Differential Relations for Fluid Flow

In this approach, we apply our four basic conservation laws to an infinitesimally small control volume. The differential approach provides point-by-point details of a flow pattern as oppose to control volume technique that provide gross-average information about the flow.

Acceleration field of a fluid

The Cartesian vector form of a velocity filed can be written as:

$$V(r,t) = \vec{i}\, u(x,y,z,t) + \vec{j}\, v(x,y,z,t) + \vec{k}\, w(x,y,z,t)$$

The flow filed is the most important variable in the fluid mechanics, i.e., knowledge of the velocity vector filed is equivalent to solving a fluid flow problem.

The acceleration vector field can be calculated:

$$\frac{du(x,y,z,t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

$$= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= \frac{\partial u}{\partial t} + (V.\nabla)u$$

where the compact dot products is:

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = V.\nabla \quad and \quad \nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

With a similar approach, we obtain the total acceleration vector:

$$a = \frac{dV}{dt} = \frac{\partial V}{\partial t} + \left(u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial V}{\partial t} + (V.\nabla)V$$

The term $\partial V/\partial t$ is called the <u>local acceleration</u> and vanishes if the flow is steady. The three terms in the parentheses are called the <u>convective acceleration</u> and rises when the particles move through regions of spatially varying velocity, e.g. nozzle.

The total time derivative (d/dt) is sometimes called the <u>substantial</u> or <u>material derivative</u> can be applied any variable such as pressure. This operator sometimes assigned a special symbol D/Dt.

The differential equation of mass conservation

All basic equations can be derived by considering an elemental system. Figure 1 shows the control volume (*dx*, *dy*, *dz*) in which flow through each side of the element is approximately one-dimensional. Since the size of the element is so small, we can assume that all the fluid properties are uniform and constant within the element.

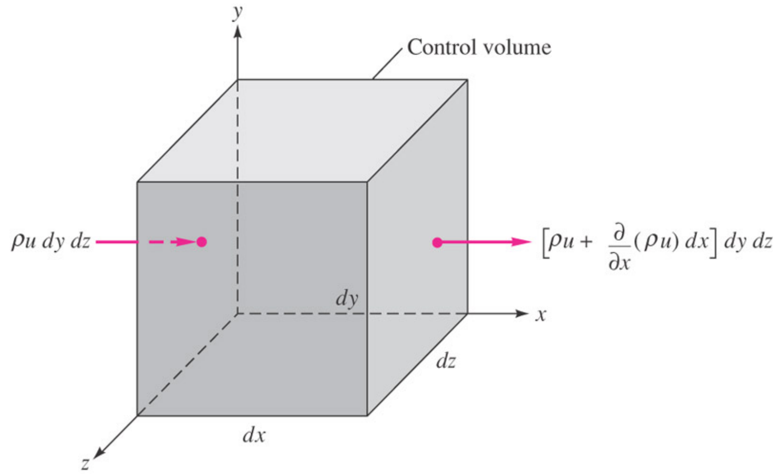

Fig.1: Fixed Cartesian element showing inlet and outlet flows on the x-direction.

The conservation of mass for the element can be written as:

$$\frac{\partial \rho}{\partial t} dx\, dy\, dz + \frac{\partial}{\partial x}(\rho u) dx\, dy\, dz + \frac{\partial}{\partial y}(\rho v) dx\, dy\, dz + \frac{\partial}{\partial z}(\rho w) dx\, dy\, dz = 0$$

After dividing by the volume of the element:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}$$

$$= \frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0$$

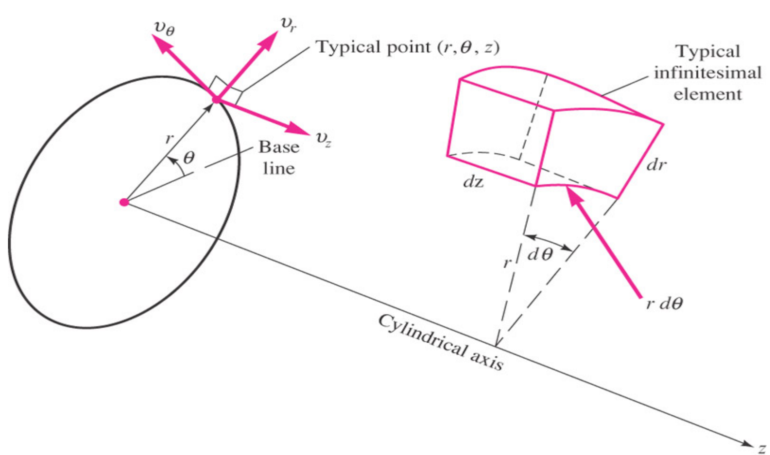

Fig. 2: Cylindrical polar coordinate.

