Dimensional Analysis & Similarity

- Dimensional analysis is very useful for planning, presentation, and interpretation of experimental data.

- Dimensional analysis is a method for reducing the number and complexity of experimental variables that affect a given physical phenomena.

Dimensional Analysis

- Reduce the number of variables: Suppose the force *F* on a particular body shape immersed in a stream of fluid depends only on the body length *L*, velocity *V*, fluid density and viscosity µ:

$$F = f(L, V, \rho, \mu)$$

- Using dimensional analysis, we can reduce the parameters to only one:

$$\frac{F}{\rho L^2 V^2} = g\left(\frac{\rho V L}{\mu}\right) = g(Re)$$

- Non-dimensional equations: that will provide insight on controlling parameters and the nature of the problem.

- Scaling laws: that allows testing models instead of expensive large full-scale prototypes.

$$Re_m = Re_p$$

$$\frac{F_p}{F_m} = \frac{\rho_p}{\rho_m}\left(\frac{V_p}{V_m}\right)^2 \left(\frac{L_p}{L_m}\right)^2$$

Dimensional homogeneity

- In a dimensionally homogenous relationship for a physical process, each term will have the same dimensions.

- For example in the Bernoulli equation, each term, including the constant, has dimensions of velocity squared $[L^2T^{-2}]$.

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = const$$

- The <u>variables</u> are the things we wish to plot, the basic output of the experiment or theory. The <u>parameters</u> are those quantities whose effects on the variables we wish to know.

- For example, consider the relation that expresses the displacement of a falling body. Variables (S, t) and parameters (S_0, V_0, g)

$$S = S_0 + V_0 t + \frac{1}{2} g t^2$$

To non-dimensionalize our results, we need to know how many dimensions are contained among our variables (S, t) and parameters (V_0, S_0, and g); in this case only two, length {L} and time {T}:

$$\{S\} = \{S_0\} = \{L\} \qquad \{t\} = \{T\} \qquad \{V_0\} = \{LT^{-1}\} \qquad \{g\} = \{LT^{-2}\}$$

• We select any two of the three parameters to be <u>scaling parameters</u>, thus we have three choices:
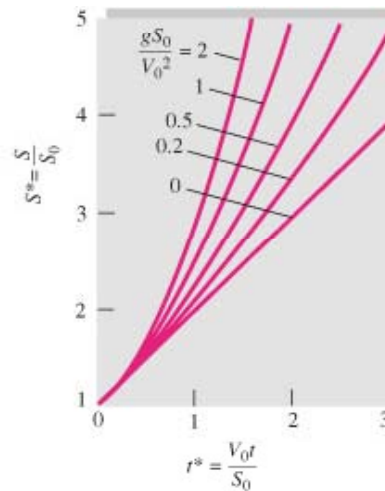
- Using these scaling parameters, we define non-dimensional displacement and time as:

$$S^* = \frac{S}{S_0} \qquad t^* = \frac{V_0 t}{S_0}$$

- Substituting these variables into the falling-body equation, after simplifications:

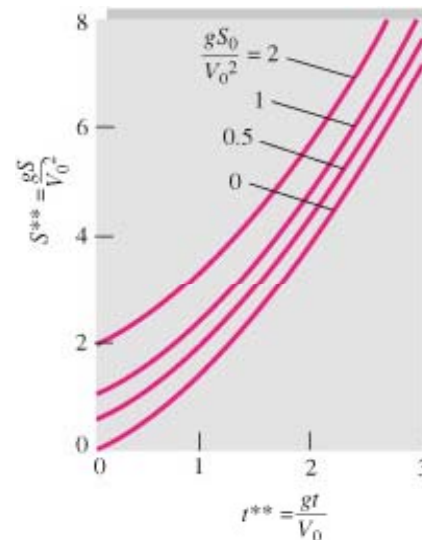$$S^* = 1 + t^* + \frac{1}{2}\alpha t^{*2} \qquad \alpha = \frac{g S_0}{V_0^2}$$

