Viscous Flow in Ducts

We want to study the viscous flow in ducts with various velocities, fluids and duct shapes. The basic problem is this: Given the pipe geometry and its added components (e.g. fittings, valves, bends, and diffusers) plus the desired flow rate and fluid properties, what pressure drop is needed to drive the flow?

Reynolds number regimes

The most important parameter in fluid mechanics is the Reynolds number:

$$Re = \frac{\rho UL}{\mu}$$

where ρ is the fluid density, U is the flow velocity, L is the characteristics length scale related to the flow geometry, and μ is the fluid viscosity. The Reynolds number can be interpreted as the ratio of momentum (or inertia) to viscous forces.

A profound change in fluid behavior occurs at moderate Reynolds number. The flow ceases being smooth and steady (laminar) and becomes fluctuating (turbulent). The changeover is called transition.

Transition depends on many effects such as wall roughness or the fluctuations in the inlet stream, but the primary parameter is the Reynolds number. Turbulence can be measured by sensitive instruments such as a hot-wire anemometer or piezoelectric pressure transducer.
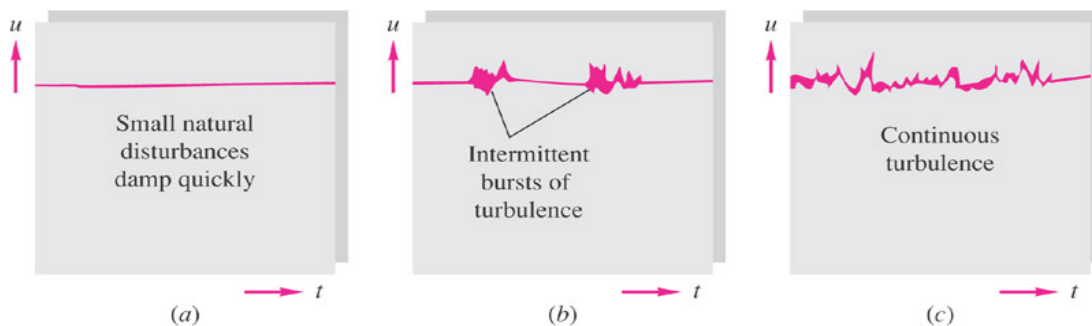

Fig.1: The three regimes of viscous flow: a) laminar (low Re), b) transition at intermediate Re, and c) turbulent flow in high Re.

The fluctuations, typically ranging from 1 to 20% of the average velocity, are not strictly periodic but are random and encompass a continuous range of frequencies. In a typical wind tunnel flow at high Re, the turbulent frequency ranges from 1 to 10,000 Hz.

The higher Re turbulent flow is unsteady and irregular but, when averaged over time, is steady and predictable. For a fluid flow with average velocity U which its shear layer has the characteristics length scale L, $Re = \rho UL/\mu$, the following approximate ranges occur:

0 < Re < 1: highly viscous laminar "creeping" motion

1 < Re < 100: laminar strong Re dependence

100 < Re < 10^3: laminar, boundary layer theory useful

10^3 < Re < 10^4: transition to turbulence

10^4 < Re < 10^6: turbulent, moderate Re dependence

10^6 < Re <∞: Turbulent, slight Re dependence

Note: The representative ranges vary with flow geometry, surface roughness, and the level of fluctuations in the inlet stream.
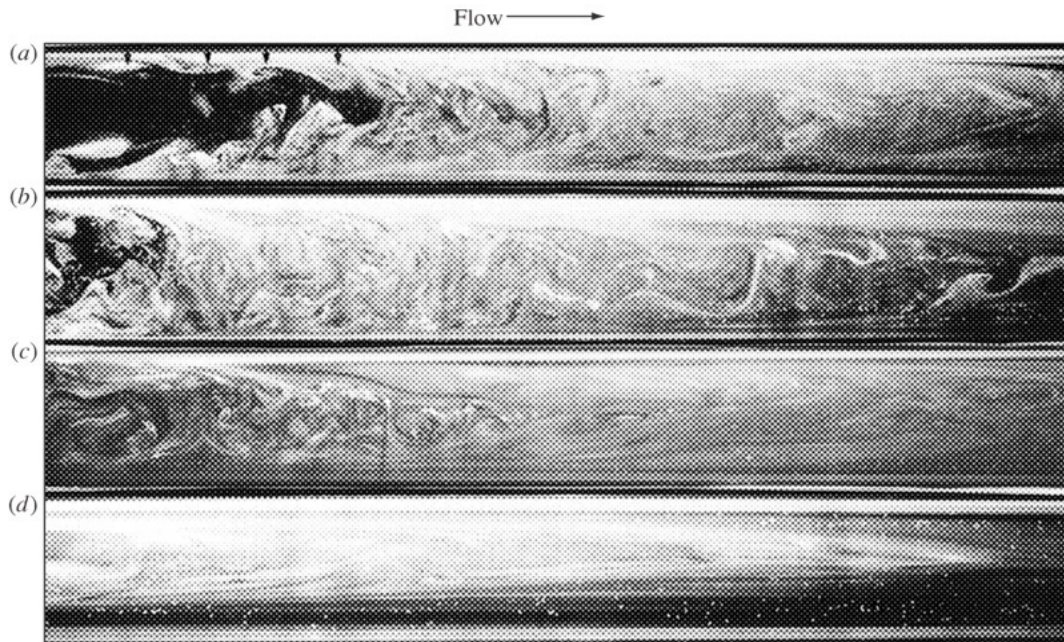
Fig. 2: Formation of turbulence in pipe flow, a0 and b) near entrance; c) somewhat downstream; d) far downstream.

Most of our knowledge on turbulent flow is empirical; a general solution for the turbulent flow, that can predict the transition and the fluctuation in the flow parameters, does not exist. In 1883 Osborne Reynolds, a British engineering professor, observed the transition to turbulence in a pipe flow by introducing a dye streak in the flow.

Reynolds experimentally showed that the transition occurs in a pipe flow at:

$$Re_d = \frac{\rho U d}{\mu} \approx 2300$$

where, d is the diameter of a circular tube and is considered as the characteristic length scale for the fluid flow.

The value 2300 is for transition in pipes. Other geometries, such as plates, airfoils, cylinders, and spheres have completely different transition Re numbers.
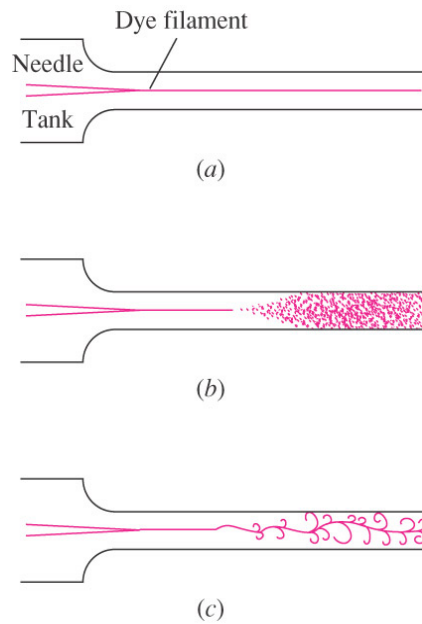
Fig. 4: Reynolds's sketches of pipe transition.

Internal and external viscous flow

Both laminar and turbulent flow may be internal or external.

Internal flow: An internal flow is contained (or bounded) by walls and the viscous flow will grow and meet and permeate the entire flow. As a result, there is an <u>entrance region</u> where nearly inviscid upstream flow converges and enters the duct.
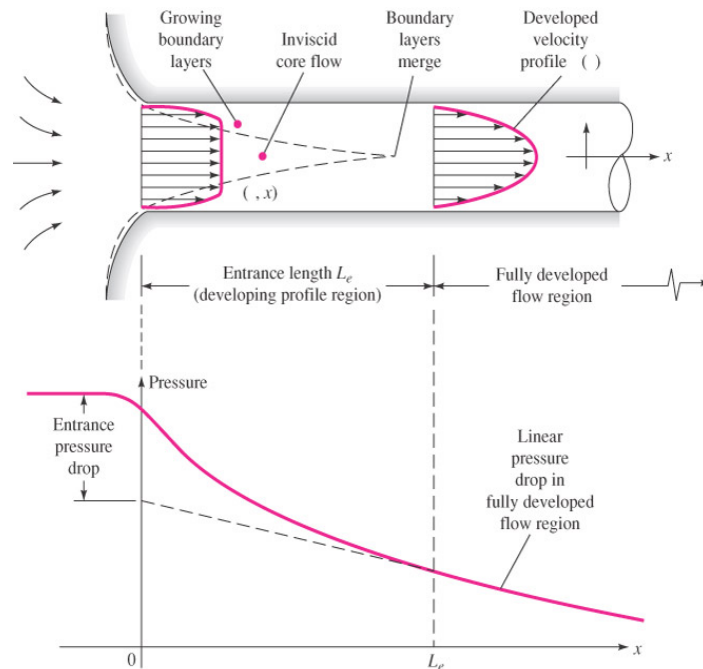

Fig. 5: Developing velocity profiles and pressure changes in the entrance of a duct.

