

## The angular momentum

A control volume analysis can be applied to the angular momentum, by letting  $\mathbf{B}$  equal to angular-momentum vector  $\mathbf{H}$ . If  $O$  is the point about which moments are desired, the angular moment about  $O$  is given:

$$H_O = \int_{System} (\vec{r} \times \vec{V}) dm$$

where  $r$  is the position vector from  $O$  to the elemental mass  $dm$  and  $V$  is velocity of that element. The angular momentum per unit mass then becomes:

$$\beta = \frac{dH_O}{dm} = \vec{r} \times \vec{V}$$

The Reynolds transport equation can be written as follows:

$$\frac{dH_O}{dt} = \sum M_O = \sum (\vec{r} \times \vec{F})_O = \frac{d}{dt} \left[ \int_{CV} (\vec{r} \times \vec{V}) \rho dV \right] + \int_{CS} (\vec{r} \times \vec{V}) \rho (\vec{V} \cdot \vec{n}) dA$$

Recall that the total moment equals the summation of moments of all applied forces about point  $O$ .

For one-dimensional inlets and exits, the angular momentum flux terms evaluated on the control surface become:

$$\int_{CS} (\vec{r} \times \vec{V}) \rho (\vec{V} \cdot \vec{n}) dA = \sum (\vec{r} \times \vec{V})_{out} \dot{m}_{out} - \sum (\vec{r} \times \vec{V})_{in} \dot{m}_{in}$$

## Energy equation

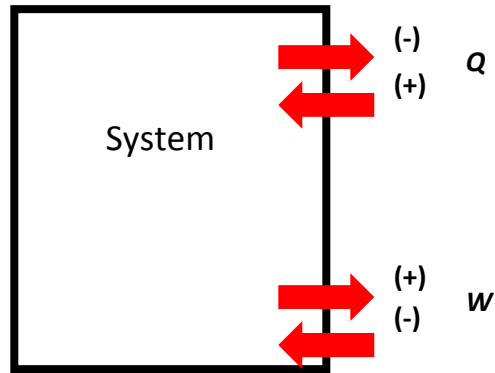
One of the most fundamental laws of nature is the conservation of energy principle or the first law of thermodynamics. It simply states that during an interaction, energy can change from one form to another but the total amount of energy remains constant.

A system can exchange energy with its surroundings through heat and work transfer. In other words, work and heat are the forms that energy can be transferred across the system boundary.

Sign convention: work done by a system is positive, and the work done on a system is negative. Heat transfer to the system is positive and from a system will be negative.

If  $\delta Q$  heat is added to a system and  $\delta W$  work is done by the system, the system energy  $dE$  must change based on the first law of thermodynamics:

$$\delta Q - \delta W = dE$$



Or in a rate form:

$$\dot{Q} - \dot{W} = \frac{dE}{dt}$$

Note that it is a scalar relationship and has only a single component.

Applying the Reynolds transport theorem to the first law of thermodynamics, the dummy variable  $B=E$  (energy) and  $\beta = dE/dm = e$ . The Reynolds transport equation for a fixed control volume becomes:

$$\frac{dQ}{dt} - \frac{dW}{dt} = \frac{dE}{dt} = \frac{d}{dt} \left( \int_{CV} e \rho dV \right) + \int_{CS} e \rho (\mathbf{V} \cdot \mathbf{n}) dA$$

Recall that energy per unit mass  $e$  may be of several types:

$$e = e_{internal} + e_{kinetic} + e_{potential} + e_{others}$$

Here, we neglect  $e_{others}$  and consider only the first three terms, with  $z$  defined as “up”:

$$e = u + \frac{1}{2} V^2 + gz$$

where  $u$  is internal energy.

$\dot{W}$  or work rate can be divided into three parts:

$$\dot{W} = \dot{W}_{shaft} + \dot{W}_{press} + \dot{W}_{viscous\ stresses} = \dot{W}_s + \dot{W}_p + \dot{W}_v$$

Note that the work of gravitational forces has been included in the potential energy and other types of work (e.g. electromagnetic) are excluded here.

Shaft work: is the work done by a machine (e.g. pump impeller, fan blade, piston, etc) and it involves a shaft that crosses the control surface.

