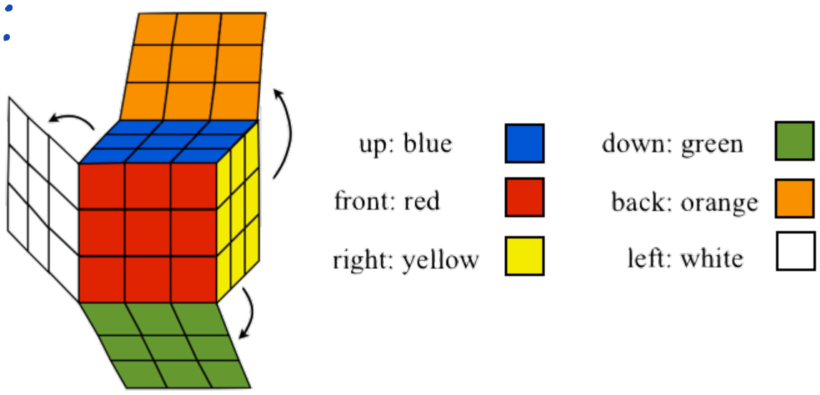
Chapter 20 – Fundamental Theorem of Cubology

Fix an orientation of the cube in space, we'll take blue up and red front.

Standard orientation:



Each configuration (scrambling) of the cube defines a 4-tuple:

$$(\rho, \sigma, \vec{v}, \vec{w})$$

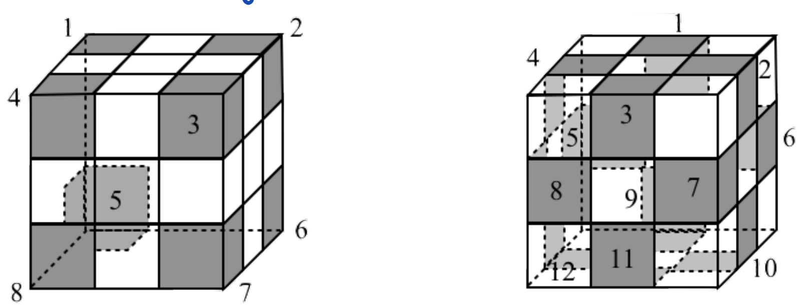
ρ : permutation of corner cubies, i.e. $\rho \in S_8$

σ : permutation of edge cubies, i.e. $\sigma \in S_{12}$

$\vec{v}$: encoding of orientation of corner cubies

$\vec{w}$: encoding of orientation of edge cubies

Permutations of corner and edge cubies:



Label every cubie and cubicle with a number:

corners : 1 – 8

edges : 1 – 12

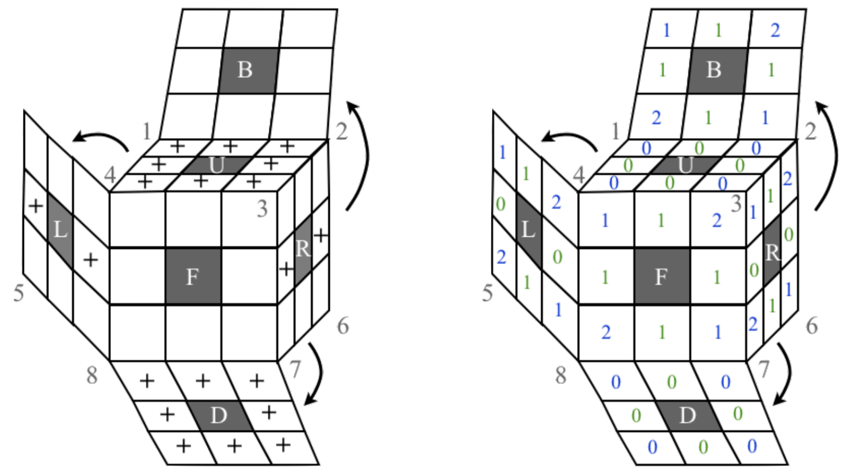
Orientation of corner and edge cubies:

1. Choose a primary facet for each cubicle, mark it with a "+".

   Imagine marking is on a thin layer of skin surrounding the puzzle and not on the pieces, i.e. markings do not move.

2. With the puzzle in the solved state, label all stickers with orientation markings (numbers 0,1,2):

   - These labels are on the stickers so they move with the pieces.
   - "0" is label of sticker in facet marked with a "+".