Beyond the entrance region, which is a finite distance from the entrance $x = L_e$, the velocity profile becomes constant, i.e. it no longer changes with x and is said to be <u>fully developed</u>, $u \approx u(r)$.

For <u>laminar flow</u>, the entrance region can be found form the following empirical correlation:

$$\frac{L_e}{d} = 0.06 Re_d \quad (Laminar)$$

Assuming the maximum Re for the laminar flow in a duct is $Re_{d,crit}$ =2300 the longest laminar developing region becomes: L_e = 138d, i.e. 138 times of the tube diameter.

Turbulent flow boundary layers grow faster, and the entrance region (L_e) is relatively shorter:

$$\frac{L_e}{d} = 4.4 Re_d^{1/6} \quad (Turbulent)$$

Assuming Re = 10^6 for turbulent flow in a duct, a typical turbulent developing region becomes: L_e = 44d.

Note: the entrance region length can vary from *40d* to *100d* for turbulent and laminar regimes, respectively. However, for typical pipe flow application, the pipe length is 1000 of its diameter in which case the <u>entrance effect</u> may be neglected. Therefore a simple analysis for fully-developed region can be used for the entire pipe length.

External flow: Has no restraining walls and is free to expand no matter how thick the viscous layers on the immersed body may become. As a result, far from the body the flow is nearly inviscid and there is no external equivalent of fully developed internal flow.

Flow head loss (the friction factor)

Consider incompressible steady flow between sections 1 and 2 of the inclined constant area pipe shown in Fig. 6.
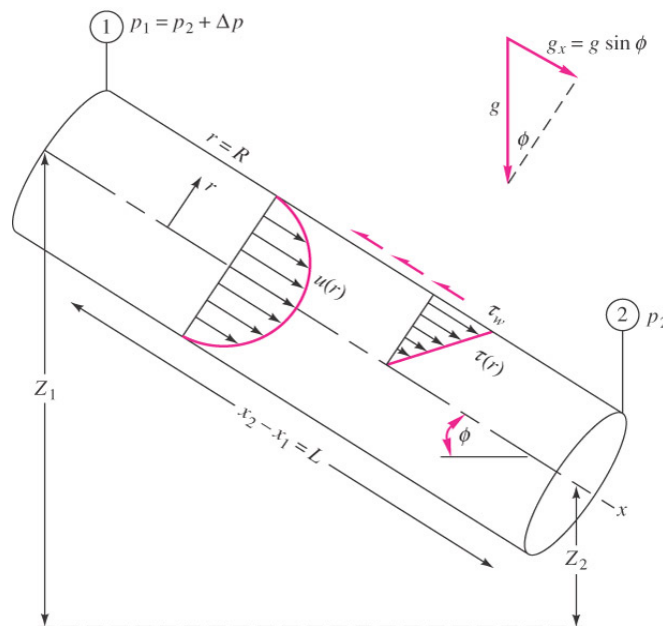

Fig. 6: Steady fully developed flow in an inclined pipe.

The 1-D continuity equation:

$$Q_1 = Q_2 = const \quad or \quad V_1 = V_2 = V$$

For a constant area pipe, the steady energy equation becomes:

$$\left(\frac{p}{\gamma} + \frac{V^2}{2g} + z\right)_1 = \left(\frac{p}{\gamma} + \frac{V^2}{2g} + z\right)_2 + h_f$$

Since $V_1 = V_2$, the friction head lost is:

$$h_f = (z_1 - z_2) + \left(\frac{p_1}{\rho g} - \frac{p_2}{\rho g}\right) = \Delta z + \frac{\Delta p}{\rho g}$$

Applying the momentum equation, the summation on the forces along the x-direction, we find:

$$\sum F_x = \Delta p(\pi R^2) + \rho g(\pi R^2)L \, sin\varphi - \tau_w(2\pi R)L = \dot{m}(V_2 - V_1) = 0$$

