Fluid Statics

When the fluid velocity is zero, called the *hydrostatic condition*, the pressure variation is due only to the weight of the fluid.

Consider a small wedge of fluid at rest of size Δ*x*, Δ*z*, Δ*s* and depth *b* into the paper. There is no shear stress by definition, and pressure is assumed to be identical on each face (small element).

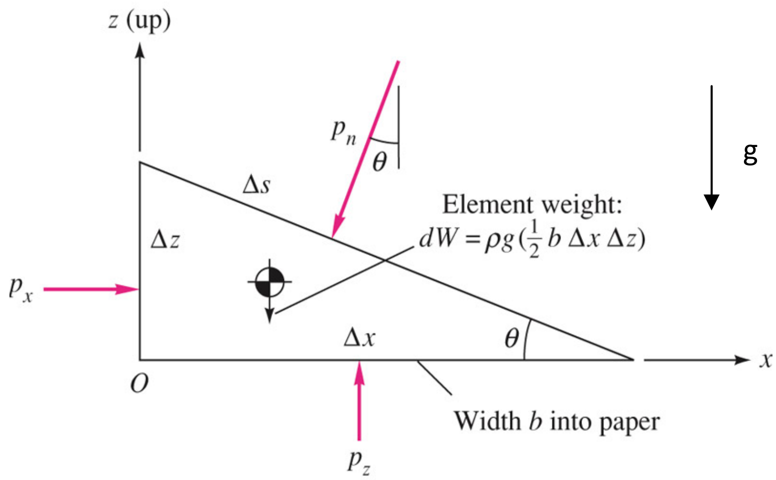


Fig. 1: Equilibrium of a small Fluid element at rest.

Since the element is at rest, summation of all forces must equal zero.

$$\sum F_x = 0 = p_x\, b\, \Delta z - p_n b\, \Delta s\, sin\theta$$

$$\sum F_z = 0 = p_z\, b\, \Delta x - p_n b\, \Delta s\, cos\theta - \frac{1}{2}\rho g\, b\, \Delta x\, \Delta z$$

From geometry, $\Delta s\; sin\theta = \Delta z \qquad \Delta s\; cos\theta = \Delta x$. After substitution in above equations, one finds:

$$p_x = p_n \qquad p_z = p_n + \frac{1}{2}\rho g\, \Delta z$$

This means:

1) There is no pressure change in the horizontal direction.
2) There is a vertical change in pressure proportional to the density, gravity and depth change in the fluid (i.e. the weight of the column of the fluid above the point).

Note: in the limit as the fluid wedge shrinks to a point, Δ*z* goes to zero, we have: $p_x = p_z = p_n = p$. Thus, pressure in a static fluid is a point property.

Pressure force on a fluid element

Assume the pressure vary arbitrarily in a fluid, *p*=*p*(x,y,z,t). Consider a fluid element of size Δ*x*, Δy, Δz as shown in Fig. 2. The net force in the x-direction is given by:

$$dF_x = pdydz - \left(p + \frac{\partial p}{\partial x}dx\right)dydz = -\frac{\partial p}{\partial x}dx\,dy\,dz$$

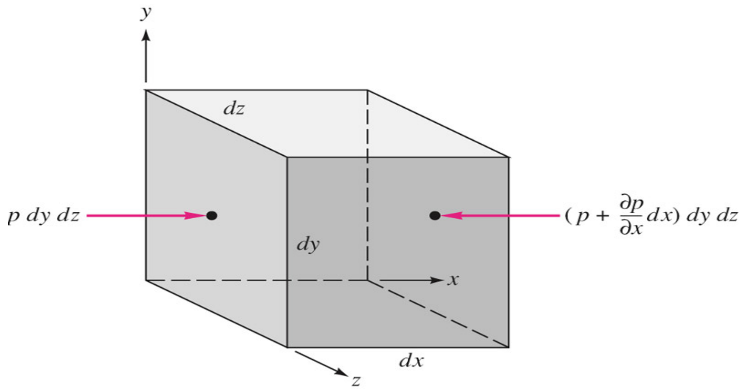


Fig. 2: Net force in the x-direction due to pressure variation.

In a similar manner, net forces acting in y- and z-directions can be calculated. The total net force vector, due to pressure, is:

$$dF_{press} = -\left(\frac{\partial p}{\partial x}i + \frac{\partial p}{\partial y}j + \frac{\partial p}{\partial z}k\right)dx\,dy\,dz$$

Notice that the term in the parentheses is the negative vector gradient of pressure and the term *dx dy dz =dV*, is the volume of the element. Therefore, one can write:

$$\boldsymbol{f}_{press} = -\nabla p$$

where $\boldsymbol{f}_{press}$ is the net force per volume. Notice that the pressure gradient (not pressure) causing a net force that must be balanced by gravity or acceleration and/or other effects in the fluid.