- corner stickers are labelled "0" under "+", and then "1", "2" in clockwise order around corner

(a) Marking the primary facets of a cubicle.

(b) Numbering the stickers of a cubie.

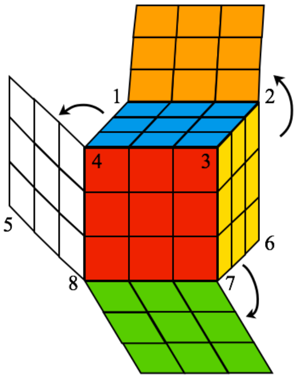
Figure 20.3: Orientation markings.

Definition 20.1.1 — Position vector of a configuration of cube pieces.. If X is any configuration of Rubik's cube the **position vector** is a 4-tuple (ρ, σ, v, w) where $\rho \in S_8$, $\sigma \in S_{12}$ encode the permutations of the cubies, and $v \in \mathbb{Z}_3^8$ and $w \in \mathbb{Z}_2^{12}$ encode the orientations of the cubies.

$$
\begin{array}{rll}
\rho \in S_8 : & \rho(i) = j & \text{if corner cubie } i \text{ is in cubicle } j. \\
\sigma \in S_{12} : & \sigma(i) = j & \text{if edge cubie } i \text{ is in cubicle } j. \\
v = (v_1, v_2, \ldots, v_8) \in \mathbb{Z}_3^8 = \{0,1,2\}^8 : & \multicolumn{2}{l}{v_i \text{ is the number on the } i^{\text{th}} \text{ corner cubie beneath}} \\
& \multicolumn{2}{l}{\text{the ``+" mark of the cubicle } \rho(i) \text{ it occupies.}} \\
w = (w_1, w_2, \ldots, w_{12}) \in \mathbb{Z}_2^{12} = \{0,1\}^{12} : & \multicolumn{2}{l}{w_i \text{ is the number on the } i^{\text{th}} \text{ edge cubie beneath}} \\
& \multicolumn{2}{l}{\text{the ``+" marking of the cubicle } \sigma(i) \text{ it occupies.}}
\end{array}
$$

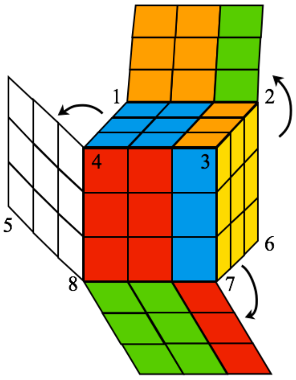
Examples: Determine the position vectors for each of the following configurations.

1.



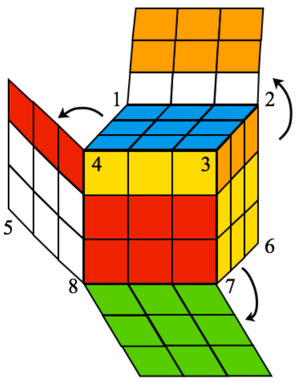
solved state

2.



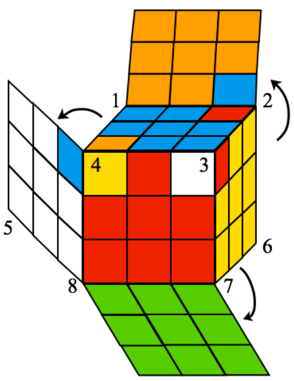
move R^{-1}

3.



move U

4.

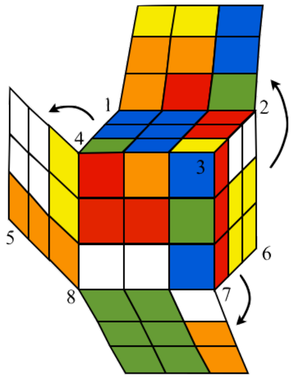


Move

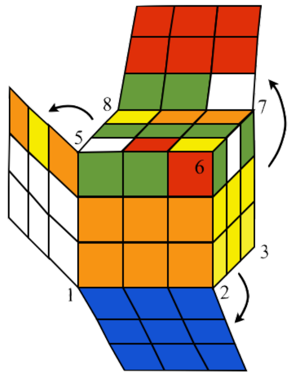
$B^{-1}[LD^2L^{-1}, U]B$

Additional Examples:

4.



5.



