# SFU Differential relations for fluid flow

- In this approach, we apply basic conservation laws to an infinitesimally small control volume.

- The differential approach provides point-by-point details of a flow pattern as oppose to control volume technique that provide gross-average information about the flow.

# Acceleration field of a fluid

The Cartesian vector form of a velocity field in general is:

$$V(r,t) = \vec{i}\,u(x,y,z,t) + \vec{j}\,v(x,y,z,t) + \vec{k}\,w(x,y,z,t)$$

The acceleration vector field can be calculated:

$$\frac{du(x,y,z,t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

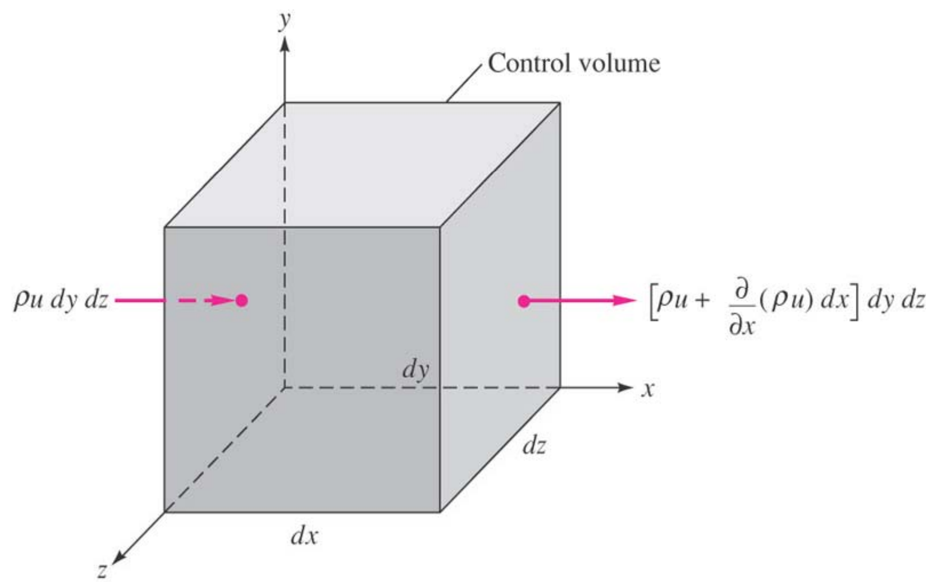
$$= \frac{\partial u}{\partial t} + (V.\nabla)u$$

The total acceleration vector:

$$a = \frac{dV}{dt} = \frac{\partial V}{\partial t} + \left(u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial V}{\partial t} + (V.\nabla)V$$

# Conservation of mass

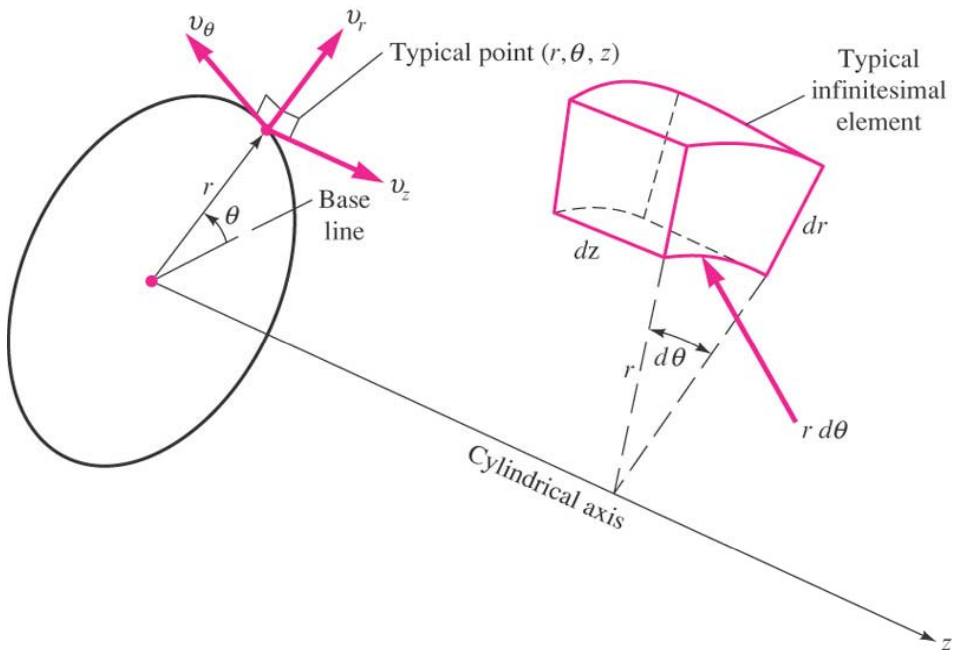



The conservation of mass for the element can be written as:

$$\frac{\partial \rho}{\partial t} dx\, dy\, dz + \frac{\partial}{\partial x}(\rho u) dx\, dy\, dz + \frac{\partial}{\partial y}(\rho v) dx\, dy\, dz + \frac{\partial}{\partial z}(\rho w) dx\, dy\, dz = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = \frac{\partial \rho}{\partial t} + \nabla . (\rho V) = 0$$