- The non-dimensional parameters will be:

$$S^{**} = \frac{Sg}{V_0^2} \qquad t^{**} = \frac{tg}{V_0}$$

- Substituting these variables into the falling-body equation, after simplifications:

$$S^{**} = \alpha + t^{**} + \frac{1}{2}t^{**2} \qquad \alpha = \frac{gS_0}{V_0^2}$$
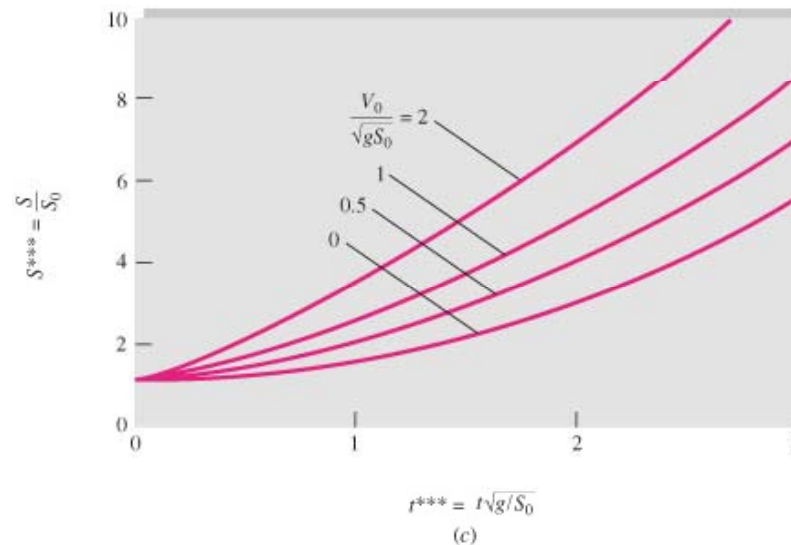
- The non-dimensional parameters will be:

$$S^{***} = \frac{S}{S_0} \qquad t^{***} = t\left(\frac{g}{S_0}\right)^{1/2}$$

- Substituting these variables into the falling-body equation, after simplifications:

$$S^{***} = 1 + \beta t^{***} + \frac{1}{2}t^{***2} \qquad \beta = \frac{1}{\sqrt{\alpha}} = \frac{V_0}{\sqrt{gS_0}}$$

$$t^{***} = t\sqrt{g/S_0}$$

(c)

SFU   Selection of scaling parameters

- In all three options, the same parameters α appears but has a different meaning: dimensionless gravity, initial displacement, and initial velocity.

Guidelines:

The scaling variables <u>must not form a dimensionless group</u> among themselves, but adding one more variable will form a dimensionless quantity. For example, in the above fluid flow problem, test the powers of

$$\rho^a V^b L^c = (ML^{-3})^a (LT^{-1})^b (L)^c = M^0 L^0 T^0 \quad only\ if \quad a = b = c = 0$$

<u>Do not select output variables</u> for your scaling parameters.

If convenient, <u>select popular</u>, not obscure, <u>scaling variables</u> because they will appear in all of your dimensionless groups.

The Pi theorem

- The Buckingham theorem provides a method for computing sets of dimensionless parameters from the given variables, even if the form of the equation is still unknown.

- Let $q_1, q_2, q_3, \ldots q_n$ be **n** dimensional variables that are physically relevant in a given problem and that are inter-related by an (unknown) <u>dimensionally homogeneous</u> set of equations.

$$F(q_1, q_2, q_3, \ldots q_n) = 0 \quad or \quad q_1 = f(q_2, q_3, \ldots q_n)$$

- If **k** is the number of fundamental dimensions required to describe the **n** variables, then there will be **k** primary variables and the remaining **j = k − n** variables can be expressed as **n − k** dimensionless and independent quantities or 'Pi groups'

$$\Phi(\Pi_1, \Pi_2, \Pi_3, \ldots \Pi_{n-k}) = 0 \quad or \quad \Pi_1 = \phi(\Pi_2, \Pi_3, \ldots \Pi_{n-k})$$

- **Note:** this set of non-dimensional parameters is not unique. They are however independent and form a complete set.

Applying Pi theorem

1) Clearly define the problem and think about which variables are important. Identify which is the main variable of interest $q_1 = f(q_2, q_3, \dots q_n)$

It is important to physically think about the problem. Are there any constrains, i.e. can I vary all these variables independently?

