Dimensional Analysis and Similarity

Dimensional analysis is very useful for planning, presentation, and interpretation of experimental data. As discussed previously, most practical fluid mechanics problems are too complex to solve analytically and must be tested by experiment or approximated by computational fluid dynamics (CFD). These data have much more generality if they are expressed in compact, economic non-dimensional form.

Dimensional analysis is a method for reducing the number and complexity of experimental variables that affect a given physical phenomena.

Advantages of dimensional analysis

1) <u>Reduce the number of variables</u>: Suppose the force F on a particular body shape immersed in a stream of fluid depends only on the body length L, velocity V, fluid density ρ and viscosity μ:

$$F = f(L, V, \rho, \mu)$$

In general, it takes about 10 points to define a curve. To find the effects of each parameter on the force, we need to perform 10X10X10X10=10^4 tests! However, using dimensional analysis, we can reduce the parameters to only one:

$$\frac{F}{\rho L^2 V^2} = g\left(\frac{\rho V L}{\mu}\right) = g(Re)$$

That is, the non-dimensional force is a function of the dimensionless parameter Reynolds number. So, we do not have to vary L, V, ρ, μ separately but only the grouping $\rho V L/\mu$. This can be done by varying velocity V in the wind tunnel or water channel.

2) <u>Non-dimensional equations</u>: that will provide insight on controlling parameters and the nature of the problem.

3) <u>Scaling laws</u>: that allows testing models instead of expensive large full-scale prototypes. There are rules for finding scaling laws or conditions of similarity. In our force example, if the similarity condition exists:

$$Re_m = Re_p$$

Then one can write:

$$\frac{F_p}{F_m} = \frac{\rho_p}{\rho_m}\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2$$

where subscripts *m* and *p* indicate model and prototype, respectively. This equation is the scaling law: *if you measure the model force at the model Reynolds number, the prototype force at the same Reynolds number equals the model force times the density ratio times the velocity ratio squared times the length ratio squared*.

Principle of dimensional homogeneity: in a dimensionally homogenous relationship for a physical process, each term will have the same dimensions. As an example, consider the Bernoulli's equation:

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = const$$

Each term, including the constant, has dimensions of velocity squared [L^2T^{-2}].

Variables and scaling parameters

The variables are the things we wish to plot, the basic output of the experiment or theory. The parameters are those quantities whose effects on the variables we wish to know. For example, consider the relation that expresses the displacement of a falling body:

$$S = S_0 + V_0 t + \frac{1}{2}gt^2$$

To non-dimensionalize our results, we need to know how many dimensions are contained among our variables (S, t) and parameters (V_0, S_0, and g); in this case only two, length {L} and time {T}:

$$\{S\} = \{S_0\} = \{L\} \qquad \{t\} = \{T\} \qquad \{V_0\} = \{LT^{-1}\} \qquad \{g\} = \{LT^{-2}\}$$

Among our parameters, we therefore select two to be <u>scaling parameters</u> (or repeating parameters), used to define dimensionless variables. For the falling-body problem, we select any two of the three parameters to be scaling parameters, thus we have three choices:

<u>Option 1</u>: Scaling parameters S_0 and V_0; the effect of gravity g. Using these scaling parameters, we define non-dimensional displacement and time as:
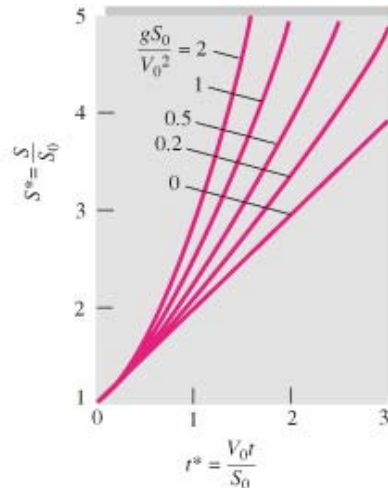
$$S^* = \frac{S}{S_0} \qquad\qquad t^* = \frac{V_0 t}{S_0}$$

Substituting these variables into the falling-body equation, after simplifications, we find the non-dimensional equation:

$$S^* = 1 + t^* + \frac{1}{2}\alpha t^{*2}$$

where

$$\alpha = \frac{gS_0}{V_0^2}$$

There is a single dimensionless parameter α. Note that this plot cannot show the effect of S_0 and V_0 since they are hidden in the ordinate and abscissa.

