Integral Relations for a Control Volume

Control volume approach is accurate for any flow distribution but is often based on the "one-dimensional" property values at the boundaries. It gives useful engineering estimates.

System and control volume

A system is defined as a fixed quantity of matter or a region in space chosen for study. The mass or region outside the system is called the surroundings.

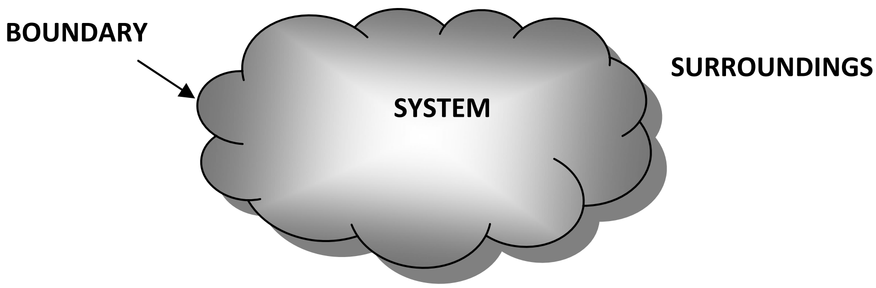


Fig.1: System, surroundings, and boundary.

$$m_{sys} = const. \qquad \frac{dm_{sys}}{dt} = 0$$

System boundary: the real or imaginary surface that separates the system from its surroundings. The boundaries of a system can be fixed or movable. Mathematically, the boundary has zero thickness, no mass, and no volume.

Open system or control volume is a properly selected region in space. It usually encloses a device that involves mass flow such as a compressor. Both mass and energy can cross the boundary of a control volume.

Note: control volume is an abstract concept and does not hinder the flow in any way.

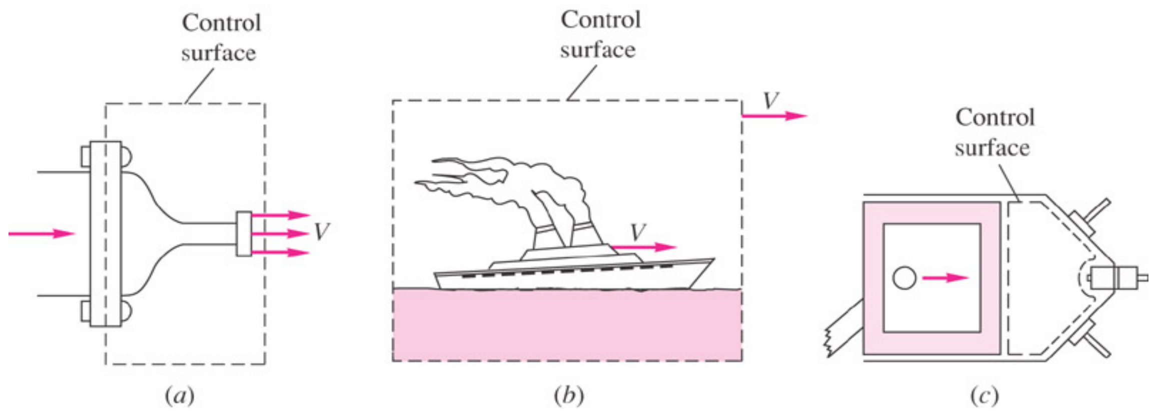


Fig. 2: Examples of fixed, moving, and deformable control volume.

Volume and mass flow rate

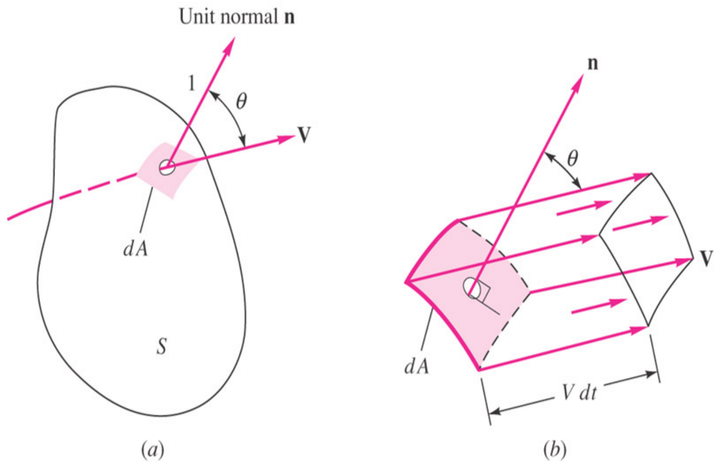


Fig.3: Volume flow rate through arbitrary surface.