Pressure work: is done by pressure forces and occurs at the surface only; all work on internal portions of the material in the control volume is by equal and opposite forces and is self-canceling. The pressure work rate can be calculated from:

$$d\dot{W}_p = -(pdA)\mathbf{V}_{n,in} = -p(-\mathbf{V}\cdot\mathbf{n})dA$$

To find the total work, we should integrate it over the entire control surface:

$$\dot{W}_p = \int_{CS} p(\mathbf{V}\cdot\mathbf{n})dA$$

Shear work: is due to viscous stresses occurs at the control surface and consists of the product of each viscous stress (one normal and two tangential) and the respective velocity component:

$$d\dot{W}_v = -\boldsymbol{\tau}\cdot\mathbf{V}dA \quad \text{or} \quad \dot{W}_v = - \int_{CS} \boldsymbol{\tau}\cdot\mathbf{V}dA$$

where  $\boldsymbol{\tau}$  is the stress vector on the elemental surface  $dA$ . This term is rarely important and may vanish or be negligible according to the particular type of surface.

Note 1: that for all parts of control surface that are solid containing walls,  $\mathbf{V}=0$  (no-slip condition) thus  $\dot{W}_v = 0$ .

Note 2: for surface of a machine, the viscous work is contributed to the machine and we absorb it in the shaft work  $\dot{W}_s$ .

Note 3: the viscous work at inlet or outlets can be neglected due to negligible amount of  $\tau_{nn}V_n dA$ .

As a result, the net rate-of-work simplifies to:

$$\dot{W} = \dot{W}_s + \int_{CS} p(\mathbf{V}\cdot\mathbf{n})dA - \int_{CS} (\boldsymbol{\tau}\cdot\mathbf{V})_{stream\ surface}dA$$

The last term in above equation should be considered if the control volume is a streamline and in case shear stresses are significant along the streamline, for example the upper curve in the boundary layer analysis. The control volume energy equation becomes:

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{d}{dt} \left( \int_{CV} e\rho dV \right) + \int_{CS} \left( e + \frac{p}{\rho} \right) \rho(\mathbf{V}\cdot\mathbf{n})dA$$

From definition of enthalpy  $h = u + p/\rho$ , the final general form of the energy equation for a fixed control volume becomes:

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{d}{dt} \left( \int_{CV} \left( u + \frac{1}{2} V^2 + gz \right) \rho dV \right) + \int_{CS} \left( h + \frac{1}{2} V^2 + gz \right) \rho (\mathbf{V} \cdot \mathbf{n}) dA$$

### One-dimensional energy-flux terms

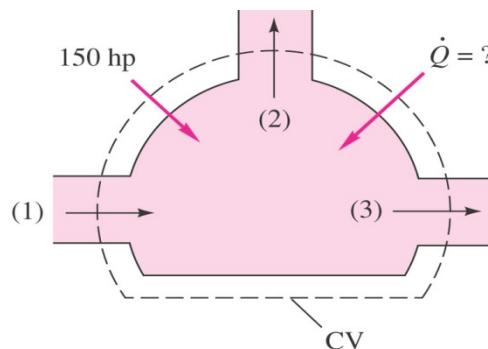
If the control volume has a series of one-dimensional inlets and outlets, the surface integral in the energy equation reduces to a summation of outlet fluxes minus inlet fluxes:

$$\int_{CS} \left( h + \frac{1}{2} V^2 + gz \right) \rho (\mathbf{V} \cdot \mathbf{n}) dA = \sum_{out} \left( h + \frac{1}{2} V^2 + gz \right) \dot{m}_{out} - \sum_{in} \left( h + \frac{1}{2} V^2 + gz \right) \dot{m}_{in}$$

### Example 3

A steady flow machine, shown in figure below, takes in air at section 1 and discharges it at section 2 and 3 the properties at each section are as follows:

section	A, ft <sup>2</sup>	Q, ft <sup>3</sup> /s	T, F	p, lbf/in <sup>2</sup> , abs	z, ft
1	0.4	100	70	20	1.0
2	1.0	40	100	30	4.0
3	0.25	50	200	?	1.5



Work is provided to the machine at the rate of 150 hp. Find the pressure  $p_3$  in lbf/in<sup>2</sup> absolute and heat transfer in Btu/s. Assume air is a perfect gas with  $R=1716$  and  $c_p = 6003$  ft. lbf/(slug.R).