Note: the pressure gradient is a *surface force* that acts on the sides of the element. That must be balanced by gravity force, or weight of the element, in the fluid at rest.

In addition to gravity, a fluid in motion will have surface forces due to viscous stresses. Viscous forces, however, for a fluid at rest are zero.

The gravity force is a body force, acting on the entire mass of the element:

$$dF_{gravity} = \rho g dx\,dy\,dz \qquad \boldsymbol{f}_{gravity} = \rho g$$

Gage pressure and vacuum pressure

The actual pressure at a given position is called the *absolute pressure*, and it is measured relative to absolute vacuum.

The measure pressure may be either lower (called vacuum pressure) or higher (gage pressure) than the local atmosphere.

$$p > p_a \quad Gage\ pressure \qquad p_{gage} = p - p_a$$

$$p < p_a \quad Vacuum\ pressure \qquad p_{vacuum} = p_a - p$$

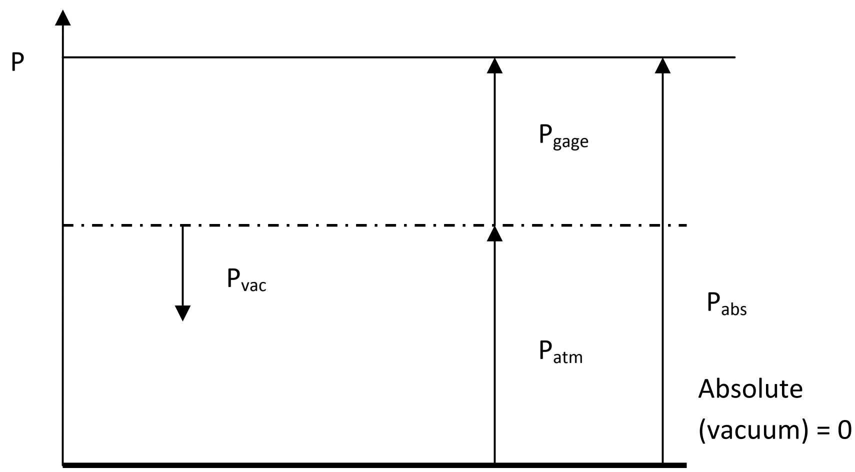


Fig. 3: Absolute, gage, and vacuum pressures.

Hydrostatic pressure distribution

For a fluid at rest, the summation of forces acting on the element must be balanced by the gravity force.

$$\overrightarrow{\nabla p} = \rho \vec{g}$$

This is a hydrostatic distribution and is correct for all fluids at rest, regardless of viscosity.

Recall that the vector operator ∇ expresses the magnitude and direction of the maximum spatial rate of increase of the scalar property (in this case pressure).

Note: ∇p is perpendicular everywhere to surface of constant pressure *p*. In other words, in a fluid at rest will align its constant-pressure surfaces everywhere normal to the local-gravity vector. Or, the pressure increase will be in the direction of gravity (downward). However, in our customary coordinate z is "upward" and the gravity vector is:

$$\vec{\boldsymbol{g}} = -gk$$

where $g=9.807\ m/s^2$. For this coordinate, the pressure gradient vector becomes:

$$\frac{\partial p}{\partial x} = 0 \quad \frac{\partial p}{\partial y} = 0 \quad \frac{\partial p}{\partial z} = -\rho g = -\gamma$$

Since pressure is only a function of *z* (independent of x and y), we can write:

$$\frac{dp}{dz} = -\gamma \qquad p_2 - p_1 = -\int_1^2 \gamma dz$$

As a result, we can conclude: pressure in a continuously distributed uniform static fluid **varies only with vertical distance** and is independent of the shape of the container. The pressure is the same at all points on a given horizontal plane in a fluid.

