ENSC-283

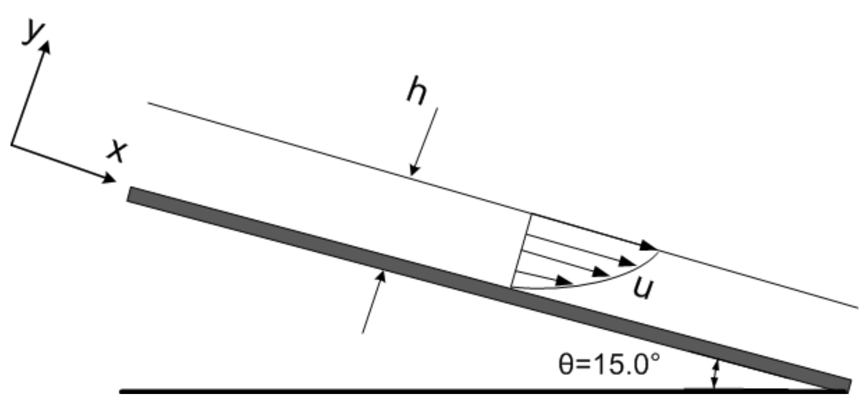
Assignment #7

Assignment date: Monday Mar. 9, 2009

Due date: Monday Mar. 16, 2009

Problem1: (Fully developed laminar flow down an inclined plane surface)

A liquid flows down an inclined plane surface in a steady, fully developed laminar film of thickness h. Simplify the continuity and Navier-Stokes equations to model this flow field. Obtain expressions for the liquid velocity profile, the shear stress distribution, the volume flow rate, and the average velocity. Relate the liquid film thickness to the volume flow rate per unit depth of surface normal to the flow. Calculate the volume flow rate in a film of water $h = 1\ mm$ thick, flowing on a surface $b = 1\ m$ wide, inclined at $\theta = 15°$ to the horizontal.



Solution:

Assumptions:

1. Steady flow.
2. Incompressible flow.
3. No flow or variation of properties in the z direction; $w = 0$ and $\frac{\partial}{\partial z} = 0$.
4. Fully developed flow, so no properties vary in the x direction; $\frac{\partial}{\partial x} = 0$.

Using assumptions (1) to (4) and continuity equation

$$\frac{\partial v}{\partial y} = 0$$

Since $v = 0$ at the solid surface (no-slip condition), it should be zero everywhere. The fact that $v = 0$, reduces the Navier-Stokes equation as follows:

$$0 = \rho g_x + \mu \frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$0 = \rho g_y - \frac{\partial p}{\partial y} \quad (2)$$

And all terms of momentum equation in z-direction cancel. Since u is only a function of y (why?), Equation (1) can be written as

$$-\frac{\rho g sin\theta}{\mu} = \frac{\partial^2 u}{\partial y^2} \quad (3)$$

By integrating twice

$$u = -\frac{\rho g sin\theta}{\mu}\frac{y^2}{2} + c_1 y + c_2 \quad (4)$$

Applying the boundary conditions, i.e. no-slip

$$u = 0 \;\; at \; y = 0$$

and zero shear stress at the free surface (why?)

$$\frac{du}{dy} = 0 \quad at\ y = h$$

gives

$$u = \frac{\rho g sin\theta}{\mu} y \left(h - \frac{y}{2}\right) \tag{5}$$

and the shear stress would be

$$\tau_{xy} = \mu \frac{du}{dy} = \rho g sin\theta (h - y) \tag{6}$$

which gives the maximum shear at the wall surface and zero shear at the free surface.

To find the volumetric flow rate

$$Q = \int_0^h u dy = \frac{\rho g sin\theta b}{\mu} \frac{h^3}{3} \tag{7}$$

The average velocity can then be found from the following equation

$$u_{ave} = \frac{Q}{bh} = \frac{\rho g sin\theta}{\mu} \frac{h^2}{3} \tag{8}$$

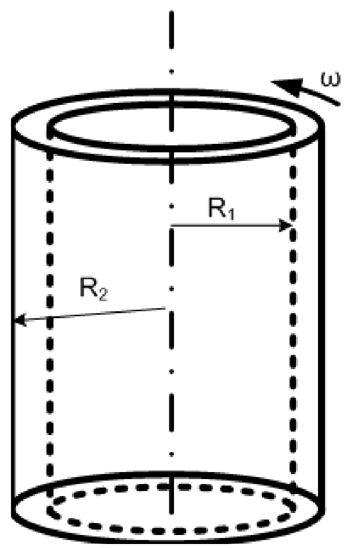
Solving for the film thickness gives

$$h = \left(\frac{3\mu Q}{\rho g sin\theta b}\right)^{\frac{1}{3}} \tag{9}$$

A film of water $h = 1mm$ thick on a plane with $b = 1m$ width, inclined at $\theta = 15°$, would carry