2) Express each of **n** variables in terms of its fundamental dimensions, {MLTθ} or {FLTθ}. It is often useful to use one system to solve the problem, and then check that groups you obtain are dimensionless by converting to other system.

3) Determine the number of Pi groups, **j = n − k** variables, where **k** is the number of reference dimensions and select **k** primary or repeating variables. Typically pick variables which characterize the <u>fluid properties</u>, <u>flow geometry</u>, <u>flow rate</u>…

4) Form **j** dimensionless **Π** groups and check that they are all indeed dimensionless.

Applying Pi theorem

5) Express result in form $\Pi_1 = \phi(\Pi_2, \Pi_3, \ldots \Pi_{n-k})$ where contains the quantity of interest and interpret your result physically!

6) Make sure that your groups are indeed independent; i.e. can I vary one and keep others constant?

7) Compare with experimental data!

Dimensionless basic equations

- Another useful application of pi theorem is application to the basic (governing) equations.

- Even though these equations cannot be solved in general, they will reveal basic dimensionless parameters, such as Reynolds number, in their proper form and proper position, giving clues when they are negligible.

$$\nabla . V = 0$$

$$\rho \frac{dV}{dt} = \rho g - \nabla p + \mu \nabla^2 V$$

- These equations contain the three basic dimensions M, L, and T. All variables *p, V, x ,y ,z,* and *t* can be non-dimensionalized by using density and two reference constants that might be characteristic of the particular fluid flow:

     Reference velocity = *U*   Reference length (or characteristics length) = *L*

Dimensionless basic equations

- As an example, *U* can be the inlet or upstream velocity and *L* the diameter of a body immersed in the stream. We can define all dimensionless variables:

$$V^* = \frac{V}{U} \qquad \nabla^* = \frac{\nabla}{L}$$

$$x^* = \frac{x}{L}, \qquad y^* = \frac{y}{L}, \qquad z^* = \frac{z}{L}, \qquad R^* = \frac{R}{L}$$

$$t^* = \frac{tU}{L}, \qquad p^* = \frac{p + \rho g z}{\rho U^2}$$

- After substitution and simplifications, we obtain:

$$\nabla^* . V^* = 0$$

$$\frac{dV^*}{dt^*} = -\nabla^* p^* + \frac{\mu}{\rho U L} \nabla^{*2}(V^*)$$

Non-dimensional Parameters

The continuity equation contains no parameter; however, the momentum (Navier-Stokes) equation reveals the **Reynolds number**:

$$Re = \frac{\rho U L}{\mu}$$

Euler number (pressure coefficient) is rarely important unless the pressure drops low enough to cause vapor formation (cavitation) in a liquid:

$$Eu = \frac{\Delta p}{\rho U^2}$$

Froude number is the dominant effect in free-surface flows and it does not appear if there is no free surface:

$$Fr = \frac{U^2}{gL}$$

Mach number has a strong effect on compressible flow properties if it is greater than 0.3:

$$Ma = \frac{U}{a}$$

Modeling and its pitfalls

Performing scale analysis is mathematically straightforward; however, there are several issues to consider:

1) We took for granted the important variables for the phenomena. Selection of important variables is not a trivial task and requires considerable judgment and experience.

2) Typically the Reynolds number of the model is too small (by a factor of 10 to 1000). As a result, we end up estimating the desired high-Reynolds-number prototype data, by extrapolating the model data in low Re. This may result in high uncertainty in the analysis; but there is no other practical alternative in hydraulic model testing.

Model vs. prototype

Typically the Reynolds number of the model is too small (by a factor of 10 to 1000.