# Cylindrical polar coordinates




The continuity equation in cylindrical coordinates become:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

# Special cases

For steady compressible flow, continuity equation simplifies to:

$$\begin{cases} Cartesian, & \dfrac{\partial}{\partial x}(\rho u) + \dfrac{\partial}{\partial y}(\rho v) + \dfrac{\partial}{\partial z}(\rho w) = 0 \\ Cylinderical, & \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\rho v_r) + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}(\rho v_\theta) + \dfrac{\partial}{\partial z}(\rho v_z) = 0 \end{cases}$$

For incompressible flow, continuity equation can be further simplified since density changes are negligible:

$$\nabla . V = 0$$

$$\begin{cases} Cartesian, & \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \\ Cylinderical, & \dfrac{1}{r}\dfrac{\partial}{\partial r}(r v_r) + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}(v_\theta) + \dfrac{\partial}{\partial z}(v_z) = 0 \end{cases}$$

Note: the continuity equation is always important and must always be satisfied for a rational analysis of a flow pattern.

# Linear momentum equation

In a Cartesian coordinates, the momentum equation can be written as:

$$\sum F = \rho \frac{dV}{dt} dx\ dy\ dz$$

There are types of forces: <u>body forces</u> and <u>surface forces</u>.

**Body forces** are due to external fields such as gravity and magnetism fields. We only consider gravity forces:

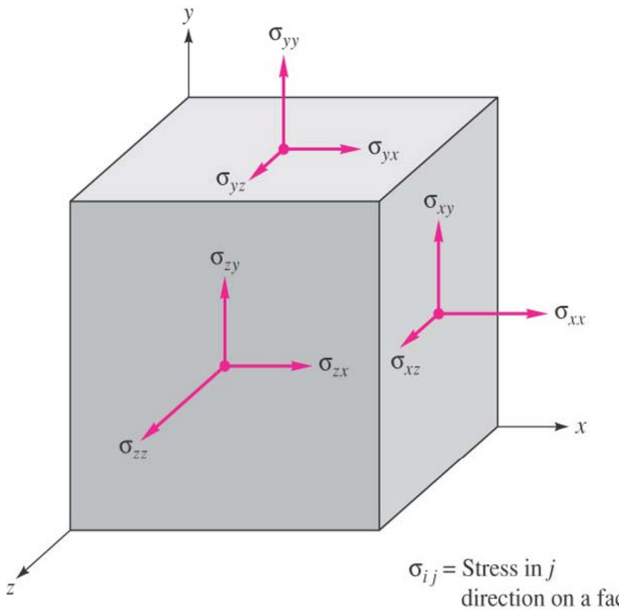
$$dF_{grav} = \rho \vec{g} dx\ dy\ dz \qquad where \qquad \vec{g} = -g\vec{k}$$

**Surface forces** are due to the stresses on the sides of the control surface.

These stresses are the sum of hydrostatic pressure plus viscous stresses which arise from the motion of the fluid.

# Stress tensor

Unlike velocity, stresses and strains are nine-component tensors and require two subscripts to define each component.




$\sigma_{ij}$ = Stress in $j$ direction on a face normal to $i$ axis

$$\sigma_{ij} = \begin{vmatrix} -p+\tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p+\tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p+\tau_{zz} \end{vmatrix}$$

The net surface force due to stresses in the x-direction can be found as:

$$dF_{x,surf} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dx\, dy\, dz$$

# Momentum equation cont'd.

Similarly we can find the net surface force in *y* and *z* direction. After summing them up and dividing through by the volume:

$$\left(\frac{dF}{dxdydz}\right)_{Viscous} = i\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)+j\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)+k\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right)$$

$$= \nabla.\tau_{ij}$$

Therefore the linear momentum equation for an infinitesimal element becomes:

$$\rho g - \nabla p + \nabla.\tau_{ij} = \rho \frac{dV}{dt} \qquad \frac{dV}{dt} = \frac{\partial V}{\partial t} + \left(u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}\right)$$

This is a vector equation, and can be written as:

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

# Special cases of momentum eq.

**Euler's equation** (inviscid flow), when the viscous terms are negligible:

$$\rho g - \nabla p = \rho \frac{dV}{dt}$$

**Navier-Stoke equation** (Newtonian fluid), For a Newtonian fluid, the viscous stresses are proportional to the element strain rates and the coefficient of viscosity.