Not every 4-tuple $(\rho, \sigma, \vec{v}, \vec{w})$ corresponds to a soluble configuration of Rubik's cube. For example, a single edge flip (say the ur edge) corresponds to

$$(\varepsilon, \varepsilon, \vec{0}, (0,1,0,0,0,0,0,0,0,0,0,0))$$

and this configuration is not soluble.

The set

$$RC_3^* = S_8 \times S_{12} \times \mathbb{Z}_3^8 \times \mathbb{Z}_2^{12} = \{ (\rho, \sigma, \vec{v}, \vec{w}) : \rho \in S_8, \sigma \in S_{12}, \vec{v} \in \mathbb{Z}_3^8, \vec{w} \in \mathbb{Z}_2^{12} \}$$

is bigger than RC_3. This set is called the <u>Illegal Cube group</u>

RC_3^* is the set of ways to reassemble the cube (w/o taking apart the internal mechanism or peeling stickers).

The Fundamental Theorem of Cubology:

Theorem 20.2.1 — Fundamental Theorem of Cubology. A position vector $(\rho, \sigma, v, w) \in S_8 \times S_{12} \times \mathbb{Z}_3^8 \times \mathbb{Z}_2^{12}$ corresponds to a legal configuration of Rubik's cube if and only if the following three conditions are satisfied.

(a) $\operatorname{sign}(\rho) = \operatorname{sign}(\sigma)$

(b) $v_1 + v_2 + \cdots + v_8 = 0 \pmod 3$

(c) $w_1 + w_2 + \cdots + w_{12} = 0 \pmod 2$

In words,

- permutation of edges & permutation of corners have the same parity
- number of cw corner twists is equal to number of ccw corner twists modulo 3
- edge flips occur in pairs.

Corollary 20.2.2 — Impossible Configurations. Each of the following configurations cannot be obtained from the solved state cube through legal cube moves.

(a) Exactly two edge cubies are swapped.

(b) Exactly two corner cubies are swapped.

(c) Exactly one edge cubie is flipped.

(d) Exactly one corner cubie is twisted.

(e) Exactly two corner cubies are twisted in the same direction.

Corollary 20.2.3 — The Size of the Cube Group. The number of legal and illegal configurations of Rubik's cube are:

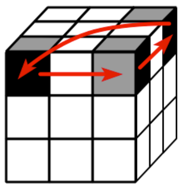
$$|RC_3| = |\mathscr{C}| = \frac{|RC_3^*|}{12} = 2^{27} \cdot 3^{14} \cdot 5^3 \cdot 7^2 \cdot 11 = 43{,}252{,}003{,}274{,}489{,}856{,}000 \approx 4.3 \cdot 10^{19}.$$

$$|RC_3^*| = |\mathscr{A}| = 8! \cdot 12! \cdot 3^8 \cdot 2^{12}.$$

Proof:

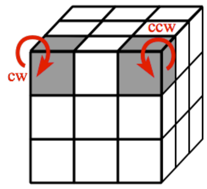
Recall some basic moves:

C1′



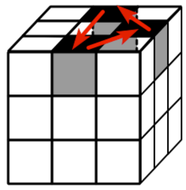
corner 3-cycle

C2



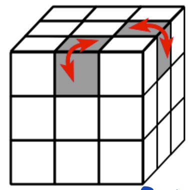
corner twist

E1′



edge 3-cycle

E2



edge flip

Proof of FTC:

($\Rightarrow$) First we show conditions (a)–(c) are satisfied for every legal configuration. To do this we just need to show

① conditions hold for the solved state

$$(\rho, \sigma, \vec{v}, \vec{w}) = (\varepsilon, \varepsilon, \vec{0}, \vec{0}) \quad \checkmark$$

② conditions remain invariant under cube moves

(a) remains satisfied since each move simultaneously multiplies ρ & σ by a 4-cycle.

(b) remains satisfied since

- U & D doesn't change $\vec{v}$
- R, L, F, B simultaneously increases 2 components of $\vec{v}$ by 1 and decreases 2 components by 1 (mod 3).

(c) remains satisfied since

- U, D, F, B doesn't change $\vec{w}$
- R, L increases two components by 1 (mod 2) and decreases two components by 1 (mod 2)

$\therefore$ X a legal configuration $\Rightarrow$ (a)–(c) satisfied.

($\Leftarrow$) Let $X = (\rho, \sigma, \vec{v}, \vec{w}) \in RC_3^*$ satisfy (a)–(c). We want to show X can be returned to the solved state by cube moves.