Modeling and its pitfalls cont'd.
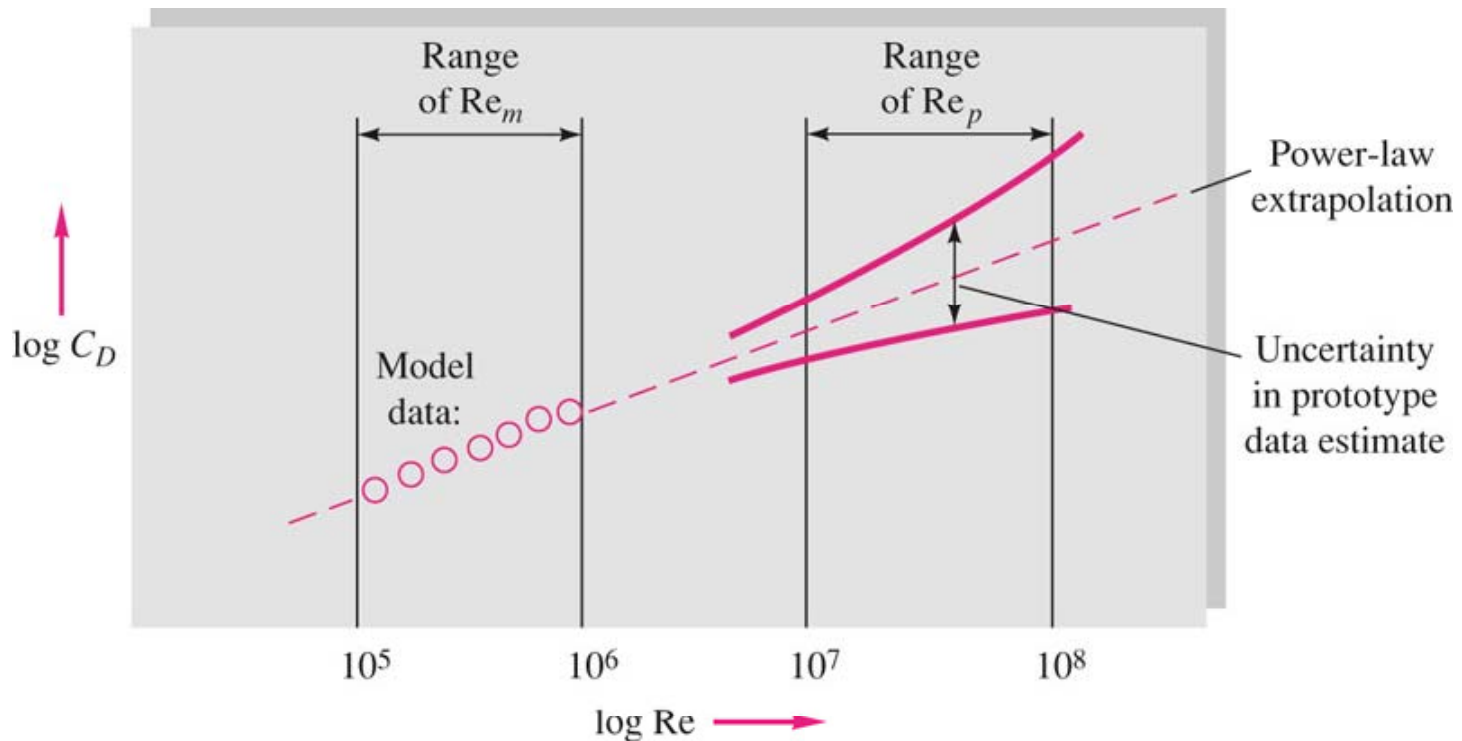
3) After selecting important parameters and determining pi groups, we should seek to achieve similarity between the model tested and the prototype to be designed. In general:

"Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for the model and the prototype."

Establishing complete similarity is highly unlikely. Therefore, we seek particular types of similarity.

Geometric similarity

Geometrical similarity is established when:

A model and prototype are geometrically similar if and only if all body dimensions in all 3 coordinates have the same linear scale ratio.