The continuity equation for the cylindrical polar coordinates is:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

where velocity vector $V = (v_r, v_\theta, v_z)$.

For steady compressible flow, continuity equation simplifies to:

$$\begin{cases} Cartesian, & \dfrac{\partial}{\partial x}(\rho u) + \dfrac{\partial}{\partial y}(\rho v) + \dfrac{\partial}{\partial z}(\rho w) = 0 \\ Cylinderical, & \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\rho v_r) + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}(\rho v_\theta) + \dfrac{\partial}{\partial z}(\rho v_z) = 0 \end{cases}$$

For incompressible flow, continuity equation can be further simplified since density changes are negligible:

$$\nabla . V = 0$$

$$\begin{cases} Cartesian, & \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0 \\ Cylinderical, & \dfrac{1}{r}\dfrac{\partial}{\partial r}(r v_r) + \dfrac{1}{r}\dfrac{\partial}{\partial \theta}(v_\theta) + \dfrac{\partial}{\partial z}(v_z) = 0 \end{cases}$$

A flow can be considered incompressible when $Ma \leq 0.3$.

Note: the continuity equation is always important and must always be satisfied for a rational analysis of a flow pattern.

The differential equation of linear momentum

In a Cartesian coordinates, the momentum equation can be written as:

$$\sum F = \rho \frac{dV}{dt} dx\, dy\, dz$$

There are types of forces: body forces and surface forces.

Body forces are due to external fields such as gravity and magnetism fields. We only consider gravity forces:

$$dF_{grav} = \rho \vec{g} dx\, dy\, dz \qquad where \qquad \vec{g} = -g\vec{k}$$

The **surface forces** are due to the stresses on the sides of the control surface. These stresses are the sum of hydrostatic pressure plus viscous stresses τ_{ij} which arise from the motion of the fluid:

$$\sigma_{ij} = \begin{vmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{vmatrix}$$

Unlike velocity, stresses and strains are nine-component tensors and require two subscripts to define each component.

The net surface force due to stresses in the x-direction can be found as:

$$dF_{x,surf} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dx\, dy\, dz$$

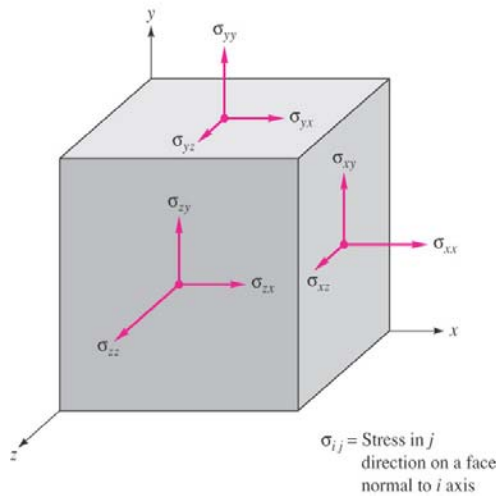

Fig.2: stress tensor.

Similarly we can find the net surface force in *y* and *z* direction. After summing them up and dividing through by the volume, we get:

$$\left(\frac{dF}{dxdydz}\right)_{Viscous} = i\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)+j\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)+k\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right) = \nabla.\tau_{ij}$$

Note: the surface force is the sum of the <u>pressure gradient</u> and the <u>divergence of the viscous stress tensor</u>.

Therefore the linear momentum equation for an infinitesimal element becomes:

$$\rho g - \nabla p + \nabla.\tau_{ij} = \rho\frac{dV}{dt}$$

where

$$\frac{dV}{dt} = \frac{\partial V}{\partial t}+\left(u\frac{\partial V}{\partial x}+v\frac{\partial V}{\partial y}+w\frac{\partial V}{\partial z}\right)$$

This is a vector equation, and can be written as:

$$\rho g_x - \frac{\partial p}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)$$

Special cases of momentum equation:

Inviscid flow: Euler's equation

When the viscous terms are negligible, i.e. $\tau_{ij} = 0$

$$\rho g - \nabla p = \rho \frac{dV}{dt}$$

Euler's equation can be integrated along a streamline to yield the frictionless Bernoulli equation.

