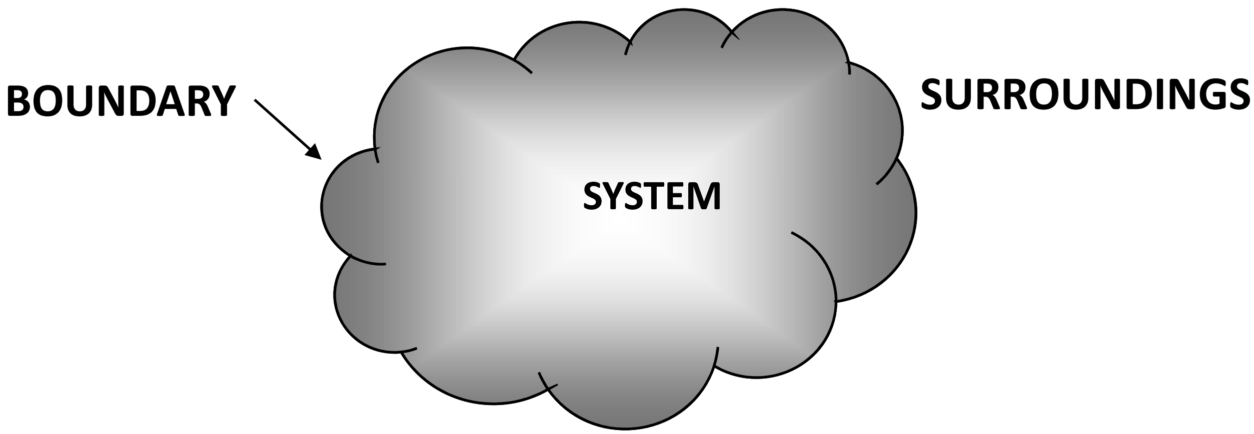
Integral relations for CV

- Control volume approach is accurate for any flow distribution but is often based on the “one-dimensional” property values at the boundaries.
- It gives useful engineering estimates.

System and control volume

- A system is defined as a fixed quantity of matter or a region in space chosen for study. The mass or region outside the system is called the surroundings.

