ_3 or σ_3^{-1}

(b) α is odd:

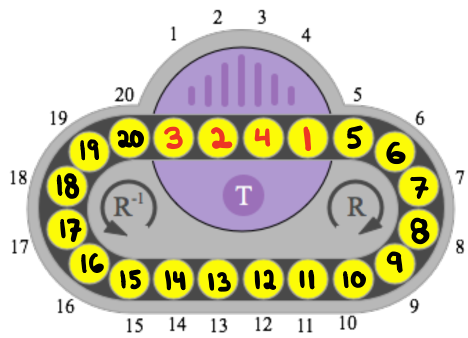
(i) α is a 2-cycle → use a conjugate of σ_2

(ii) α is a 4-cycle → check if T reduces it to a 2-cycle.
Otherwise, there is a 3-cycle that does;
then use a conjugate of σ_2.



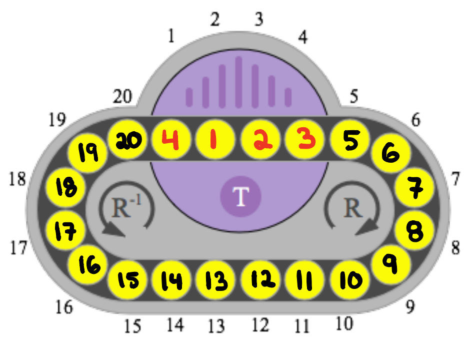
Examples:

1.



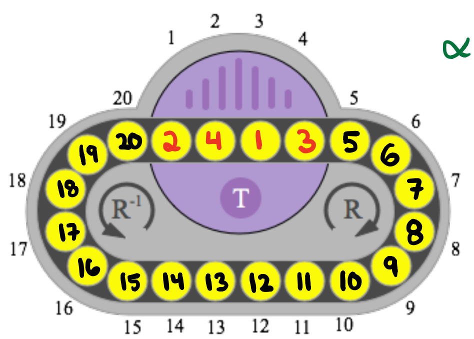
$\alpha = (1\ 4\ 3)$

2.



$\alpha = (1\ 2\ 3\ 4)$

3.



$\alpha = (1\ 3\ 4\ 2)$