① Assume $\text{sign}(\rho) = \text{sign}(\sigma) = 1$ (otherwise apply R). Since both ρ & σ are even then all corner and edge cubies can be returned home using 3-cycles. Let $X' = (\varepsilon, \varepsilon, \vec{v}', \vec{w}')$ be the resulting configuration. By above $\vec{v}'$ & $\vec{w}'$ still satisfy (a)–(c).

② Since $\vec{v}'$ satisfies (b) then the number of cw twists equals the number of ccw twists (mod 3). First untwist pairs, then what is left is triple corners twisted in the same direction. These can be solved using corner twists as well.

③ Since $\vec{w}'$ satisfies (c) there are an even number of edges flipped. Flip them in pairs to solve.

$\therefore X \in RC_3$. $\square$

When are two assembled cubes equivalent? (requires Chapters 17 & 18)

Consider the equivalence relation $\sim_{RC_3}$ on RC_3^* :

$$X \sim_{RC_3} Y \iff X^{-1}Y \in RC_3$$

$$\iff X \text{ can be taken to } Y \text{ via cube moves}$$

$\sim_{RC_3}$ partitions RC_3^* into equivalence classes : the left cosets of RC_3 in RC_3^*.

$$[\#\text{ equivalence classes}] = [RC_3^* : RC_3] = |RC_3^*| / |RC_3|$$
$$= 12$$

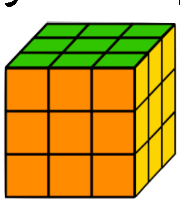
Let $C_{i,j,k}$ denote the conditions on elements in RC_3^* :

(a) $\operatorname{sign}(\rho) = \operatorname{sign}(\sigma) = i$ (where $i = 1$ or -1)
(b) $v_1 + \cdots + v_8 \equiv j \pmod 3$ (where $j = 0, 1$ or 2)
(c) $w_1 + \cdots + w_{12} = k \pmod 2$ (where $k = 0$ or 1)

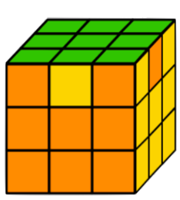
For each i,j,k, let $X_{i,j,k} \in RC_3^*$ be a configuration which satisfies $C_{i,j,k}$. Then

$$X_{i,j,k}RC_3, \quad i \in \{-1,1\},\ j \in \{0,1,2\},\ k \in \{0,1\}$$

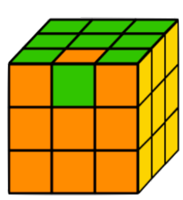
is the collection of all cosets of RC_3 in RC_3^*.



(a) $X_{(1,0,0)}$: solved



(b) $X_{(-1,0,0)}$: edge swap



(c) $X_{(1,0,1)}$: edge flip

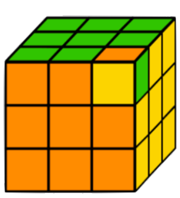
(d) $X_{(-1,0,1)}$: edge swap & edge flip

(e) $X_{(1,1,0)}$: counter-clockwise corner twist

(f) $X_{(-1,1,0)}$: ccw corner twist & edge swap

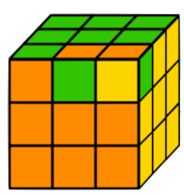
(g) $X_{(1,1,1)}$: ccw corner twist & edge flip

(h) $X_{(-1,1,1)}$: ccw corner twist & edge swap & edge flip

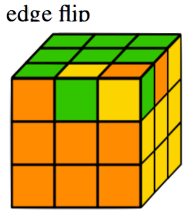


(i) $X_{(1,2,0)}$: clockwise corner twist

(j) $X_{(-1,2,0)}$: cw corner twist & edge swap

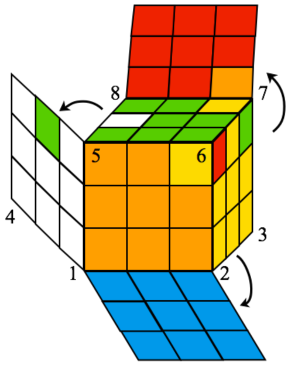


(k) $X_{(1,2,1)}$: cw corner twist & edge flip



(l) $X_{(-1,2,1)}$: cw corner twist & edge swap & edge flip

Example: Which equivalence class does this configuration belong to?



$\rho =$

$\sigma =$

$\vec{v} =$

$\vec{w} =$