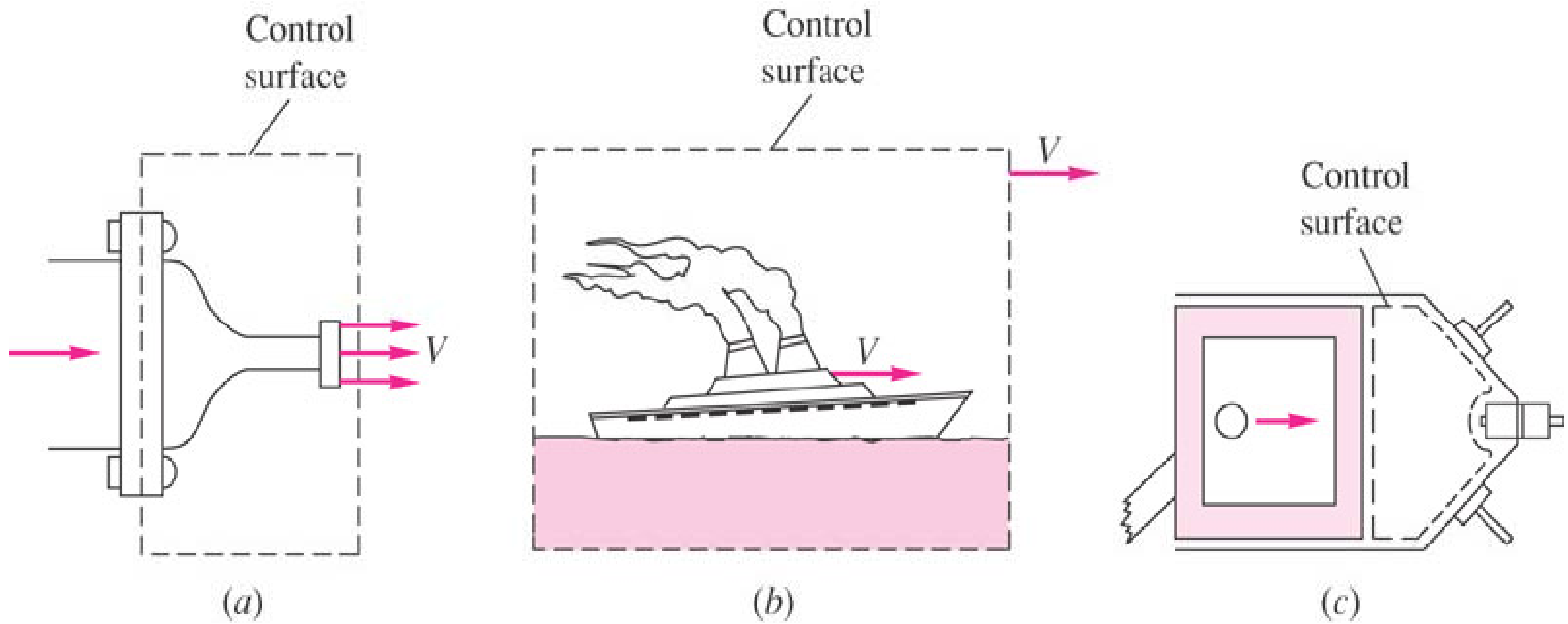


- System boundary: the real or imaginary surface that separates the system from its surroundings. The boundaries of a system can be fixed or movable.

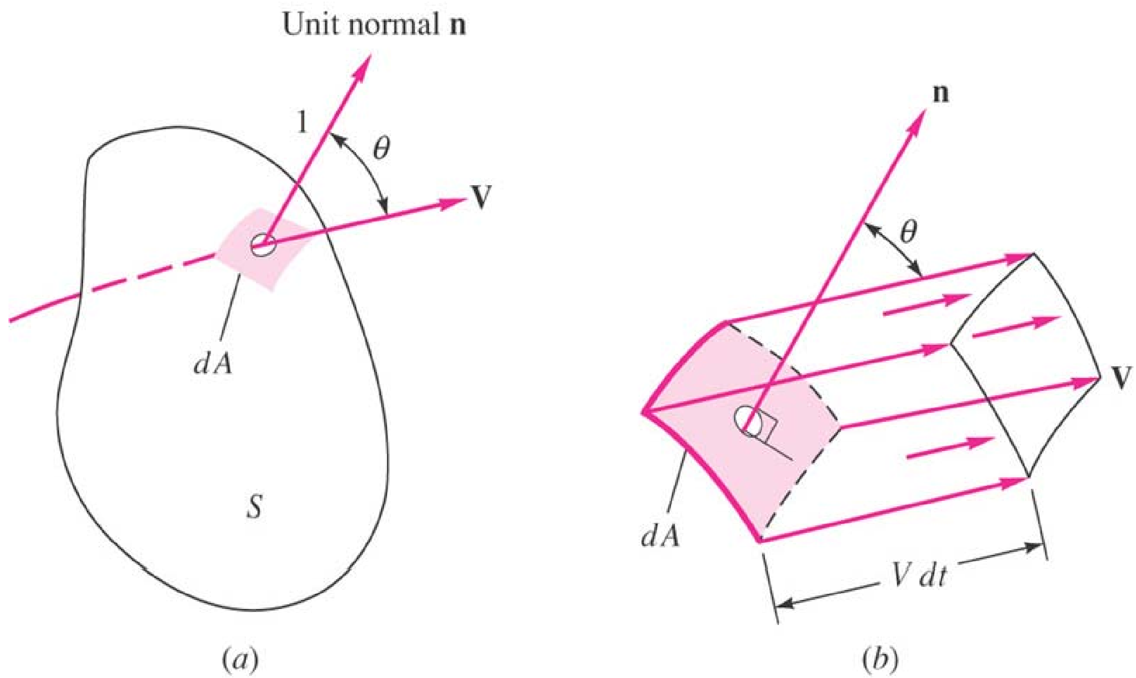
- Open system or control volume is a properly selected region in space. It usually encloses a device that involves mass flow such as a compressor.

Control volume

- control volume is an abstract concept and does not hinder the flow in any way.