<u>Option 2</u>: Scaling parameters V_0 and g: the effect of initial displacement S_0. The non-dimensional parameters will be:
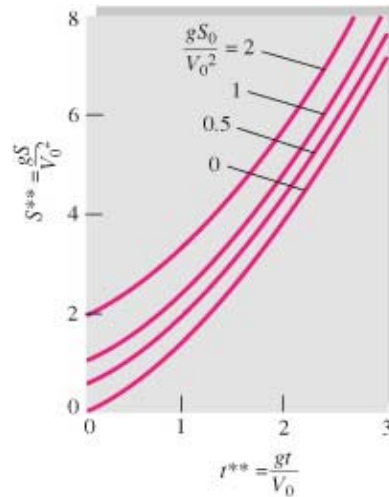
$$S^{**} = \frac{Sg}{V_0^2} \qquad\qquad t^{**} = \frac{tg}{V_0}$$

Substituting these variables into the falling-body equation, after simplifications, we find the non-dimensional equation:

$$S^{**} = \alpha + t^{**} + \frac{1}{2}t^{**2}$$

where

$$\alpha = \frac{gS_0}{V_0^2}$$

The same single parameter α appears again.

Option 3: Scaling parameter S_0 and g: the effect of initial speed V_0. Using the scaling parameters (S_0, g), dimensionless displacement and time can be found:
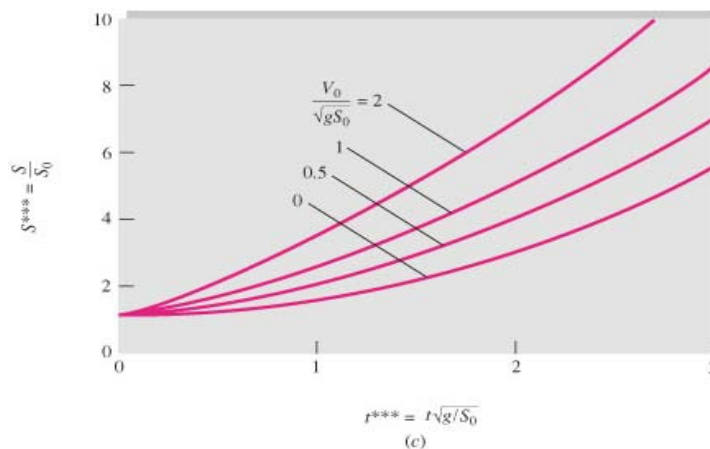
$$S^{***} = \frac{S}{S_0} \qquad t^{***} = t\left(\frac{g}{S_0}\right)^{1/2}$$

Substituting these variables into the falling-body equation, after simplifications, we find the non-dimensional equation:

$$S^{***} = 1 + \beta t^{***} + \frac{1}{2}t^{***2}$$

where

$$\beta = \frac{1}{\sqrt{\alpha}} = \frac{V_0}{\sqrt{gS_0}}$$

(c)

Note that, in all three options, the same parameters α appears but has a different meaning: dimensionless gravity, initial displacement, and initial velocity. So, in general, one can write:

$$S' = f(t', \alpha)$$

The selection of scaling parameters is left to the user, but there are some guidelines:

1) The scaling variables must not form a dimensionless group among themselves, but adding one more variable will form a dimensionless quantity. For example, in the above fluid flow problem, test the powers of $\rho, V, and\ L$:

$$\rho^a V^b L^c = (ML^{-3})^a (LT^{-1})^b (L)^c = M^0 L^0 T^0 \quad only\ if\ \ a = b = c = 0$$

2) Do not select output variables for your scaling parameters.
3) If convenient, select popular, not obscure, scaling variables because they will appear in all of your dimensionless groups. For example, select density not surface tension.