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \qquad \tau_{yy} = 2\mu \frac{\partial v}{\partial y} \qquad \tau_{zz} = 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

For a Newtonian fluid with constant density and viscosity, we get:

$$\rho \frac{dV}{dt} = \rho g - \nabla p + \mu \nabla^2 V$$

# Navier-Stokes equation cont'd

Incompressible flow Navier-stokes equations with constant density.

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

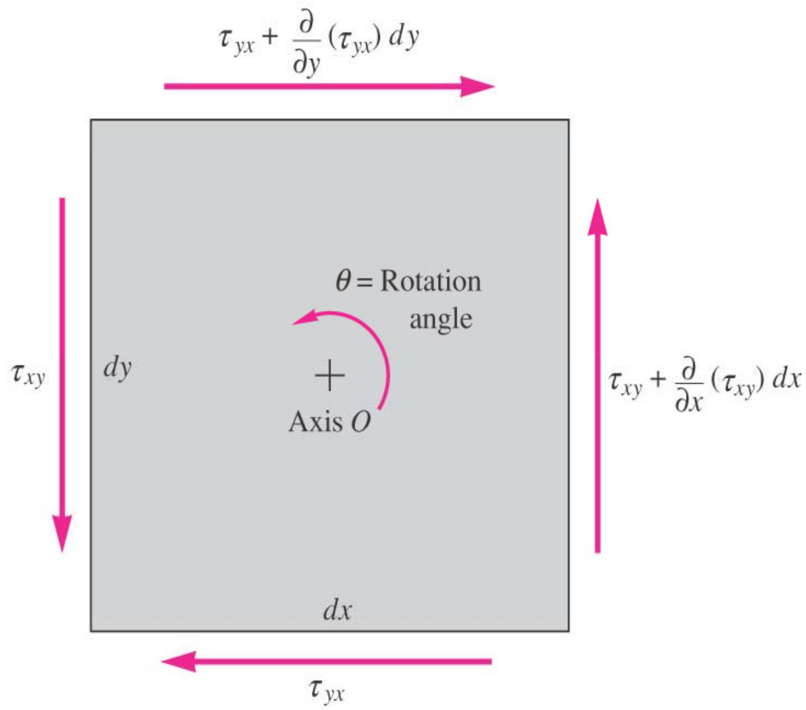
Navier-Stokes equations have 4 unknowns: *p, u, v*, and *w.* They should be combined with the continuity equation to form four equations for theses unknowns.

Navier-Stokes equations have a limited number of analytical solutions; these equations typically are solved numerically using computational fluid dynamics (CFD) software and techniques.

# Angular momentum equation

Application of the integral theorem to a differential element gives that the shear stresses are symmetric:

$$\tau_{ij} = \tau_{ji}$$




Therefore, there is no differential angular momentum equation.

# Boundary conditions

We have 3 equations to solve: i) continuity equation, ii) momentum, and iii) energy with 5 unknowns: *ρ, V, p, u* and *T*.

We use data or algebraic expressions for state relations of thermodynamic properties such as ideal gas equation of state:

$$\rho = \rho(p,T) \quad and \quad \hat{u} = \hat{u}(p,T)$$

# Important boundary conditions

At solid wall: $V_{fluid} = V_{wall}$ (no-slip condition) $T_{fluid} = T_{wall}$ (no-temperature jump)

At inlet or outlet section of the flow: *V, p, T* are known

At a liquid-gas interface: equality of vertical velocity across the interface (kinematic boundary condition)

Mechanical equilibrium at liquid-gas interface $(\tau_{zx})_{liq} = (\tau_{zx})_{gas}$ $(\tau_{zy})_{liq} = (\tau_{zy})_{gas}$

At a liquid-gas interface: heat transfer must be the same $(q_z)_{liq} = (q_z)_{gas}$

$$\left(k\frac{\partial T}{\partial z}\right)_{liq} = \left(k\frac{\partial T}{\partial z}\right)_{gas}$$

# SFU Incompressible flow const properties

Flow with constant *ρ, μ*, and *k* is a basic simplification that is very common in engineering problem that leads to:

Continuity equation:

$$\nabla . V = 0$$

Momentum equation:

$$\rho \frac{dV}{dt} = \rho g - \nabla p + \mu \nabla^2 V$$

For frictionless or inviscid flows in which *μ=0*. The momentum equation reduces to Euler's equation:

$$\rho \frac{dV}{dt} = \rho g - \nabla p$$