Volume and mass flow rate

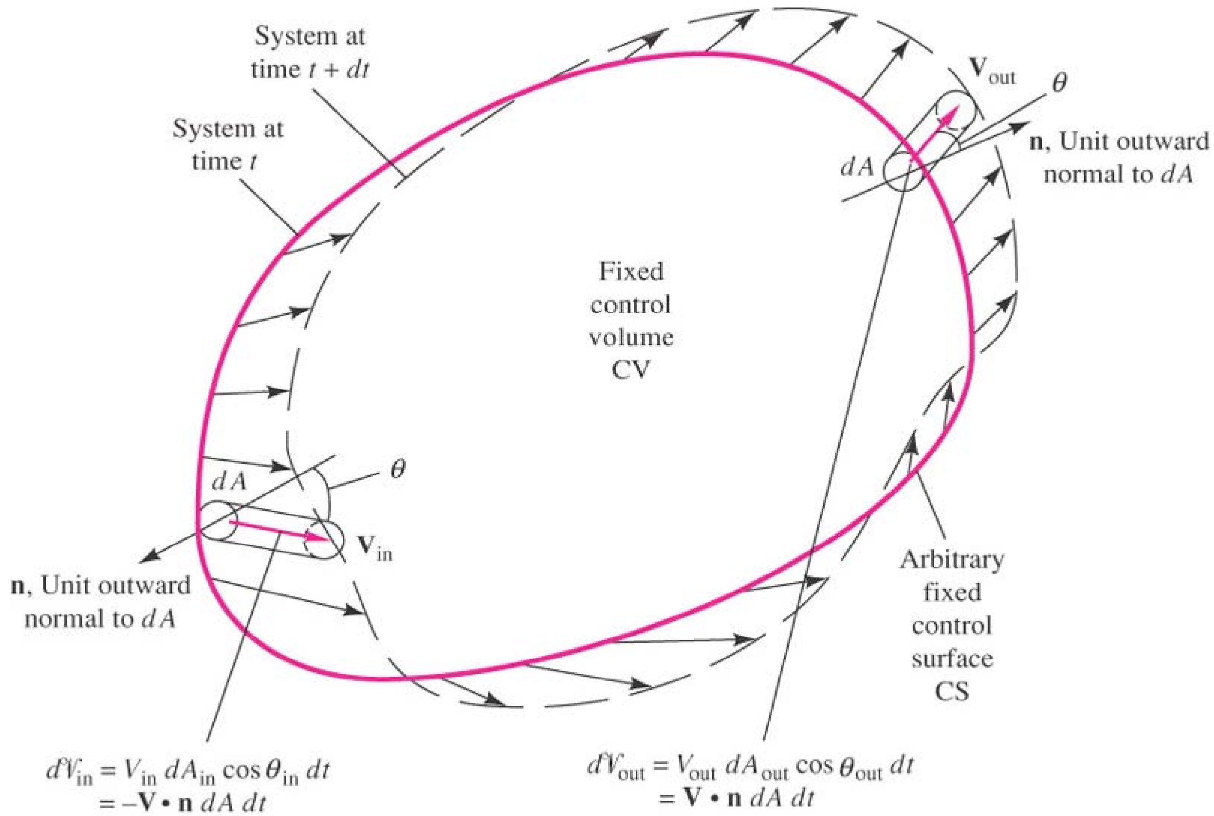


$$Q = \int_S (V.n)dA = \int_S V_n \, dA$$

$$\dot{m} = \int_S \rho(V.n)dA = \int_S \rho V_n \, dA$$

$$\dot{m} = \rho Q = \rho A V$$

Reynolds transport theorem



$$\frac{d}{dt}(B_{sys}) = \frac{d}{dt}\left(\int_{CV} \beta \rho dV\right) + \int_{CS} \beta \rho (\vec{V}.\vec{n})\, dA$$

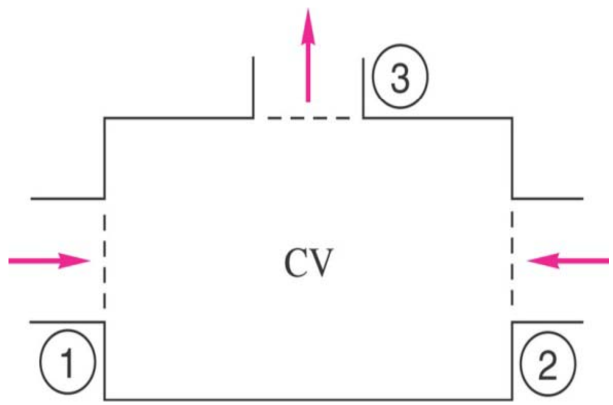
One-dimensional approximation

- In many situations, the flow crosses the boundaries of the control surface at simplified inlets and exits that are approximately one-dimensional (the velocity can be considered uniform across each control surface).