Note: the two following criteria must be satisfied before performing dimensional analysis: 1) the proposed physical relation is dimensionally homogenous, and 2) all the relevant variables have been included in the proposed relation.

The Pi theorem

The Buckingham π theorem is a key theorem in dimensional analysis. This provides a method for computing sets of dimensionless parameters from the given variables, even if the form of the equation is still unknown. However, the choice of <u>dimensionless parameters is not unique</u>: Buckingham's theorem only provides a way of generating sets of dimensionless parameters, and will not choose the most 'physically meaningful'.

Let $q_1, q_2, q_3, \dots q_n$ be **n** dimensional variables that are physically relevant in a given problem and that are inter-related by an (unknown) <u>dimensionally homogeneous</u> set of equations. These can be expressed via a functional relationship of the form:

$$F(q_1, q_2, q_3, \dots q_n) = 0 \quad or \quad q_1 = f(q_2, q_3, \dots q_n)$$

If **k** is the number of fundamental dimensions required to describe the **n** variables, then there will be **k** primary variables and the remaining **j = k − n** variables can be expressed as **n − k** dimensionless and independent quantities or 'Pi groups' $\Pi_1, \Pi_2, \Pi_3, \dots \Pi_{n-k}$. The functional relationship can thus be reduced to the much more compact form:

$$\Phi(\Pi_1, \Pi_2, \Pi_3, \dots \Pi_{n-k}) = 0 \quad or \quad \Pi_1 = \phi(\Pi_2, \Pi_3, \dots \Pi_{n-k})$$

Note: this set of non-dimensional parameters is not unique. They are however independent and form a complete set.

Application:

1) Clearly define the problem and think about which variables are important. Identify which is the main variable of interest, i.e., $q_1 = f(q_2, q_3, \dots q_n)$. It is important to physically think about the problem. Are there any constrains, i.e. can I vary all these variables independently? For example, weight of an object $W = \rho V g$ (only two of these are independent, unless g is also variable).

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2) Express each of **n** variables in terms of its fundamental dimensions, {MLTθ} or {FLTθ}. It is often useful to use one system to solve the problem, and then check that groups you obtain are dimensionless by converting to other system.

3) Determine the number of Pi groups, **j** = **n** − **k** variables, where **k** is the number of reference dimensions and select **k** primary or repeating variables. Typically pick variables which characterize the fluid properties, flow geometry, flow rate…

4) Form **j** dimensionless **Π** groups and check that they are all indeed dimensionless.

5) Express result in form $\Pi_1 = \phi(\Pi_2, \Pi_3, \dots \Pi_{n-k})$ where Π_1 contains the quantity of interest and interpret your result physically!

6) Make sure that your groups are indeed independent; i.e. can I vary one and keep others constant?

7) Compare with experimental data!

Example 1:

Use the pi theorem to find the dimensionless parameters in the drag problem.

$$F = F(L, V, \rho, \mu)$$

Non-dimenalization of the basic equations

Another useful application of pi theorem is application to the basic (governing) equations. Even though these equations cannot be solved in general, they will reveal basic dimensionless parameters, such as Reynolds number, in their proper form and proper position, giving clues when they are negligible.

Consider incompressible steady flow governing equations:

$$\nabla . V = 0$$

$$\rho \frac{dV}{dt} = \rho g - \nabla p + \mu \nabla^2 V$$

With boundary conditions. These equations contain the three basic dimensions *M, L*, and *T*. All variables *p, V, x ,y ,z*, and *t* can be non-dimensionalized by using density and two reference constants that might be characteristic of the particular fluid flow:

Reference velocity = *U*   Reference length (or characteristics length) = *L*

As an example, *U* can be the inlet or upstream velocity and *L* the diameter of a body immersed in the stream. We can define all dimensionless variables:

$$V^* = \frac{V}{U} \quad \nabla^* = \frac{\nabla}{L}$$

$$x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad z^* = \frac{z}{L}, \quad R^* = \frac{R}{L}$$

$$t^* = \frac{tU}{L}, \qquad p^* = \frac{p + \rho gz}{\rho U^2}$$

Note that for the pressure, we considered the piezometeric pressure, assuming that z is up (Bernoulli equation). Since $\rho, U, and\ L$ are constants, the derivatives in the governing equations can all be handled in dimensionless form, for example:

$$\frac{\partial u}{\partial x} = \frac{\partial (U u^*)}{\partial (L x^*)} = \frac{U}{L} \frac{\partial u^*}{\partial x^*}$$

After substitution and simplifications, we obtain:

$$\nabla^* . V^* = 0$$

$$\frac{dV^*}{dt^*} = -\nabla^* p^* + \frac{\mu}{\rho UL} \nabla^{*2}(V^*)$$

The boundary conditions should also be non-dimensionalized in a similar manner.

Non-dimensional Parameters

The continuity equation contains no parameter; however, the momentum (Navier-Stokes) equation reveals the **Reynolds number**:

$$Re = \frac{\rho UL}{\mu}$$

The Reynolds number is always important, with or without a free surface, and can be neglected only in flow regions away from high-velocity gradients, e.g. away from solid surfaces, jets or wakes.

Euler number (pressure coefficient) is rarely important unless the pressure drops low enough to cause vapor formation (cavitation) in a liquid:

$$Eu = \frac{\Delta p}{\rho U^2}$$

Froude number is the dominant effect in free-surface flows and it does not appear if there is no free surface:

$$Fr = \frac{U^2}{gL}$$

Mach number has a strong effect on compressible flow properties if it is greater than 0.3:

$$Ma = \frac{U}{a}$$

where a is the speed of sound in the fluid.

Table 5-2, in the textbook, provides a comprehensive list of important dimensionless groups in fluid mechanics.

Modeling and its pitfalls

Performing scale analysis is mathematically straightforward; however, there are several issues to consider:

1) We took for granted the important variables for the phenomena. Selection of important variables is not a trivial task and requires considerable judgment and experience. Note that each extra pi group that is retained increases the expense and effort and level of complexity of the solution.

2) Typically the Reynolds number of the model is too small (by a factor of 10 to 1000). As a result, we end up estimating the desired high-Reynolds-number prototype data, by extrapolating the model data in low Re, as shown in Fig.1. This may result in high uncertainty in the analysis; but there is no other practical alternative in hydraulic model testing.
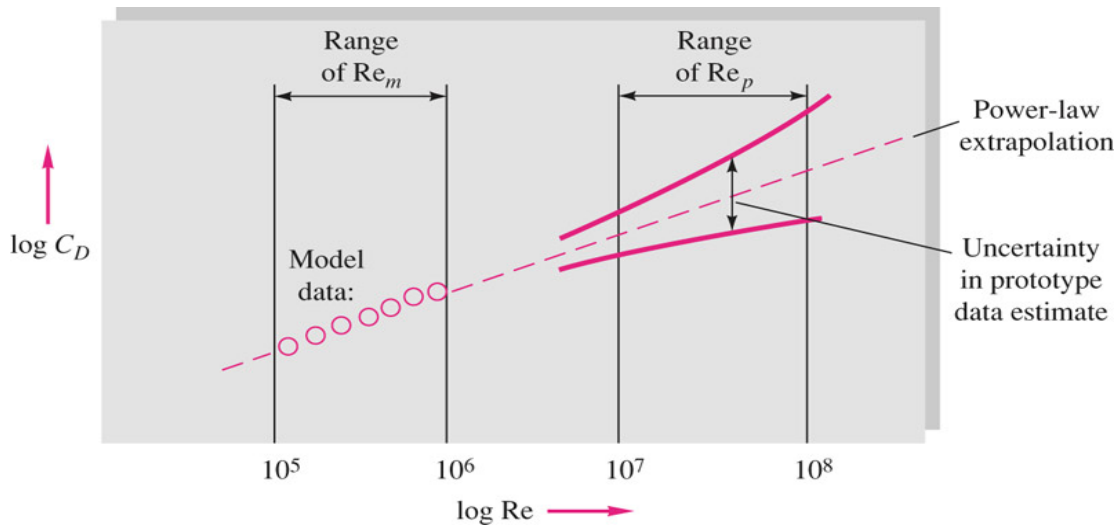

Fig.1: Reynolds number extrapolation.