Newtonian fluid: Navier-Stokes equation

For a Newtonian fluid, the viscous stresses are proportional to the element strain rates and the coefficient of viscosity.

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \qquad \tau_{yy} = 2\mu \frac{\partial v}{\partial y} \qquad \tau_{zz} = 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \qquad \tau_{xz} = \tau_{zx} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$$

where μ is the viscosity coefficient. Substituting shear stresses in the momentum equation, for a Newtonian fluid with constant density and viscosity, we get:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) = \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) = \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)$$

These are the <u>incompressible flow Navier-stokes equations</u>.

Note: Navier-Stokes equations have four unknowns: *p, u, v*, and *w.* They should be combined with the continuity equation to form four equations for theses unknowns.

Navier-Stokes equations have a limited number of analytical solutions; these equations typically are solved numerically using <u>computational fluid dynamics</u> (CFD) software and techniques.

The differential equation of angular momentum

Application of the integral theorem to a differential element gives that the shear stresses are symmetric:

$$\tau_{ij} = \tau_{ji}$$

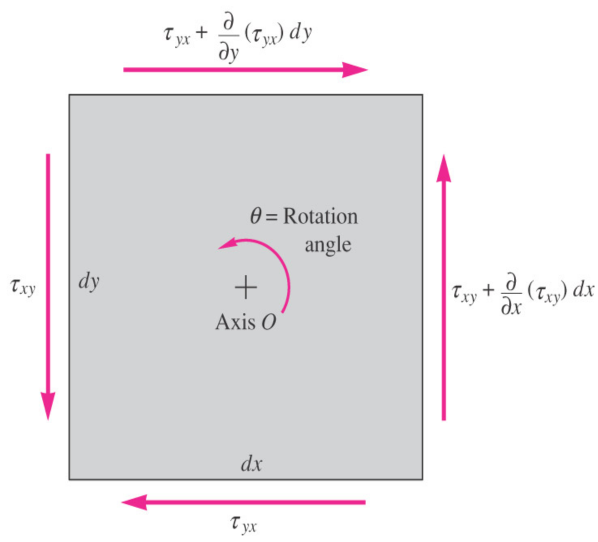

Fig. 3: Fixed Cartesian element showing shear stresses that may cause a net angular acceleration about O.

Therefore, there <u>is no differential angular momentum equation</u>.

The boundary conditions for the basic equations

We have 3 differential equations to solve: i) continuity equation, ii) momentum, and iii) energy. Typically, the density is variable, so the three equations contain 5 unknowns: $\rho, V, p, \hat{u}$ *and* T. Therefore, we need 2 additional relations to complete the system of equations. These are provided by data or algebraic expressions for state relations of thermodynamic properties such as ideal gas equation of state:

$$\rho = \rho(p, T) \quad and \quad \hat{u} = \hat{u}(p, T)$$

For ideal gas, we have: $\rho = p/RT$ *and* $\hat{u} = c_v T$. So, we need to set proper, initial and boundary conditions for each variable.

Some important boundary conditions:

- At solid wall: $V_{fluid} = V_{wall}$ (no-slip condition) $T_{fluid} = T_{wall}$ (no-temperature jump)
- At inlet or outlet section of the flow: *V, p, T* are known
- At a liquid-gas interface: equality of vertical velocity across the interface (kinematic boundary condition).
- Mechanical equilibrium at liquid-gas interface, $(\tau_{zx})_{liq} = (\tau_{zx})_{gas}$ and $(\tau_{zy})_{liq} = (\tau_{zy})_{gas}$
- At a liquid-gas interface: heat transfer must be the same, $(q_z)_{liq} = (q_z)_{gas}$, or

$$\left(k\frac{\partial T}{\partial z}\right)_{liq} = \left(k\frac{\partial T}{\partial z}\right)_{gas}$$

Incompressible flow with constant properties

Flow with constant $\rho, \mu, and\ k$ is a basic simplification that is very common in engineering problem that leads to:

Continuity $$\nabla . V = 0$$

Momentum $$\rho \frac{dV}{dt} = \rho g - \nabla p + \mu \nabla^2 V$$

For frictionless or inviscid flows in which $\mu = 0$. The momentum equation reduces to Euler's equation:

$$\rho \frac{dV}{dt} = \rho g - \nabla p$$

Some illustrative incompressible viscous flows

Couette flow between a fixed and a moving plate

Consider two-dimensional incompressible plane viscous flow between parallel plates, $\frac{\partial}{\partial z} = 0$, a distance *2h* apart, as shown in Fig. 4. We assume that the plates are very wide and very long and that the flow is essentially axial.