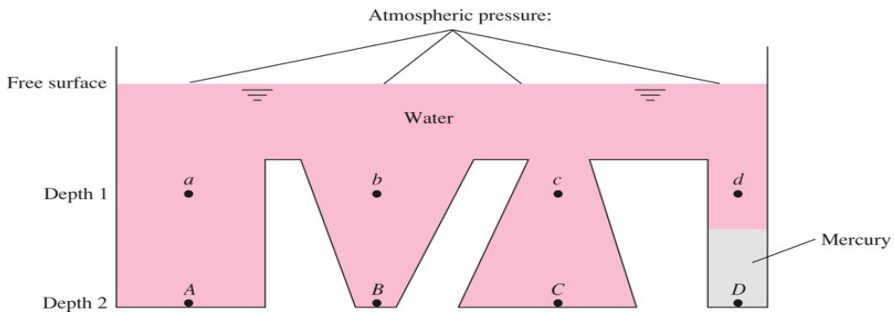


Fig. 4: Hydrostatic pressure is only a function of the depth of the fluid, $p_a = p_b = p_c$. However, $p_A = p_B = p_C \neq p_D$. Because point *D*, although at the same level, lies beneath a different fluid, mercury. The free surface of the container is atmospheric and forms a horizontal line.

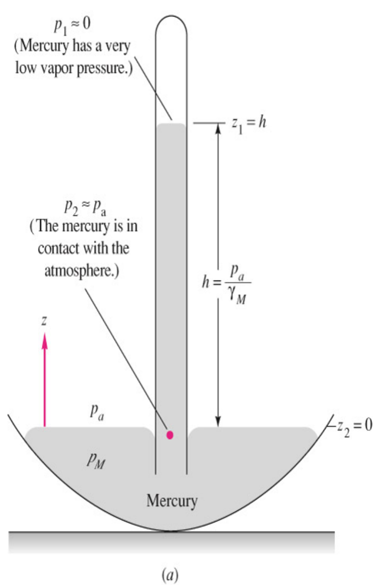
<u>Note</u>: In most engineering applications, the variation in acceleration of gravity (g) due to different heights is less than 0.6% and can be neglected.

For liquids, which are incompressible, we have:

$$p_2 - p_1 = -\gamma(z_2 - z_1) \quad or \quad z_1 - z_2 = \frac{p_2}{\gamma} - \frac{p_1}{\gamma}$$

The quantity, p/γ is a length called the *pressure head* of the fluid.

The mercury barometer



Mercury has an extremely small vapor pressure at room temperature (almost vacuum), thus p_1 = 0. One can write:

$$p_a - 0 = -\gamma_{mercury}(0 - h) \quad or \quad h = \frac{p_a}{\gamma_{mercury}}$$

At the sea-level, the atmospheric pressure reads, 761 mmHg.

Hydrostatic pressure in gases

Gases are compressible with density nearly proportional to pressure, thus the variation in density must be considered in hydrostatic calculations. Using the ideal gas equation of state, $p = \rho RT$:

$$\frac{dp}{dz} = -\rho g = -\frac{p}{RT} g$$

After integration between points 1 and 2 and also assuming a constant temperature at both points $T_1 = T_2 = T_0$ (isothermal atmosphere), we find:

$$p_2 = p_1 exp\left[-\frac{g(z_2 - z_1)}{RT_0}\right]$$

The isothermal assumption is a fair assumption for earth. However, for higher altitudes the atmospheric temperature drops off nearly linearly with *z*, i.e. $T \approx T_0 - Bz$, where T_0 is the sea-level temperature (in Kelvin) and *B=0.00650 K/m*, we find:

$$p = p_a \left(1 - \frac{Bz}{T_0}\right)^{g/RB} \qquad \text{for air } \frac{g}{RB} = 5.26$$

Note that the atmospheric pressure is nearly zero (vacuum condition) at *z = 30 km*.

Manometry

It is shown that a change in elevation of a liquid is equivalent to a change in pressure, $\Delta h = \Delta p/\gamma$. Thus a static column of one or multiple fluids can be used to measure pressure difference between 2 points. Such a device is called *manometer*.

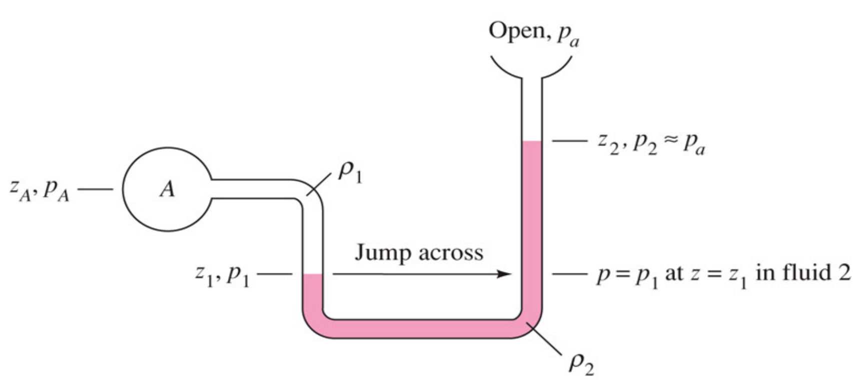


Fig. 5: Simple open manometer.

$$p_A + \gamma_1|z_A - z_1| - \gamma_2|z_1 - z_2| = p_2 = p_{atm}$$

Two roles for manometer analysis:

1) <u>Adding/ subtracting $\gamma\Delta z$</u> as moving down/up in a fluid column.