3) After selecting important parameters and determining pi groups, we should seek to achieve similarity between the model tested and the prototype to be designed. In general:

"Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for the model and the prototype."

Establishing complete similarity is highly unlikely. Therefore, we seek particular types of similarity.

Geometric similarity: concerns the length dimension {L} and must be ensured before any model testing can proceed. In general geometrical similarity is established when:

A model and prototype are geometrically similar if and only if all body dimensions in all 3 coordinates have the same linear scale ratio.

This means all the angles, flow directions, and the orientations of model and prototype with respect to surroundings must be identical.
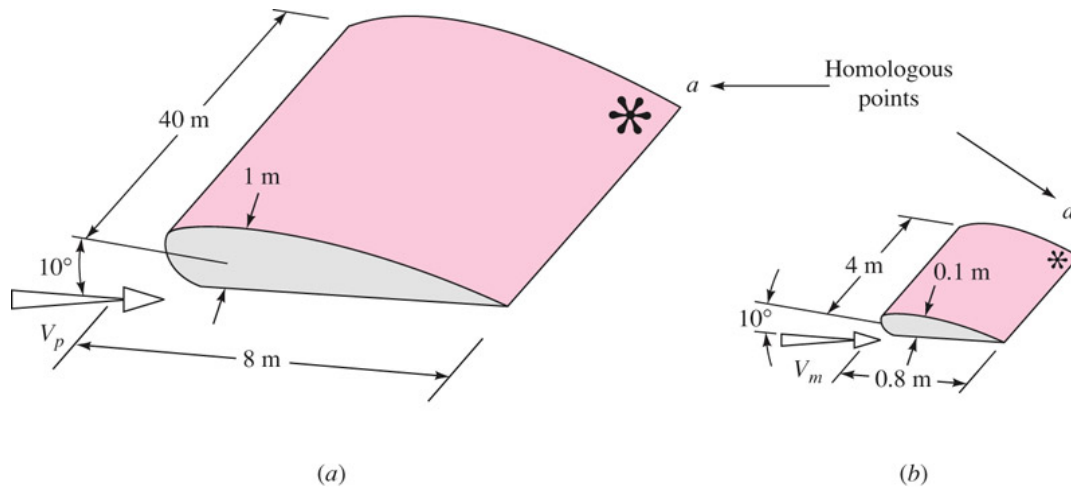

Fig. 2: Geometric similarity in model testing.

Kinematic similarity: requires that the model and prototype have the same length scale ratio and the same time scale ratio; thus the velocity scale ratio will be the same for both.

Length scale equivalence simply implies geometric similarity, but time scale equivalence may require additional dynamic considerations such as equivalence of the Reynolds and Mach numbers.
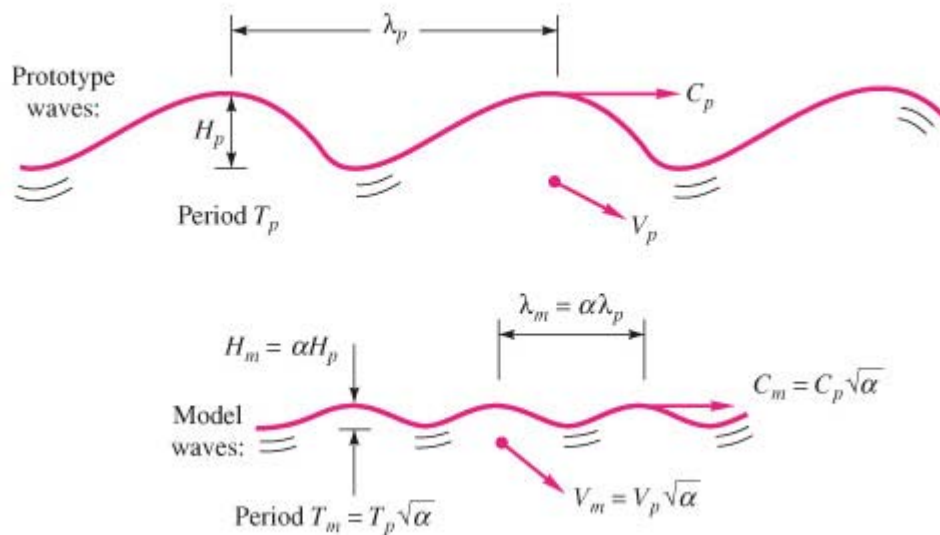