(a)

(b)
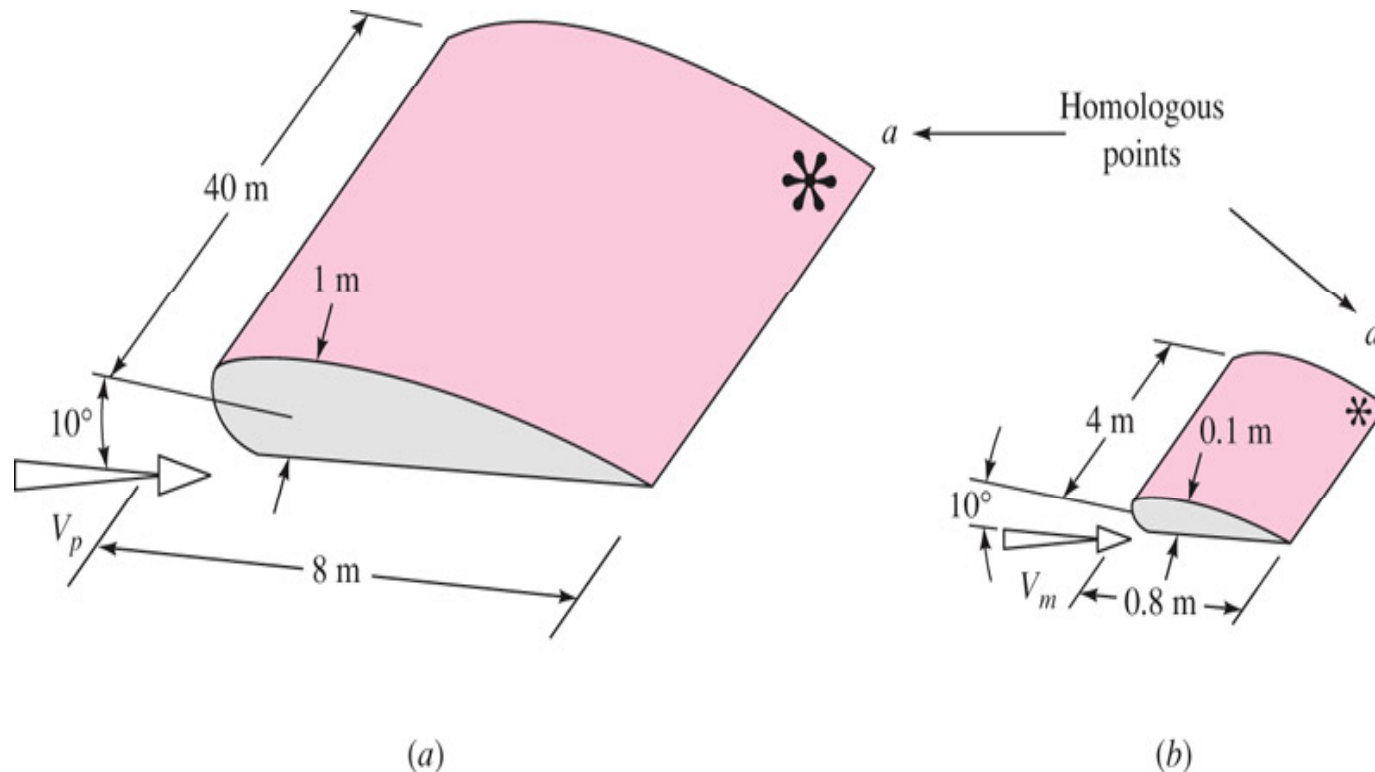
Kinematic similarity

It requires that the model and prototype have the same length scale ratio and the same time scale ratio; thus the velocity scale ratio will be the same for both.