$$v = w = 0\ and\ u \neq 0$$

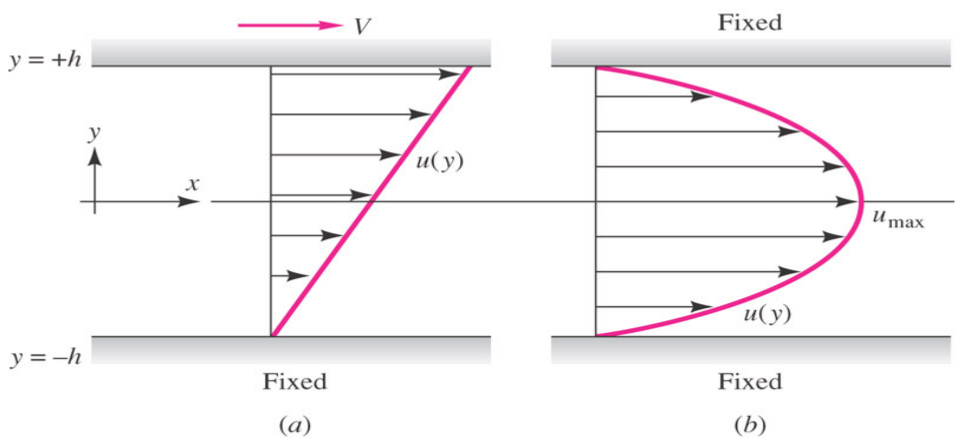

Fig. 4: Incompressible viscous flow between parallel plates, a) no pressure gradient; b) pressure gradient with both plates fixed.

From continuity equation, we learn:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \rightarrow \frac{\partial u}{\partial x} = 0\ or\ u = u(y) only$$

We neglect gravity effects and assume fully-developed flow. Substitute $u = u(y)$ in the Navier-Stokes equation for x-direction:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

After simplifications, it reduces to:

$$\frac{d^2 u}{dy^2} = 0 \;\; or \; u = C_1 y + C_2$$

We need two boundary conditions to find the constants C_1 and C_2.

At y=+h, u = V =C_1h + C_2

At y= - h, u = 0 = C_1 (-h) +C_2

Therefore, the solution for flow between plates with a moving upper wall is:

$$u = \frac{V}{2h}y + \frac{V}{2} \qquad -h \le y \le +h$$

This is called <u>Couette flow due to a moving wall</u>: a linear velocity profile with no slip at each wall.

Couette flow due to pressure gradient between two fixed plates

Let's consider case (b) in Fig. 4, where both <u>plates are fixed but pressure varies in the x-direction</u>. If $v = w = 0$, the continuity equation lead to the same result, namely:

$$u = u(y)$$

The x-momentum equation changes because the pressure is variable:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\mu\frac{d^2 u}{dy^2} = \frac{\partial p}{\partial x}$$

Also, since $v = w = 0$ and gravity is neglected, the y and z momentum equations lead to:

$$\frac{\partial p}{\partial y} = 0 \;\; and \;\; \frac{\partial p}{\partial z} = 0 \;\; \rightarrow p = p(x) \; only$$

Thus, the pressure gradient in x-direction is the only and total gradient:

$$\mu\frac{d^2 u}{dy^2} = \frac{dp}{dx} = Const < 0$$

The solution is accomplished by double integration:

$$u = \frac{1}{\mu}\left(\frac{dp}{dx}\right)\frac{y^2}{2} + C_1 y + C_2$$

The constants can be found from the no-slip boundary condition at the walls:

at $y = \pm h$: $u = 0$ Thus, C1= 0 and $C_2 = -\frac{dp}{dx}\frac{h^2}{2\mu}$.

The velocity distribution in a channel due to pressure gradient is:

$$u = -\frac{h^2}{2\mu}\frac{dp}{dx}\left(1 - \frac{y^2}{h^2}\right)$$

The flow forms a *Poiseuille* parabola of constant negative curvature, where the maximum velocity occurs at the center.

Fully-developed laminar pipe flow

This is one of the most useful exact solutions to the Navier-Stokes equation: *fully-developed incompressible flow in straight circular pipe of radius R*.