Let *n* be defined as the unit vector normal to *dA*. Then the amount of fluid swept through *dA* in time *dt* is:

$$dV = \boldsymbol{V}\, dt\, dA\, cos\theta = (\boldsymbol{V}.\boldsymbol{n})dA\, dt$$

The integral of *dV/dt* is the total volume rate of flow *Q* through the surface *S*:

$$Q = \int_S (\boldsymbol{V}.\boldsymbol{n})dA = \int_S V_n\, dA$$

where V_n is the normal component of the velocity. We consider **n** to be the outward normal unit vector. Volume can be multiplied by density to obtain the mass flow $\dot{m}$.

$$\dot{m} = \int_S \rho(\boldsymbol{V}.\boldsymbol{n})dA = \int_S \rho V_n\, dA$$

If density and velocity are constant over the surface *S*, a simple expression results:

$$\dot{m} = \rho Q = \rho A V$$

The Reynolds Transport Theorem

To convert a system analysis to control volume analysis, we must convert our mathematics to apply to a specific region rather than to individual masses. This conversion is called the Reynolds transport theorem.

Consider a fixed control volume with an arbitrary flow pattern through. In general, each differential area *dA* of surface will have a different velocity *V* with a different angle θ with the normal to *dA*. One can find:

In flow volume: $(VA\ cos\theta)_{in}dt$ Outflow volume $(VA\ cos\theta)_{out}dt$.

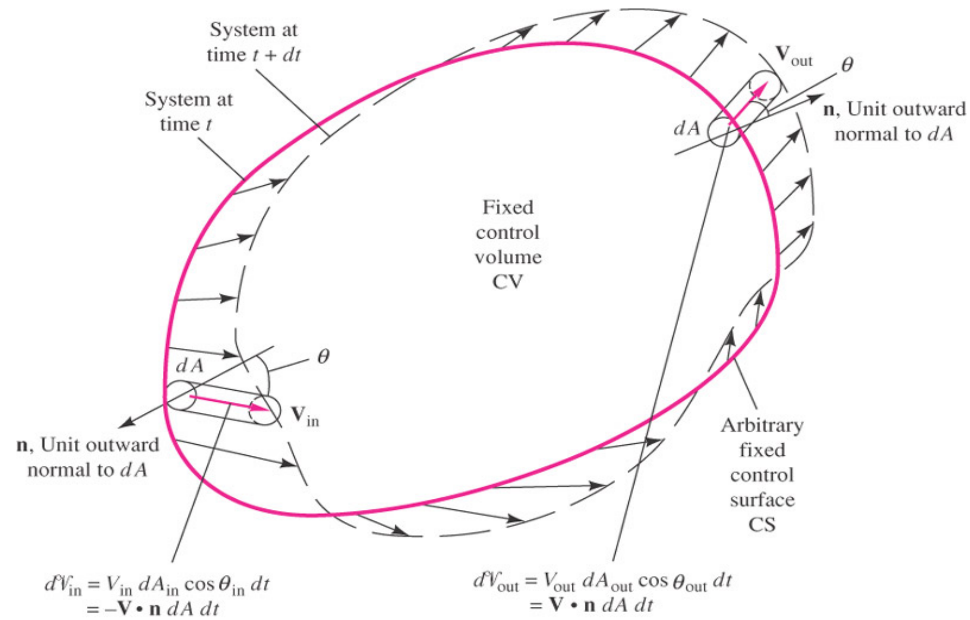


Fig. 4: Control volume, Reynolds transport theorem.

Let B be any property of the fluid (energy, momentum, enthalpy, etc.) and $\beta = dB/dm$ be the intensive value of the amount B per unit mass in any small element of the fluid.

The total amount of B in the control volume is:

$$B_{CV} = \int_{CV} \beta dm = \int_{CV} \beta\rho dV \qquad \beta = \frac{dB}{dm}$$

A change within the control volume:

$$\frac{d}{dt}\left(\int_{CV} \beta\rho dV\right)$$

Outflow of β from the control volume:

$$\int_{CS} \beta\rho V\ cos\theta\ dA_{out}$$

Inflow of β to the control volume:

$$\int_{CS} \beta\rho V\ cos\theta\ dA_{in}$$

CV and CS refer to control volume and control surface, respectively.