$$Q = 0.846 \left[\frac{Lit}{s}\right]$$

Problem2: (analysis of laminar volumetric flow between coaxial cylinders)
A viscous liquid fills the annular gap between vertical concentric cylinders. The inner cylinder is stationary, and the outer cylinder rotates at constant speed. The flow is laminar. Simplify the continuity, Navier-Stokes, and tangential shear stress equations to model this flow field. Obtain rxpressions for the liquid velocity profile and the shear stress distribution. Compare the shear stress at the surface of the inner cylinder with that computed from a planar approximation obtained by "unwrapping" the annulus into a plane and assuming linear velocity profile across the gap. Determine the ratio of cylinder radii for which the planar approximation predicts the correct shear stress at the surface of the inner cylinder within 1 percent.

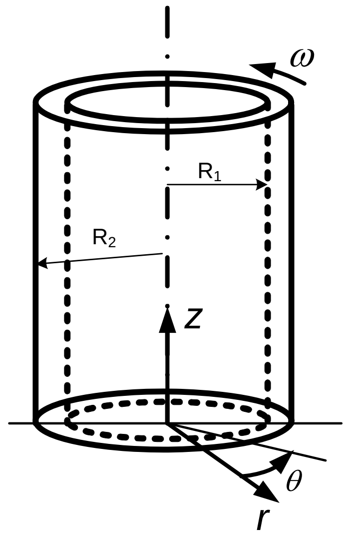


Solution:

Assumptions:

1. Steady flow.
2. Incompressible flow.
3. No flow or variation of properties in the *z* direction; $v_z = 0$ and $\frac{\partial}{\partial z} = 0$.
4. Axisymmetric flow, so no properties vary in the θ direction; $\frac{\partial}{\partial \theta} = 0$.

Navier-Stokes equation in cylindrical coordinate should be solved for this problem. Considering the coordinate system as shown, $g_r = g_\theta = 0$ and $g_z = -g$.



With the assumptions we made and continuity equation (why we can use d/dr instead of $\partial/\partial r$?)

$$\frac{1}{r}\frac{d(rv_r)}{dr} = 0$$

which gives $rv_r = constant$. Since velocity is zero at the rigid wall the constant should be zero and thus $v_r = 0$ everywhere.
The fact that $v_r = 0$ reduces the Navier-Stokes equation as follows:

$$-\rho\frac{v_\theta^2}{r} = -\frac{\partial p}{\partial r}; \quad r - momentum \tag{1}$$

$$0 = \mu\left\{\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}[rv_\theta]\right)\right\}; \quad \theta - momentum \tag{2}$$

Integrating twice with respect to r gives

$$v_\theta = c_1\frac{r^2}{2} + \frac{c_2}{r} \tag{3}$$

Applying no-slip boundary conditions at inner and outer cylinders i.e.

$v_\theta = \omega R_2 \quad at \quad r = R_2$ and $v_\theta = 0 \quad at \quad r = R_1$ gives

$$v_\theta = \frac{\omega R_1}{1-\left(\frac{R_1}{R_2}\right)^2}\left[\frac{r}{R_1}-\frac{R_1}{r}\right] \tag{4}$$

Shear stress can be obtained from the following equation

$$\tau_{r\theta} = \mu r \frac{d}{dr}\left(\frac{v_\theta}{r}\right) \tag{5}$$

Substituting v_θ from Eq. (4) into Eq. (5) at $r = R_1$ gives

$$\tau_{surface} = \mu \frac{2\omega}{1-\left(\frac{R_1}{R_2}\right)^2} \tag{6}$$

For a "planar" gap,

$$\tau_{planar} = \mu \frac{\Delta v}{\Delta y} = \mu \frac{\omega R_2}{R_2 - R_1} = \frac{\mu\omega}{1-\frac{R_1}{R_2}} \tag{7}$$

Thus

$$\frac{\tau_{surface}}{\tau_{planar}} = \frac{2}{1+\frac{R_1}{R_2}} \tag{8}$$

For 1% accuracy

$$\frac{2}{1+\frac{R_1}{R_2}} = 1.01 \; or \; \frac{R_1}{R_2} = 0.98$$

As a result the planar assumption is accurate for very narrow gaps between two rotating cylinders.