Rearranging this, we find that the head loss is related to wall shear stress:

$$\Delta z + \frac{\Delta p}{\rho g} = h_f = \frac{2\tau_w}{\rho g}\frac{L}{R} = \frac{4\tau_w}{\rho g}\frac{L}{d}$$

where $\Delta z = L \, sin\varphi$. Note that regardless of whether the pipe is horizontal or tilted, the head loss is proportional to the wall shear stress. A German professor, Julius Weisbach 1850, argued that the friction factor is proportional to *L/d* and V^2 (observed experimentally in turbulent regime). He then proposed to represent the frictionless head loss with a dimensionless parameter *f* (called the Darcy friction factor), that is defined as:

$$h_f = f\frac{L}{d}\frac{V^2}{2g}$$

where *f = f(Re$_d$, duct roughness, duct shape)*. By equating the above equations, we find:

$$f = \frac{8\tau_w}{\rho V^2}$$

Laminar fully developed pipe flow

Consider fully developed Poiseuille flow in a round duct of diameter d. The analytical velocity solution, from the previous chapter, is:

$$u(r) = u_{max}\left(1 - \frac{r^2}{R^2}\right) \quad where \quad u_{max} = \left(-\frac{dp}{dx}\right)\frac{R^2}{4\mu} \quad and \quad \left(-\frac{dp}{dx}\right) = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)$$

$$V = \frac{u_{max}}{2} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)\frac{R^2}{8\mu} \quad and \quad Q = \int u dA = \pi R^2 V = \frac{\pi R^4}{8\mu}\left(\frac{\Delta p + \rho g \Delta z}{L}\right)$$

$$\tau_w = \left|\mu \frac{du}{dr}\right|_{r=R} = \frac{4\mu V}{R} = \frac{R}{2}\left(\frac{\Delta p + \rho g \Delta z}{L}\right)$$

$$h_f = \frac{32\mu L V}{\rho g d^2} = \frac{128\mu L Q}{\pi \rho g d^4}$$

Note: the pressure drop in inversely proportional to the pipe diameter to the power 4. So, if the size of the pipe is doubled, the pressure drop will decrease by a factor of 16 for a given Q.

These formulas are valid whenever the Re <2300, i.e. laminar flow. Knowing the shear stress, the Poiseuille friction factor is easily determined:

$$f_{laminar} = \frac{8\tau_{w,lam}}{\rho V^2} = \frac{64}{Re_d}$$

Turbulence modeling

For turbulent flow, because of the fluctuations, every velocity and pressure term in momentum and energy equations is a rapidly varying random function of time and space. Currently, our mathematics cannot handle such instantaneous fluctuating variables.
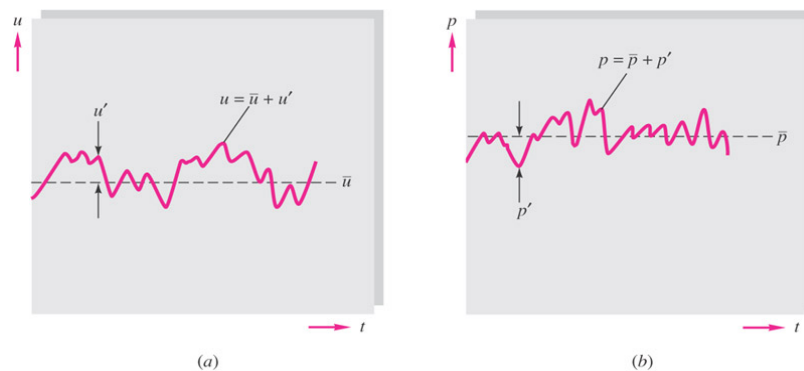

Fig. 7: Mean and turbulent variables.