For the system shown in Fig. 4, the instantaneous change of *B* in the system is sum of the change within, plus the outflow, minus the inflow:

$$\frac{d}{dt}(B_{sys}) = \frac{d}{dt}\left(\int_{CV} \beta\rho dV\right) + \int_{CS} \beta\rho V\, cos\theta\, dA_{out} - \int_{CS} \beta\rho V\, cos\theta\, dA_{in}$$

Note the control volume is fixed in space, the elemental volume do not vary with time. Also we note that V cosθ is the component of V normal to the area element of the control surface. Thus we can write:

$$Flux\ term = \int_{CS} \beta\rho V_n\, dA_{out} - \int_{CS} \beta\rho V_n\, dA_{in} = \int_{CS} \beta\, d\dot{m}_{out} - \int_{CS} \beta\, d\dot{m}_{in}$$

The vector form of the above equation is:

$$Flux\ term = \int_{CS} \beta\rho(\vec{V}.\vec{n})\, dA$$

And the Reynolds transport theorem, in the vector form, becomes:

$$\frac{d}{dt}(B_{sys}) = \frac{d}{dt}\left(\int_{CV} \beta\rho dV\right) + \int_{CS} \beta\rho(\vec{V}.\vec{n})\, dA$$

One-dimensional flux term approximation

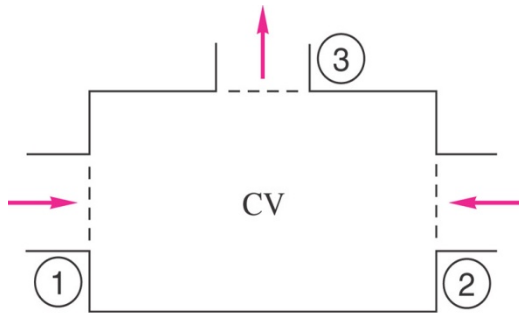
In many situations, the flow crosses the boundaries of the control surface at simplified inlets and exits that are approximately one-dimensional (the velocity can be considered uniform across each control surface). For a fixed control volume, the surface integral reduces to:

$$\frac{d}{dt}(B_{sys}) = \frac{d}{dt}\left(\int_{CV} \beta\rho dV\right) + \sum_{outlets} \beta_i \dot{m}_i|_{out} - \sum_{inlets} \beta_i \dot{m}_i|_{in} \quad where \quad \dot{m}_i = \rho_i A_i V_i$$

Example 1

A fixed control volume has three one-dimensional boundary sections, as shown in the figure below. The flow within the control volume is steady. The flow properties at each section are tabulated below. Find the rate of change of energy that occupies the control volume at this instant.

Control surface	**type**	**ρ, kg/m^3**	**V, m/s**	**A, m^2**	**e, J/kg**
1	inlet	800	5.0	2.0	300
2	inlet	800	8.0	3.0	100
3	outlet	800	17.0	2.0	150



Conservation of mass

For conservation of mass, *B*=*m* and $\beta = dm/dm = 1$. The Reynolds transport equation becomes:

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \int_{CS} \rho(\vec{V}.\vec{n})\, dA = \left(\frac{dm}{dt}\right)_{system}$$

If the control volume only has a number of one-dimensional inlets and outlets, we can write:

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

Note: for steady-state flow, $\partial\rho/\partial t = 0$, and the conservation of mass becomes:

$$\sum_i (\rho_i A_i V_i)_{out} = \sum_i (\rho_i A_i V_i)_{in}$$

This means, in steady flow, the mass flows entering and leaving the control volume must balance.

Average velocity

In cases that fluid velocity varies across a control surface, it is often convenient to define an average velocity.

$$V_{av} = \frac{Q}{A} = \frac{1}{A}\int (\vec{V}.\vec{n}) dA$$

The average velocity is only a concept, i.e., when it is multiplied by the area gives the volume flow.

If the density varies across the cross-section, we similarly can define an average density:

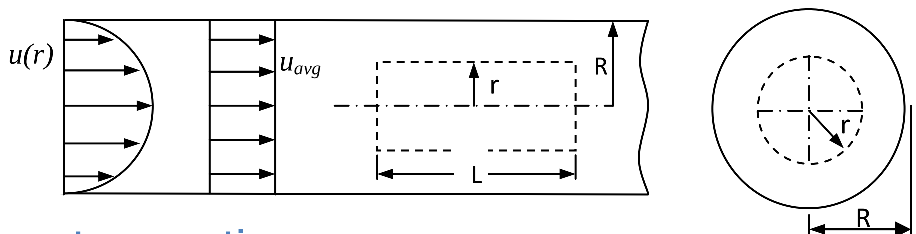
$$\rho_{av} = \frac{1}{A}\int \rho dA$$

Example 2

In a grinding and polishing operation, water at 300 K is supplied at a flow rate of 4.264×10^{-3} kg/s through a long, straight tube having an inside diameter of D=2R=6.35 *mm*. Assuming the flow within the tube is laminar and exhibits a parabolic velocity profile:

$$u(r) = u_{max}\left[1-\left(\frac{r}{R}\right)^2\right]$$

where u_{max} is the maximum fluid velocity at the center of the tube. Using the definition of the mass flow rate and the concept of average velocity, show that: $u_{avg} = \frac{u_{max}}{2}$