### The steady flow energy equation

For steady flow with one inlet and one outlet, both assumed one-dimensional, reduces to a celebrated relation used in many engineering analyses:

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \left( h + \frac{1}{2} V^2 + gz \right)_{out} \dot{m}_{out} - \left( h + \frac{1}{2} V^2 + gz \right)_{in} \dot{m}_{in}$$

From continuity,  $\dot{m}_{in} = \dot{m}_{out} = \dot{m}$ , we can re-arrange it:

$$\left( h + \frac{1}{2} V^2 + gz \right)_{in} = \left( h + \frac{1}{2} V^2 + gz \right)_{out} - q + w_s + w_v$$

where  $q = \dot{Q}/\dot{m} = dQ/dm$ , the heat transferred to the fluid per unit mass. Similarly,  $w_s = \dot{W}_s/\dot{m} = dW_s/dm$  and  $w_v = \dot{W}_v/\dot{m} = dW_v/dm$ .

The term  $H_1 = (h + 1/2 V^2 + gz)_1$  is called the stagnation enthalpy. The *steady flow energy equation* states the upstream stagnation enthalpy differs from the downstream value  $H_2$  only if there is heat transfer, shaft work, or viscous work as the fluid passes between sections 1 and 2.

If we divide the steady energy equation by  $g$ , the dimension of each term becomes a length, called head in fluid mechanics and shown by symbol  $h$ .

Not to confuse head with enthalpy, we re-write the steady flow energy equation using internal energy:

$$\frac{p_1}{\gamma} + \frac{u_1}{g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{u_2}{g} + \frac{V_2^2}{2g} + z_2 - h_q + h_s + h_v$$

where  $h_q = q/g$ ,  $h_s = w_s/g$ , and  $h_v = w_v/g$  are the head forms of the heat added, shaft work done, and viscous work done, respectively. The term  $p/\gamma$  is called pressure head, and the term  $V^2/2g$  is denoted as velocity head.

### Friction and shaft work in low speed flows

Many engineering applications can be categorized as steady, low speed, incompressible flows through a pipe or a duct. A pump or a turbine may also be included in the system. The energy equation for such system with one inlet and one outlet becomes:

$$\left( \frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 \right) = \left( \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 \right) + h_{friction} - h_{pump} + h_{turbine}$$

This is very useful relation. Note that all the head terms are positive; i.e., pump adds energy (head) and turbine extracts head.

Kinetic energy correction factor: The inlet and outlet to CV often do not have one-dimensional flow, i.e., flow varies across the cross-section. When using the average velocity in kinetic energy calculation, the velocity variation should be considered by introducing a correction factor. This is done similar to the momentum correction factor:

$$\int_{port} \frac{V^2}{2} \rho (V \cdot n) dA \equiv \alpha \frac{V_{av}^2}{2} \dot{m} \quad \text{where} \quad V_{av} = \frac{1}{A} \int u dA$$

For incompressible fluids. By letting  $u$  be the velocity normal to the port, the kinetic energy correction factor becomes:

$$\alpha = \frac{1}{A} \int \left( \frac{u}{V_{av}} \right)^3 dA$$

Using the velocity profiles for laminar and turbulent flows, we get:

Laminar flow:  $u = U_0 \left(1 - \frac{r^2}{R^2}\right)$  with  $\alpha = 2.0$

Turbulent flow:  $u = U_0 \left(1 - \frac{r}{R}\right)^m$  with  $\alpha = \frac{(2+m)^3(1+m)^3}{4(1+3m)(2+3m)}$

Introducing the kinetic correction factor, the steady energy equation for incompressible flows becomes:

$$\left(\frac{p_1}{\gamma} + \frac{\alpha V_1^2}{2g} + z_1\right) = \left(\frac{p_2}{\gamma} + \frac{\alpha V_2^2}{2g} + z_2\right) + h_{friction} - h_{pump} + h_{turbine}$$

## Bernoulli equation

If flow is assumed frictionless, the steady energy equation reduces to the Bernoulli equation. It is very famous and widely used, however, one should be wary of its restrictions, i.e., all real fluids are viscous and thus have friction to some extent.