As engineers, we are interested in the average or mean values of velocity, pressure, shear stress, etc in turbulent flow. Reynolds proposed to rewrite the governing equations in terms of average or mean values. The time mean velocity, for example, can be defined as:

$$\bar{u} = \frac{1}{T}\int_0^T u\, dT$$

where T is an averaging period taken to be longer than any significant period of the fluctuations themselves. The fluctuation u' is defined as the deviation of u from its average value:

$$u' = u - \bar{u}$$

Note that the mean (average) value of u' is zero over time; but the mean square of a fluctuation is not zero and is a measure of the intensity of the turbulence:

$$\overline{u'^2} = \frac{1}{T}\int_0^T u'^2\, dt \neq 0$$

Using the concept of mean values, the momentum equation in *x*-direction (as an example) becomes:

$$\rho\frac{d\bar{u}}{dt} = \rho g_x - \frac{\partial\bar{p}}{\partial x} + \frac{\partial}{\partial x}\left(\mu\frac{\partial\bar{u}}{\partial x} - \rho\overline{u'^2}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial\bar{u}}{\partial y} - \rho\overline{u'v'}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial\bar{u}}{\partial z} - \rho\overline{u'w'}\right)$$

The three correlation terms $-\rho\overline{u'^2}, -\rho\overline{u'v'}, -\rho\overline{u'w'}$ are called turbulent stresses because they have the same dimensions and occur right alongside the newtonian (laminar) stress terms. Actually, they are convective acceleration terms (that is why the density appears), not stresses but they have the mathematical effect of stress.

Turbulent pipe flow

There are three regions in turbulent flow, <u>inner layer</u> (or viscous sub-layer) where viscous effects are dominant (near the wall), <u>overlap layer</u> (transition to turbulent occurs) and <u>outer</u> region (the flow is completely turbulent) as shown in Fig. 8.
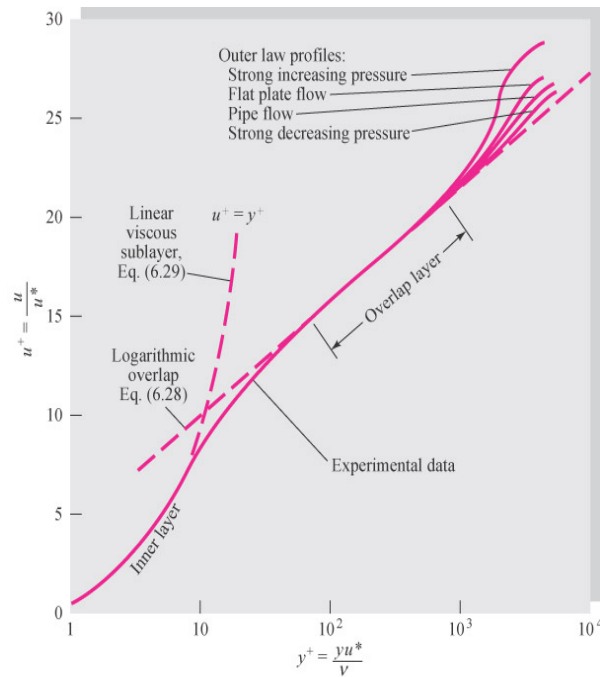

Fig. 8: turbulent flow regimes.

Since the inner wall layer forms typically only 2% of the fluid flow profile, we can neglect it. Therefore, we only consider the overlap layer velocity profile for the entire pipe flow:

$$\frac{u(r)}{u^*} = \frac{1}{\kappa}\ln\frac{(R-r)u^*}{\nu} + B$$

where $\kappa = 0.4 \; and \; B = 5.0, \; \nu = \mu/\rho$ is the kinematic viscosity of the fluid, and

$$u^* = \left(\frac{\tau_w}{\rho}\right)^{1/2}$$

u^* is called the <u>frictional velocity</u>, because it has velocity dimensions, although it is not actually a flow velocity.

The friction factor for the turbulent pipe flow can be calculated from the following correlations:

$$f = \begin{cases} 0.316Re_d^{-1/4} & 4000 < Re_d < 10^5 \\ \left(1.8log\dfrac{Re_d}{6.9}\right)^{-2} & \end{cases}$$

where $Re_d = Vd/\nu$.

For a horizontal pipe, we have:

$$\Delta p = 0.158L\rho^{3/4}\mu^{1/4}d^{-5/4}V^{7/4}$$

Or in terms of $Q = \pi d^2 V/4$, we find an alternative form:

$$\Delta p = 0.24L\rho^{3/4}\mu^{1/4}d^{-4.75}Q^{1.75}$$

Note: The pressure drop for the turbulent flow decreases with diameter even more sharply than the laminar flow. Doubling the pipe size decreases the pressure drop by a factor of 27 for a given Q.