2) Jumping across U-tubes: any two points at the same elevation in a continuous mass of the same static fluid will be at the same pressure, thus we can jump across U-tubes filled with the same fluid.

Hydrostatic forces on plane surface

Consider a plane panel of arbitrary shape completely submerged in a liquid.

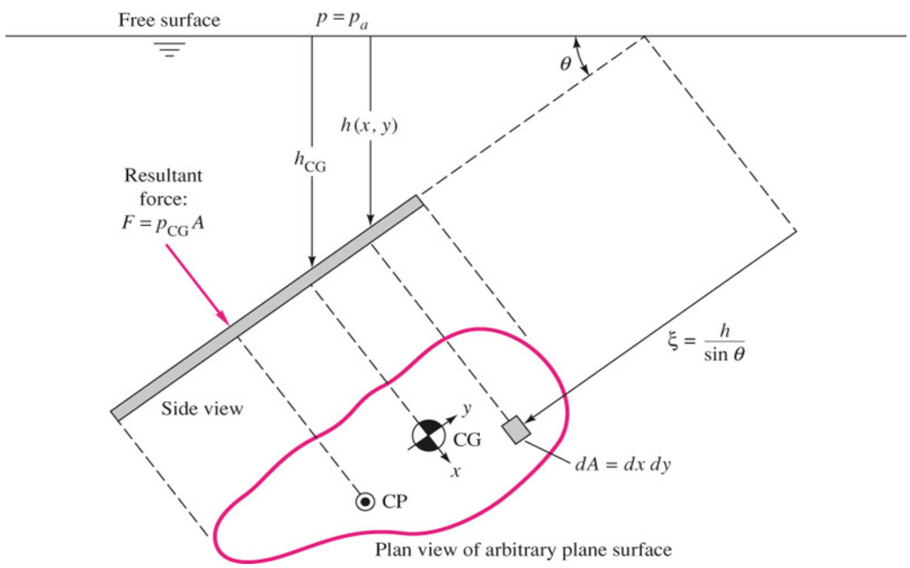


Fig. 6: hydrostatic force and center of pressure on a plane submerged in a liquid at an angle θ.

If *h* is the depth to any element area *dA*, the local pressure is:

$$p = p_a + \gamma h$$

The total hydrostatic force on one side of the plane is given by:

$$F = \int p dA = \int (p_a + \gamma h)\, dA = p_a A + \gamma \int h dA$$

We also have: $h = \xi sin\theta$. After substitution, we get:

$$F = p_a A + \gamma\ sin\theta \int \xi dA = p_a A + \gamma\ sin\theta\ \xi_{CG} A$$

Since, $h_{CG} = sin\theta\ \xi_{CG}$,

$$F = p_a A + \gamma h_{CG} A = (p_a + \gamma h_{CG})A = p_{CG} A$$

It means, the force on one side of any plane submerged surface in a uniform fluid equals the pressure at the plate centroid times the plate area, independent of the shape of the plate or angle θ.

To balance the bending-moment portion of the stress, the resultant force F acts not through the centroid but below it toward the high pressure side. Its line of action passes through the centre of pressure CP of the plate (x_{CP}, y_{CP}).

To find the center of pressure, we sum moments of the elemental force *pdA* about the centroid and equate to the moment of the resultant force, F:

$$Fy_{CP} = \int ypdA = \int y(p_a + \gamma\xi sin\theta)\, dA = \gamma sin\theta \int y\xi dA$$

The term $\int ydA = 0$, by definition of centroidal axes. Using the definition of the *area moment of inertia about centroidal x* axis, $I_{xx} = \int y^2 dA$, after some simplifications:

$$y_{CP} = -\gamma sin\theta \frac{I_{xx}}{p_{CG}A}$$

The negative sign shows that y_{CP} is below the centroid at a deeper level and depends on angle θ and the shape of the plate (I_{xx}).

Following the same procedure, we find:

$$x_{CP} = -\gamma sin\theta \frac{I_{xy}}{p_{CG}A}$$

Note: for symmetrical plates, I_{xy} = 0 and thus x_{CP} = 0. As a result, the center of pressure lies directly below the centroid on the *y* axis.