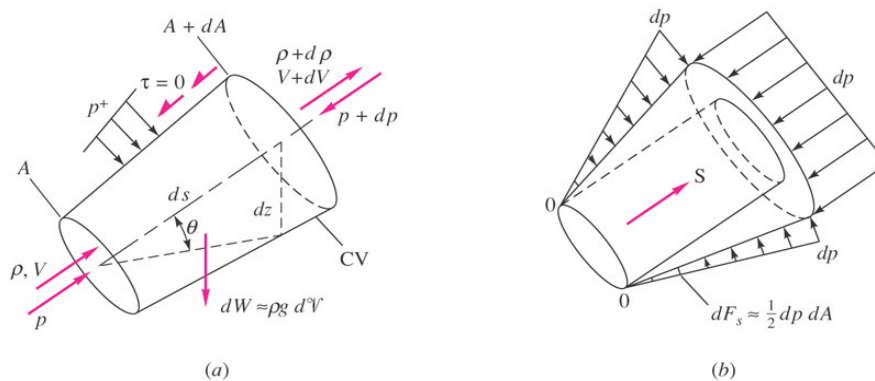


Fig. 7: The Bernoulli equation for frictionless flow along streamline.

Consider an elemental fixed streamtube control volume of variable area  $A(s)$  and length  $ds$  in the streamline direction, shown in Fig. 7. The conservation of mass for this elemental control volume yields:

$$\frac{d}{dt} \left( \int_{CV} \rho dV \right) + \dot{m}_{out} - \dot{m}_{in} = 0 = \frac{\partial \rho}{\partial t} dV + d\dot{m}$$

where  $\dot{m} = \rho AV$  and  $dV = Ads$ . The conservation of mass becomes:

$$d\dot{m} = d(\rho AV) = -\frac{\partial \rho}{\partial t} Ads$$

The friction is shown in Fig. 7 and neglected on the streamtube. The linear momentum relation for streamtube can be written as:

$$\sum dF_s = \frac{d}{dt} \left( \int_{CV} \mathbf{v} \rho dV \right) + (\dot{m}V)_{out} - (\dot{m}V)_{in} = \frac{\partial}{\partial t} (\rho V) A ds + d(\dot{m}V)$$

Assuming frictionless flow, the forces are due to pressure and gravity and can be calculated as follows:

$$dF_{s,gravity} = -dW \sin\theta = -\gamma A ds \sin\theta = -\gamma Adz$$

$$dF_{s,press} = \frac{1}{2} dp dA - dp(A + dA) \approx -Adp$$

After substituting the forces into linear momentum relation and using the conservation of mass and after simplifications, we get:

$$\frac{\partial V}{\partial t} ds + \frac{dp}{\rho} + Vdv + gdz = 0$$

This is called Bernoulli equation for unsteady frictionless flow along a streamline. It can be integrated between any two points on the streamline:

$$\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0$$

The Bernoulli for steady frictionless incompressible flow along a streamline becomes:

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2 = const.$$

steady flow energy equation	$\left(\frac{p_1}{\rho} + \frac{\alpha V_1^2}{2} + gz_1\right) = \left(\frac{p_2}{\rho} + \frac{\alpha V_2^2}{2} + gz_2\right) + (u_2 - u_1 - q) + w_s + w_v$
Bernoulli equation for steady incompressible frictionless flow	$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$

A comparison between the steady flow energy and Bernoulli equation reveals that the energy equation is much more general than Bernoulli and allows for i) friction, ii) heat transfer, iii) shaft work, and iv) various work. Bernoulli does not allow for any these!

Note that the Bernoulli equation is a momentum-based force relation and is derived assuming:

- i) Steady flow
- ii) Incompressible flow (applicable for Mach numbers less than 0.3)
- iii) Frictionless flow (restrictive- where solid walls or mixing exist)
- iv) Flow along a single streamline.

It should be noted that different streamlines may have different flow conditions and different Bernoulli constants.