Note: As depicted in Fig. 9, a turbulent velocity profile is very flat in the center and drops off sharply to zero at the wall.
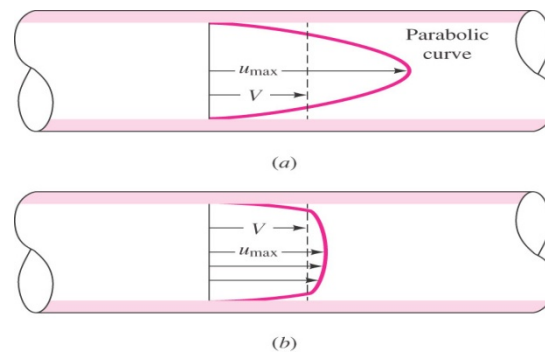

Fig. 9: Comparison of a) laminar and b) turbulent pipe flow velocity.

Effect of rough walls

The surface roughness has an effect on friction resistance; for laminar flow, however, this effect is negligible. The turbulent flow is strongly affected by roughness. Nikuradse (1933) simulated roughness by gluing uniform sand grains onto the inner walls of the pipes. He then measured the pressure drops and flow rates and correlated friction factor versus Reynolds number, as shown in Fig. 10.

Note:

- The laminar friction is unaffected,
- The turbulent friction, after an onset point, increases monotonically with the roughness ratio, ϵ/d,
- The friction factor becomes constant (fully rough) at high Reynolds numbers.

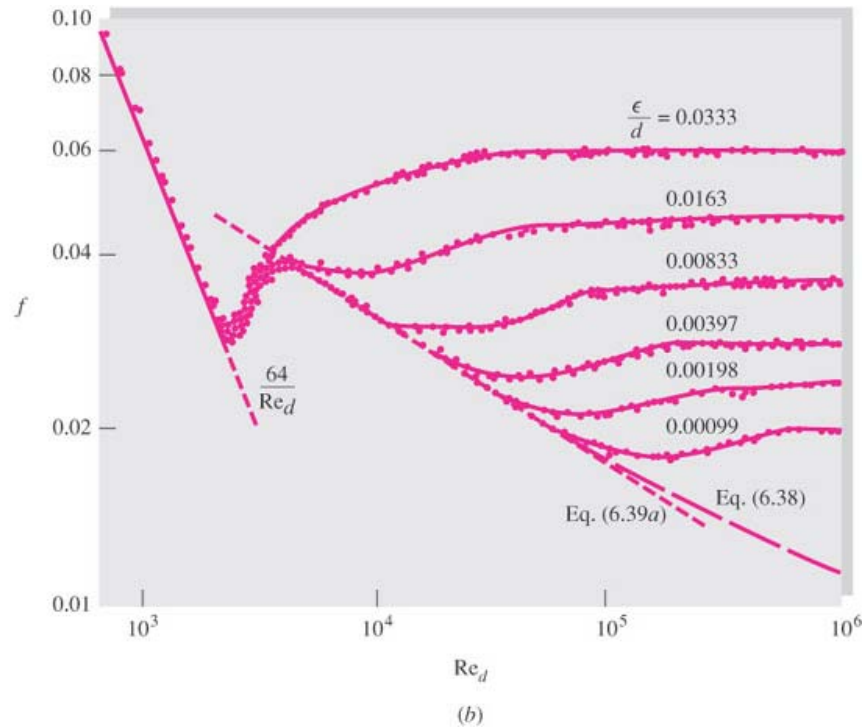Three regions can be detected in the chart:

Fig. 10: Effect of wall roughness on turbulent pipe flow.

$\frac{\epsilon u^*}{\nu} < 5$: Hydraulically smooth walls, no effect of roughness on friction

$5 \le \frac{\epsilon u^*}{\nu} \le 70$: Transitional roughness, moderate Reynolds number effect

$\frac{\epsilon u^*}{\nu} > 70$: Fully rough flow, sub-layer totally broken up and friction independent of Reynolds number.

The logarithm law modified for roughness becomes:

$$u^+ = \frac{u}{u^*} = \frac{1}{\kappa} ln\left(\frac{y}{\epsilon}\right) + 8.5$$

Integrating this equation, we can find the average velocity in the pipe:

$$\frac{V}{u^*} = 2.44 ln\left(\frac{d}{\epsilon}\right) + 3.2$$

For fully rough flow, we have:

$$\frac{1}{f^{1/2}} = -2.0 log\left(\frac{\epsilon/d}{3.7}\right)$$

Notice that there is no Reynolds number effect; hence the head loss varies exactly as the square of the velocity in this case.