Length scale equivalence simply implies geometric similarity, but <u>time scale</u> equivalence may require additional dynamic considerations such as equivalence of the <u>Reynolds</u> and <u>Mach</u> numbers.
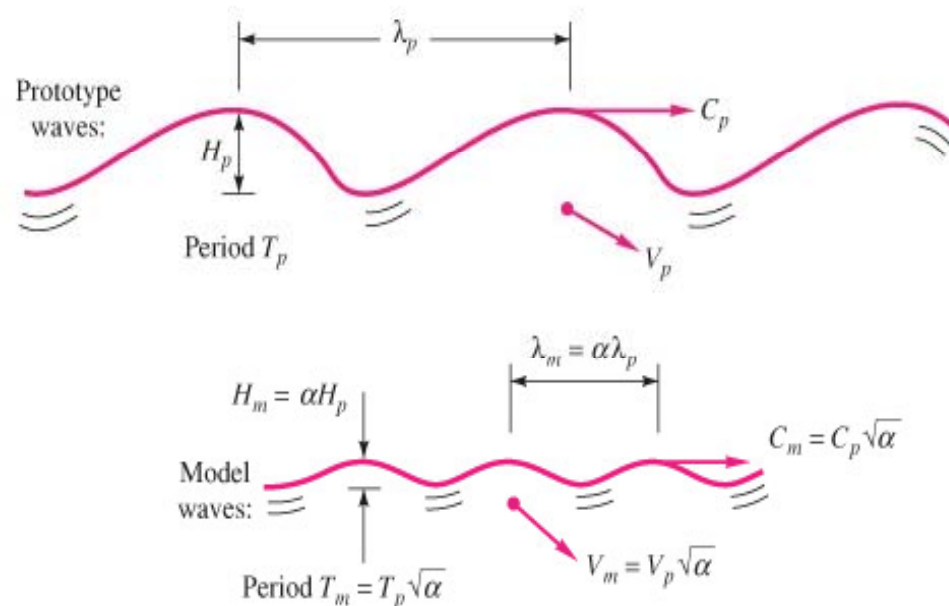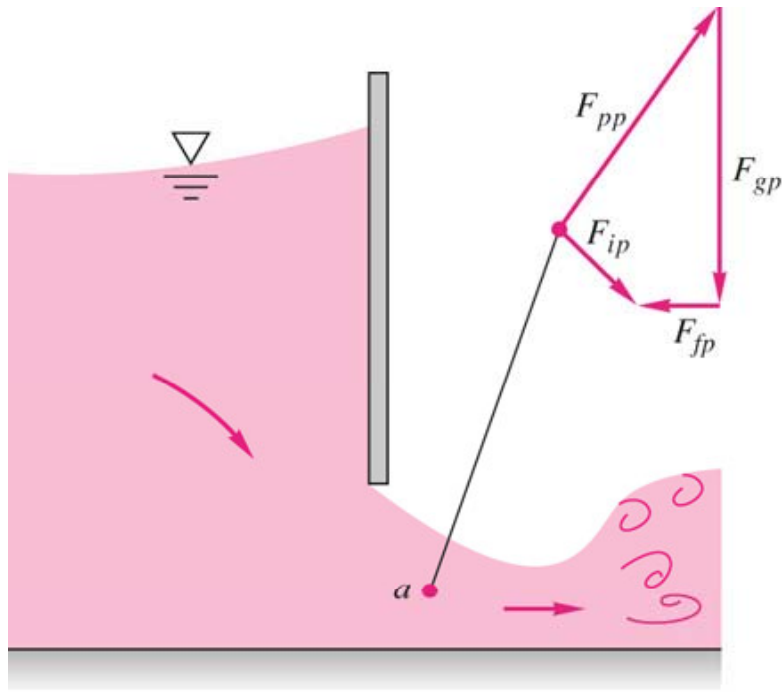
Dynamic similarity

Dynamic similarity exists when the model and the prototype have the same length scale ratio, time scale ratio, and the force scale (mass scale) ratio.
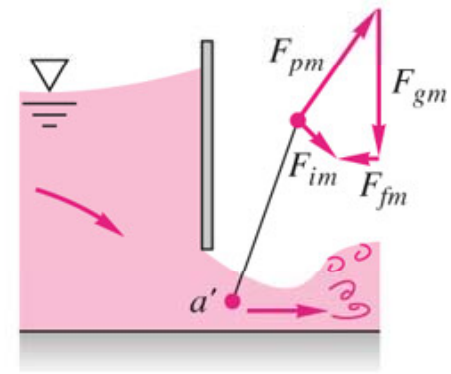
Dynamic similarity exists, simultaneously with kinematic similarity, if the model and prototype force and pressure coefficients are identical. This is established if:

1) For compressible flow, the model and prototype Reynolds number and Mach number and specific heat ratio are correspondingly equal.

2) For incompressible flow:
   a) With no free surface: model and prototype Reynolds numbers are equal.

   b) With a free surface: model and prototype Reynolds number, Froude number, and (if necessary) Weber number and cavitation number are correspondingly equal.

(a)

(b)