$$\text{Bernoulli constant} \equiv \frac{p}{\rho} + \frac{1}{2}V^2 + gz$$

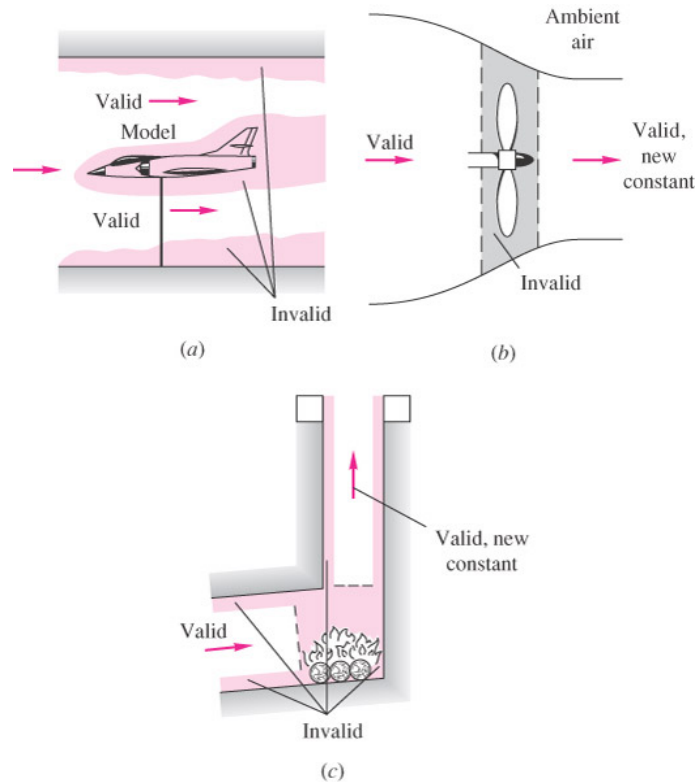


Fig. 8: regions of validity and invalidity of the Bernoulli equation.

### Hydraulic and energy grade lines

The energy grade line (EGL) shows the height of the total Bernoulli constant,  $h_0 = z + p/\gamma + V^2/(2g)$ . In frictionless flow with no work or heat transfer, the EGL has a constant height.

The hydraulic grade line (HGL) shows the height corresponding to elevation and pressure head  $z + p/\gamma$  which is EGL minus the velocity head  $V^2/(2g)$ . The HGL is the height to which liquid would rise in a piezometer tube attached to the flow. In an open-channel flow the HGL is identical to the free surface of the water.

As shown in Fig. 9, piezometer tube measures the static pressure head  $z + p/\gamma$  (corresponding to HGL) and the pitot stagnation-velocity measures the total head  $z + p/\gamma + V^2/(2g)$  which corresponds to EGL. In frictionless flow, with no work or heat exchange, EGL is constant and HGL rises due to a drop in velocity.



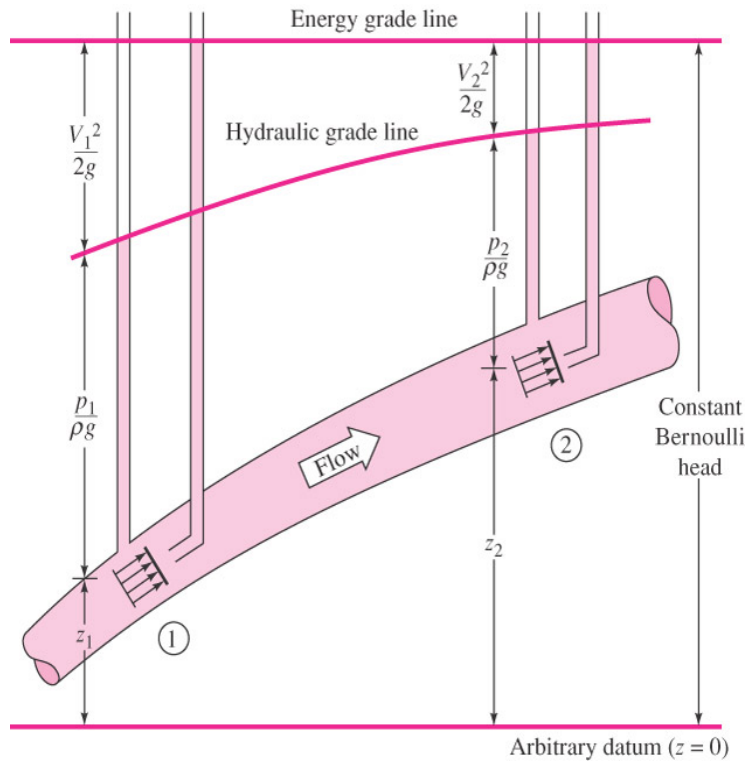


Fig. 9: Hydraulic and energy grade lines for frictionless flow in a duct.

In general flow condition, the EGL will drop slowly due to friction losses and will drop sharply due to substantial loss (e.g. a valve) or due to work extraction. The HGL generally follows the behavior of the EGL, and it raises/falls if the velocity decreases/increases respectively.

#### Example 4: Venturi tube

A constriction in a pipe will cause the velocity to rise and the pressure to fall at section 2 in the throat. The pressure difference is a measure of the flow rate through the pipe. The smoothly necked-down system shown in figure below is called venturi tube. Find an expression for the mass flux in the tube as a function of the pressure change.

