**Definition 17.1.1** A **partition** of a set $A$ is a finite collection of non-empty subsets $A_1, A_2, \ldots, A_n$ satisfying the following properties.

(a) $A$ is the union of all the $A_i$'s: $A = A_1 \cup A_2 \cup \cdots \cup A_n$,

(b) the $A_i$'s are disjoint: $A_i \cap A_j = \emptyset$ for all $i \neq j$, $1 \leq i, j \leq n$.

Example : ① $\mathcal{P}$ = set of all movable pieces on Rubik's cube.
$\mathcal{P}$ can be partitioned into three sets :

$E$ = set of edge cubies
$V$ = set of corner (vertex) cubies
$C$ = set of centre cubies

$\mathcal{P} = E \cup V \cup C$

② $\mathbb{Z}$ can be partitioned into odd & even integers :
$$\mathbb{Z} = E \cup O$$
where $E$ = set of even integers , and $O$ = set of odd integers .

**Definition 17.2.1** Let $A$ be a set. A subset $\mathscr{R} \subset A \times A$ is called a **relation on** $A$. If $(x, y) \in \mathscr{R}$ then we say $x$ is related to $y$ (and we sometimes write $x\mathscr{R}y$ for simplicity).

Example: $\mathcal{C}$ = set of all configurations of Rubik's cube . Define $\mathcal{R}$ on $\mathcal{C}$ by

$X \mathcal{R} Y \iff Y$ can be obtained from $X$ by a quarter turn of one face (either cw or ccw).

**Definition 17.3.1** Let $\mathscr{R}$ be a relation on a set $A$. We call $\mathscr{R}$ an **equivalence relation** on $A$ if it satisfies the following properties:

(a) Each element is related to itself: $(a, a) \in \mathscr{R}$ for all $a \in A$ (reflexive property)

(b) If $a$ is related to $b$ then $b$ is related to $a$: $(a, b) \in \mathscr{R}$ implies $(b, a) \in \mathscr{R}$ (symmetric property)

(c) If $a$ is related to $b$, and $b$ is related to $c$ then $a$ is related to $c$: $(a, b) \in \mathscr{R}$ and $(b, c) \in \mathscr{R}$ implies $(a, c) \in \mathscr{R}$ (transitive property).

Notation : If $\mathcal{R}$ is an equivalence relation we often write $x \sim y$ or $x \equiv y$ in place of $(x,y) \in \mathcal{R}$ .

Example: Let $\mathcal{P}$ be the set of all people alive today. Consider the following relations :

| | reflexive | symmetric | transitive | equivalence |
|---|---|---|---|---|
| $x \mathcal{R}_1 y \iff x$ is a sister of $y$ | | | | |
| $x \mathcal{R}_2 y \iff x$ is a sibling of $y$ | | | | |
| $x \mathcal{R}_3 y \iff x$ is a child of $y$ | | | | |
| $x \mathcal{R}_4 y \iff x$ lives in the same city as $y$ | | | | |

**Definition 17.3.2** Let $\sim$ be an equivalence relation on a set $A$. For each $a \in A$ the set

$$[a] = \{x \in A \mid x \sim a\}$$

is called the **equivalence class of $A$ containing $a$**. We call $a$ a **representative** of the equivalence class $[a]$.

**Lemma 17.3.1** If $\sim$ is an equivalence relation on a set $A$ and $x, y \in A$, then

(a) $x \in [x]$ (an equivalence class contains its representative)

(b) $x \sim y$ if and only if $[x] = [y]$ (if two elements are related then their equivalence classes are equal)

(c) $[x] = [y]$ or $[x] \cap [y] = \emptyset$ (equivalence classes are either equal or disjoint).

Proof :

**Theorem 17.3.2** (a) If $A$ is a set and $\mathscr{R}$ is an equivalence relation on $A$ then the set of equivalence classes form a partition of $A$.

(b) If $A_1, \ldots, A_n$ is a partition of a set $A$ then the relation $\mathscr{R}$ defined by

$$a \mathscr{R} b \quad \text{if} \quad a, b \in A_i \text{ for some } i,$$

is an equivalence relation on $A$. This relation can written as

$$\mathscr{R} = \bigcup_{i=1}^{n} A_i \times A_i.$$

The sets $A_i$ are the equivalence classes of relation $\mathscr{R}$.

**Definition 17.3.3** If $\sim$ is an equivalence relation on a set $A$, then a **set of class representatives** is a subset of $A$ which contains exactly one element from each equivalence class. We denote the set of class representative by $A/\sim$.

Example: Define a relation $\equiv$ on $\mathbb{Z}$ by

$$a \equiv b \pmod 2 \iff b-a \text{ is divisible by } 2$$

reflexive ✓
symmetric ✓
transitive ✓

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In general, for $n \in \mathbb{Z}^+$ define an equivalence relation on $\mathbb{Z}$ by

$$a \equiv b \pmod n \iff n \mid b-a$$

We say <u>$a$ is congruent to $b$ modulo $n$</u>.

Equivalence class of $a$:

$$\begin{aligned} [a] &= \{ b \in \mathbb{Z} \mid a \equiv b \pmod n \} \\ &= \{ b \in \mathbb{Z} \mid b-a = kn, \ k \in \mathbb{Z} \} \\ &= \{ b \in \mathbb{Z} \mid b = a+kn, \ k \in \mathbb{Z} \} \\ &= \{ a+kn \mid k \in \mathbb{Z} \} \end{aligned}$$

Equivalence class representatives: $\mathbb{Z}/_{\equiv} = \{0, 1, 2, \dots, n-1\}$.