$$\frac{d}{dt}\left(B_{sys}\right) = \frac{d}{dt}\left(\int_{CV} \beta \rho \, dV\right) + \sum_{outlets} \beta_i \dot{m}_i\big|_{out} - \sum_{inlets} \beta_i \dot{m}_i\big|_{in} \quad where \quad \dot{m}_i = \rho_i A_i V_i$$

Example

- A fixed control volume has three one-dimensional boundary sections, as shown in the figure below. The flow within the control volume is steady. The flow properties at each section are tabulated below. Find the rate of change of energy that occupies the control volume at this instant.



Control surface	**type**	**ρ, kg/m^3**	**V, m/s**	**A, m^2**	**e, J/kg**
1	inlet	800	5.0	2.0	300
2	inlet	800	8.0	3.0	100
3	outlet	800	17.0	2.0	150

Conservation of mass, *B=m*

- For conservation of mass, B=m and β=dm ⁄ dm=1. The Reynolds transport equation becomes:

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \int_{CS} \rho(\vec{V}.\vec{n})\, dA = \left(\frac{dm}{dt}\right)_{system}$$

- If the control volume only has a number of one-dimensional inlets and outlets, we can write:

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

- for steady-state flow, ∂ρ⁄∂t=0, and the conservation of mass becomes:

$$\sum_i (\rho_i A_i V_i)_{out} = \sum_i (\rho_i A_i V_i)_{in}$$

Average velocity

- In cases that fluid velocity varies across a control surface, it is often convenient to define an average velocity.

$$V_{av} = \frac{Q}{A} = \frac{1}{A}\int (\vec{V}.\vec{n})dA$$

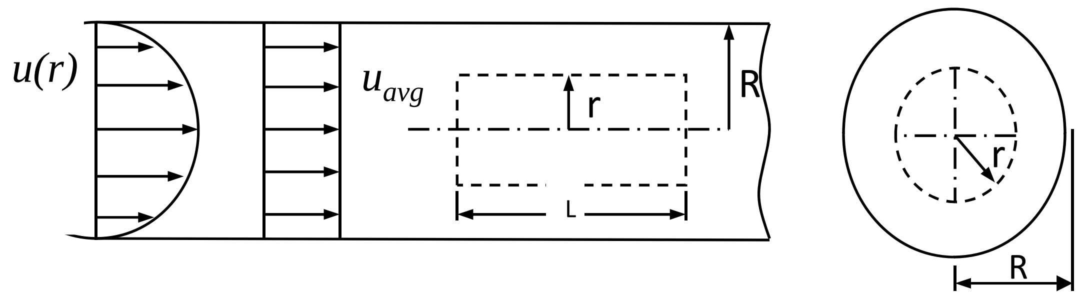
- If the density varies across the cross-section, we similarly can define an average density:

$$\rho_{av} = \frac{1}{A}\int \rho dA$$

Example 2

- In a grinding and polishing operation, water at 300 K is supplied at a flow rate of 4.264×10^{-3} kg/s through a long, straight tube having an inside diameter of D=2R=6.35 mm. Assuming the flow within the tube is laminar and exhibits a parabolic velocity profile:

$$u(r) = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right]$$



- where u_{max} is the maximum fluid velocity at the center of the tube. Using the definition of the mass flow rate and the concept of average velocity, show that

$$u_{avg} = \frac{u_{max}}{2}$$

Linear momentum equation

- For Newton's second law, the property being differentiated is the linear momentum, mV. Thus B =mV and β=dB⁄dm=V. The Reynolds transport theorem becomes:

$$\frac{d}{dt}(m\vec{V})_{sys} = \sum \vec{F} = \frac{d}{dt}\left(\int_{CV} \vec{V}\rho dV \right) + \int_{CS} \vec{V}\rho(\vec{V}.\vec{n})\, dA$$