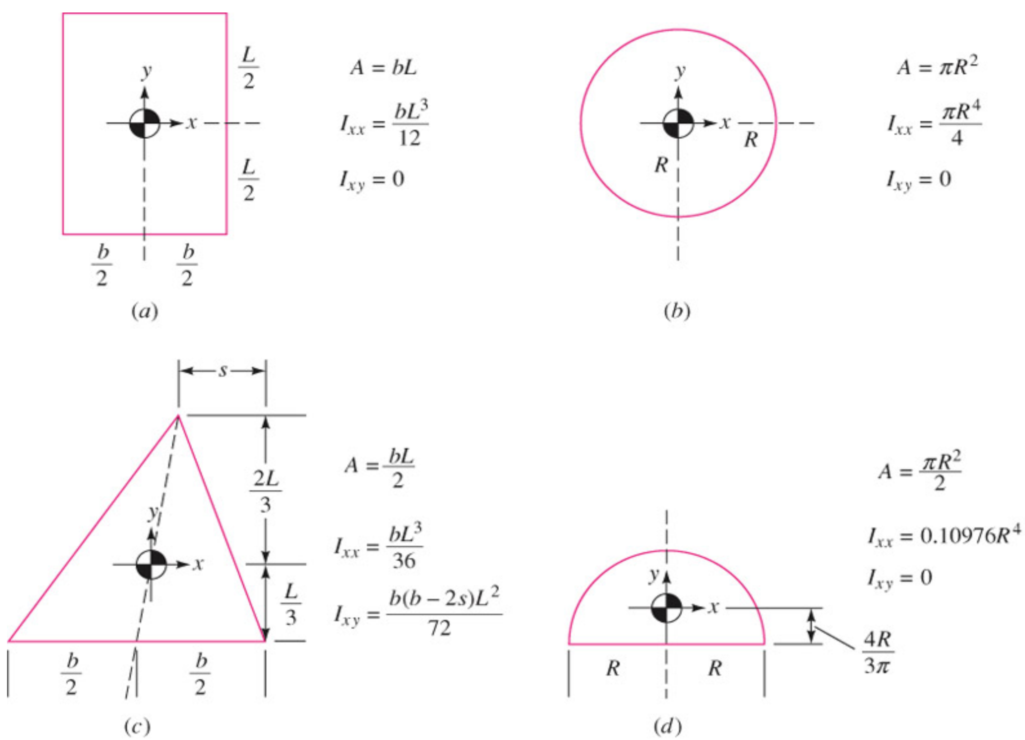


Fig. 7: Centroidal moments of inertia for various cross-sections.

Hydrostatic forces on curved surfaces

The easiest way to calculate the pressure forces on a curved surface is to compute the horizontal and vertical forces separately.

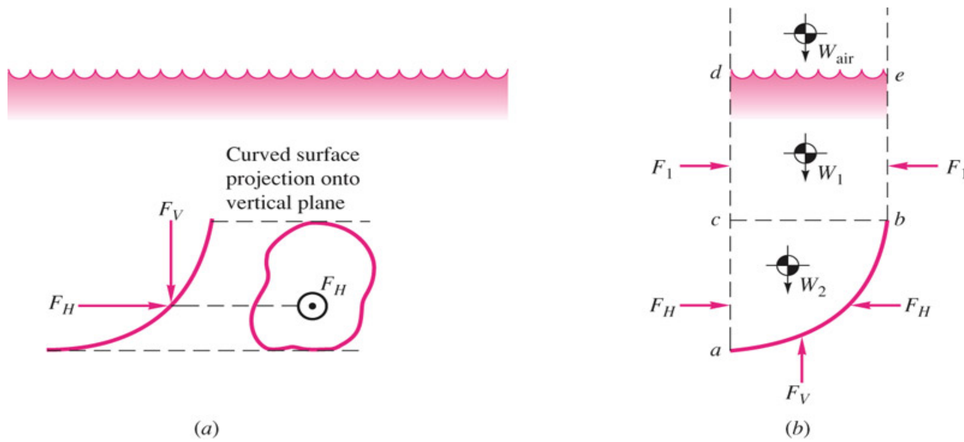


Fig. 8: Calculating horizontal and vertical pressure forces on an immersed curved surface.

Using the free-body diagram shown in Fig. 8b, one can find:

The horizontal force, F_H equals the force on the plane area formed by the projection of the curved surface onto a vertical plane normal to the component.

The vertical component equals to the weight of the entire column of fluid, both liquid and atmospheric above the curved surface. For the surface shown in Fig. 8:

$$F_V = W_2 + W_1 + W_{air}$$