The linear momentum equation

For Newton's second law, the property being differentiated is the linear momentum, *mV*. Thus *B* =*mV* and $\beta = dB/dm = V$. The Reynolds transport theorem becomes:

$$\frac{d}{dt}(m\vec{V})_{sys} = \sum \vec{F} = \frac{d}{dt}\left(\int_{CV} \vec{V}\rho dV\right) + \int_{CS} \vec{V}\rho(\vec{V}.\vec{n})\, dA$$

Note that this is a vector equation and has three components.

<u>Momentum flux term</u>,

$$\dot{M}_{CS} = \int_{CS} \vec{V}\rho(\vec{V}.\vec{n})\, dA$$

If cross-section is one-dimensional, V and ρ are uniform and over the area, momentum flux simplifies:

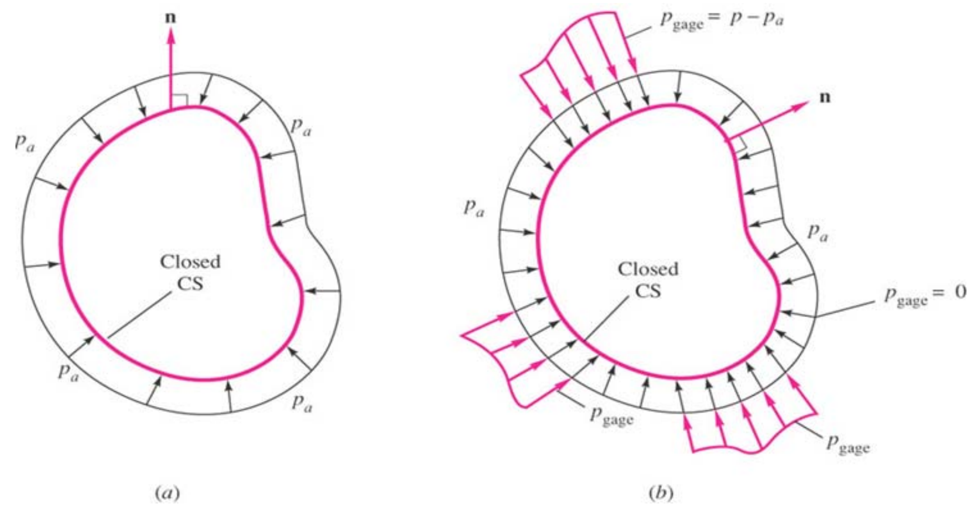
$$\dot{M}_i = V_i(\rho_i A_i V_i) \quad = \dot{m}_i V_i$$

For one-dimensional inlets and outlets, we have:

$$\Sigma\vec{F} = \frac{d}{dt}\left(\int_{CV} \vec{V}\rho\, dV\right) + \sum_i (\dot{m}_i\vec{V}_i)_{out} - \sum_i (\dot{m}_i\vec{V}_i)_{in}$$

Net pressure force on a closed CV

Recall that the external pressure force on a surface is normal and inward.



Since the unit vector n is outward, we can write:

$$F_{press} = \int_{CS} p(-n)dA$$

If the pressure has a uniform value p_a all around the surface, the net pressure force is zero.

$$F_{uniform\ press} = -p_a \int_{CS} ndA = 0$$

This is independent of the shape of the surface. Thus pressure force problems can be simplified by subtracting any convenient uniform pressure p_a and working only with the pieces of gage pressure that remain:

$$F_{press} = \int_{CS} (p - p_a)(-n)dA = \int_{CS} p_{gage}(-n)dA$$

Note: The axial velocity is non-uniform, thus the simple momentum flux calculation $\int u\rho(V.n)\,dA = \dot{m}V = \rho AV^2$ is not accurate. To consider the effects of non-uniform velocity, we introduce a correction factor β.

$$\rho \int u^2\,dA = \beta \dot{m} V_a = \beta \rho A V_{av}^2 \qquad \text{or} \qquad \beta = \frac{1}{A}\int \left(\frac{u}{V_{av}}\right)^2 dA$$

Values of β can be calculated using the duct velocity profile and the above definition:

Laminar flow: $u = U_0\left(1 - \frac{r^2}{R^2}\right) \quad with\ \ \beta = 4/3$

Turbulent flow: $u = U_0\left(1 - \frac{r}{R}\right)^m \quad with\ \ \beta = \frac{(2+m)^2(1+m)^2}{2(1+2m)(2+2m)}$