- Momentum flux term

$$\dot{M}_{CS} = \int_{CS} \vec{V}\rho(\vec{V}.\vec{n})\, dA$$

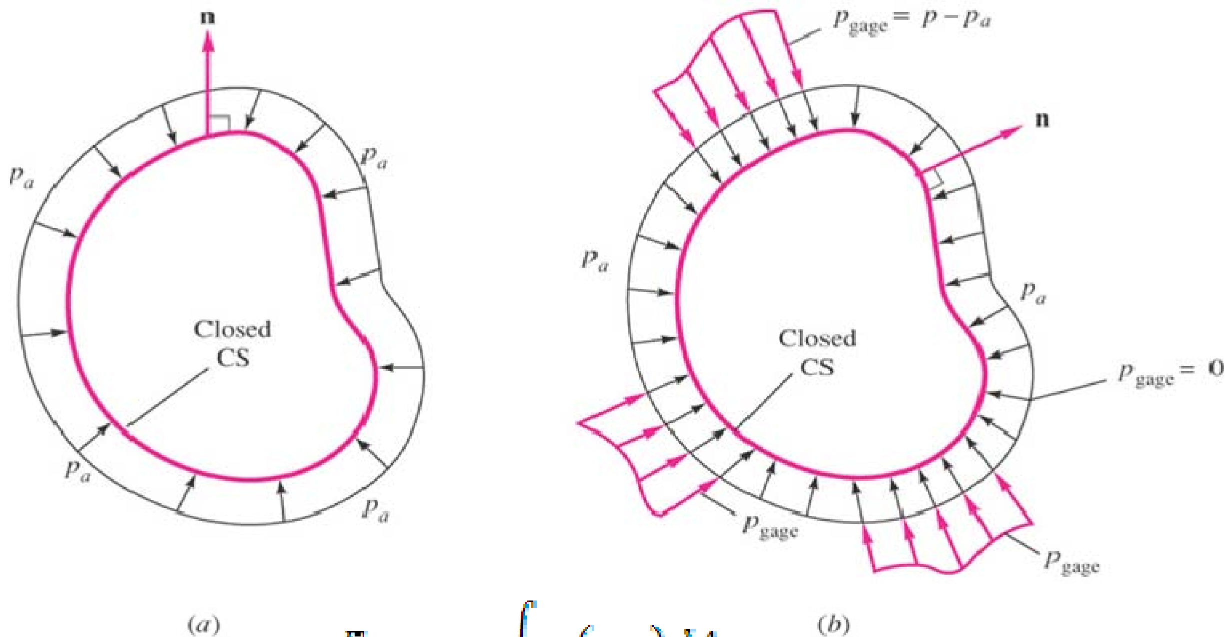
- If cross-section is one-dimensional, V and ρ are uniform and over the area, momentum flux simplifies

$$\dot{M}_i = V_i(\rho_i A_i V_i) \quad = \dot{m}_i V_i$$

- For one-dimensional inlets and outlets, we have

$$\Sigma\vec{F} = \frac{d}{dt}\left(\int_{CV} \vec{V}\rho\, dV \right) + \sum_i (\dot{m}_i \vec{V}_i)_{out} - \sum_i (\dot{m}_i \vec{V}_i)_{in}$$

Net pressure on CV



$$F_{press} = \int_{CS} p(-n)\,dA$$

- If the pressure has a uniform value p_a all around the surface, the net pressure force is zero.

$$F_{press} = \int_{CS} (p - p_a)(-n)\,dA = \int_{CS} p_{gage}(-n)\,dA$$

Momentum flux correction

- To consider the effects of non-uniform velocity, we introduce a correction factor β.

$$\rho \int u^2 \, dA = \beta \dot{m} V_a = \beta \rho A V_{av}^2$$

$$\beta = \frac{1}{A} \int \left(\frac{u}{V_{av}} \right)^2 dA$$

- Values of β can be calculated using the duct velocity profile and the above definition for
- Laminar flow: $u = U_0 \left(1 - \frac{r^2}{R^2} \right) \quad with \;\; \beta = 4/3$
- Turbulent flow: $u = U_0 \left(1 - \frac{r}{R} \right)^m \quad with \;\; \beta = \frac{(2+m)^2 (1+m)^2}{2(1+2m)(2+2m)}$