Fig.3: Free surface flows are kinematically similar with length and time scales related by the Froude number.

Frictionless flows with a free surface, see Fig. 3, are kinematically similar if their Froude numbers are equal:

$$Fr_m = \frac{V_m^2}{gL_m} = \frac{V_p^2}{gL_p} = Fr_p$$

If $L_m = \alpha L_p$ where α is a dimensionless ratio, the velocity scale becomes:

$$\frac{V_m}{V_p} = \left(\frac{L_m}{L_p}\right)^{\frac{1}{2}} = \alpha^{\frac{1}{2}}$$

$$\frac{T_m}{T_p} = \frac{L_m/V_m}{L_p/V_p} = \sqrt{\alpha}$$

Dynamic similarity: exists when the model and the prototype have the same length scale ratio, time scale ratio, and the force scale (mass scale) ratio.

Dynamic similarity exists, simultaneously with kinematic similarity, if the model and prototype force and pressure coefficients are identical. This is established if:

1) For compressible flow, the model and prototype Reynolds number and Mach number and specific heat ratio are correspondingly equal.

2) For incompressible flow

    a) With no free surface: model and prototype Reynolds numbers are equal.

    b) With a free surface: model and prototype Reynolds number, Froude number, and (if necessary) Weber number and cavitation number are correspondingly equal.
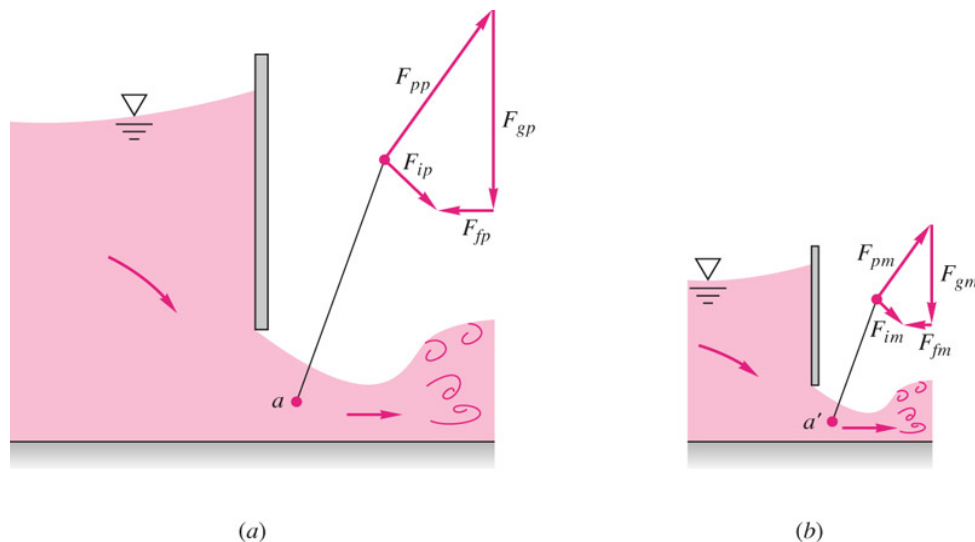

Fig. 4: dynamic similarity in sluice gate flow. Model and prototype yield identical homologous force polygons if the Reynolds and Froude numbers are the same correspondingly.