Fully-developed flow: refers to the flow in a region far enough from the entrance that the flow is purely axial. As a result, the velocity distribution in the tube is fixed (not changing along the tube).

Neglecting gravity effects and assuming axial symmetry, i.e., $\frac{\partial}{\partial \theta} = 0 \ and \ v_\theta = 0$, the continuity equation in cylindrical coordinates, reduces to:

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0 \rightarrow v_z = v_z(r)$$

It means that the flow proceeds straight down the pipe without radial motion. The r-momentum equation in cylindrical coordinates, simplifies to $\frac{\partial p}{\partial r} = 0 \ or \ p = p(z)$ only. The z-momentum equation in cylindrical coordinates, reduces to:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{dp}{dz} = Const < 0$$

This equation is linear and can be integrated twice:

$$v_z = \frac{dp}{dz}\frac{r^2}{4\mu} + C_1 \ln(r) + C_2$$

Applying, the no-slip boundary condition at the wall and finite velocity at the centerline, gives us:

No-slip at r = R: $v_z = 0 = \frac{dp}{dz}\frac{R^2}{4\mu} + C_1 \ln(R) + C_2$

Finite velocity at r = 0: $v_z = finite = 0 + C_1 \ln(0) + C_2 \rightarrow C_1 = 0$

The final solution for fully-developed Hagen-Poiseuille flow is:

$$v_z = \left(-\frac{dp}{dz}\right)\frac{1}{4\mu}(R^2 - r^2)$$

The velocity profile is a paraboloid with a maximum at the centerline.

Flow between long concentric cylinders

Consider an incompressible flow between two concentric cylinders, as shown in Fig. 5. There is no axial motion or end effect, $v_z = \partial/\partial z = 0$.

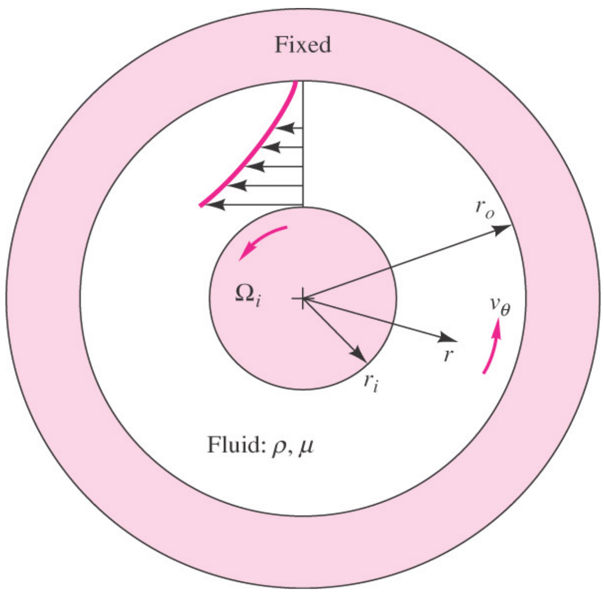

Fig. 5: Incompressible viscous flow between long cylinders.

The continuity, with $v_z = 0$, reduces:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0 \rightarrow rv_r = const.$$

Note that v_θ does not vary with θ. Also since $v_r = 0$ at both the inner and outer cylinders, it follows that $v_r = 0$ everywhere, as a result, the motion will be purely circumferential, $v_\theta = v_\theta(r)$. The momentum equation in θ direction is:

$$\rho(\nabla.V)v_\theta + \frac{\rho v_\theta v_r}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta + \mu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2}\right)$$

In cylindrical coordinates, we have:

$$\nabla^2 v_\theta = \frac{1}{r}\frac{d}{dr}\left(r\frac{dv_\theta}{dr}\right)$$

As a result, the momentum equation becomes:

$$0 = \mu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2}\right) \rightarrow \nabla^2 v_\theta = \frac{v_\theta}{r^2}$$

This is a linear 2nd order ordinary differential equation:

$$v_\theta = C_1 r + \frac{C_2}{r}$$

The constants can be found by applying boundary conditions:

At $r = r_o$: $v_\theta = 0 = C_1 r_o + C_2/r_o$

At $r = r_i$: $v_\theta = \Omega r_i = C_1 r_i + C_2/r_i$

The final solution for the velocity distribution is:

$$v_\theta = \Omega_i r_i \frac{\frac{r_0}{r} - \frac{r}{r_0}}{\frac{r_0}{r_i} - \frac{r_i}{r_0}}$$