The Moody chart

Colebrook (1939) combined the smooth wall and fully rough relations into a clever combined interpolation formula:

$$\frac{1}{\sqrt{f}} = -2.0 log \left(\frac{\epsilon/d}{3.7} + \frac{2.51}{Re_d \sqrt{f}} \right)$$

It is plotted in 1944 by Moody (1944) into what is called the Moody chart for pipe friction, Fig. 11. Moody chart is one of the most famous and useful figure in fluid mechanics. It is accurate to ±15% and can be used for circular and non-circular ducts and open-channels.
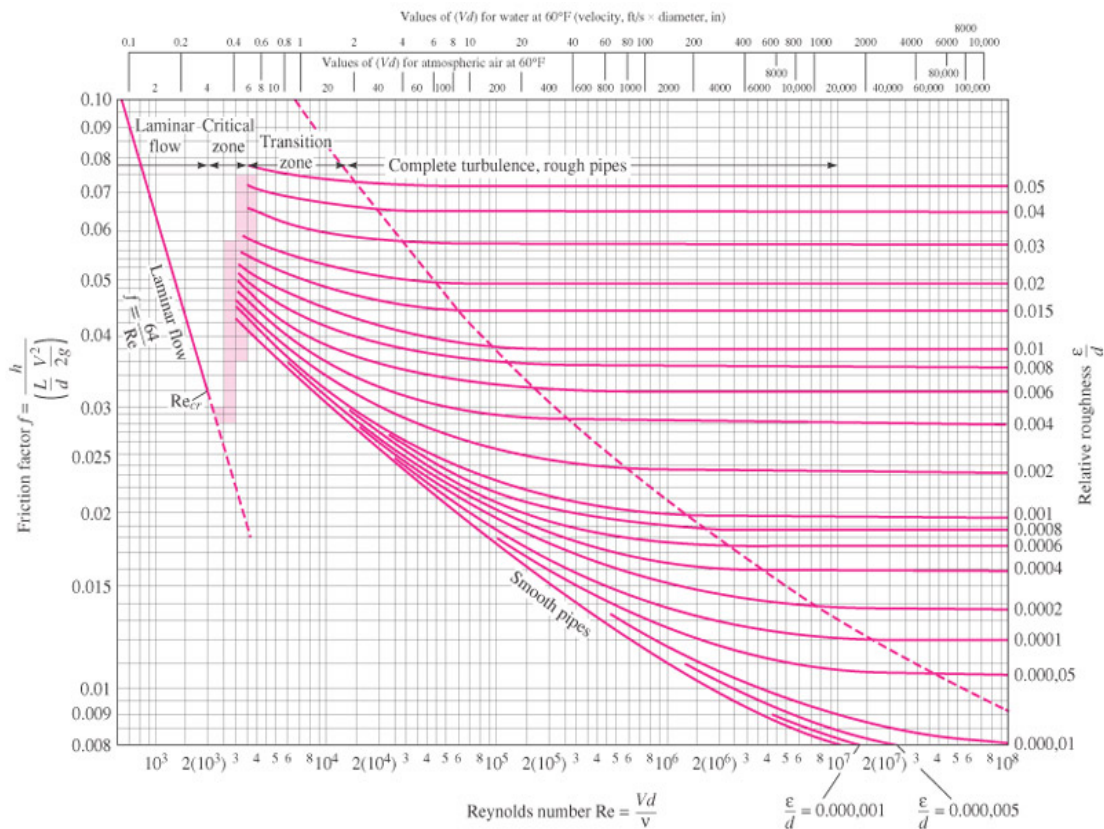

Fig. 11: The Moody chart for pipe friction with smooth and rough walls.

The Colebrook equation is cumbersome to evaluate f, Haaland (1983) proposed another correlation:

$$\frac{1}{\sqrt{f}} = -1.8 \, log \left[\frac{6.9}{Re_d} + \left(\frac{\epsilon/d}{3.7} \right)^{1.11} \right]$$

Hydraulic diameter

If the duct is noncircular, the analysis of fully-developed flow follows that of the circular pipe but is more complicated algebraically. For turbulent flow, the logarithm law velocity profile can be used. In general,

for complex geometries, it is very challenging to perform the analysis directly. Therefore, the concept of the hydraulic diameter is introduced for convenience.
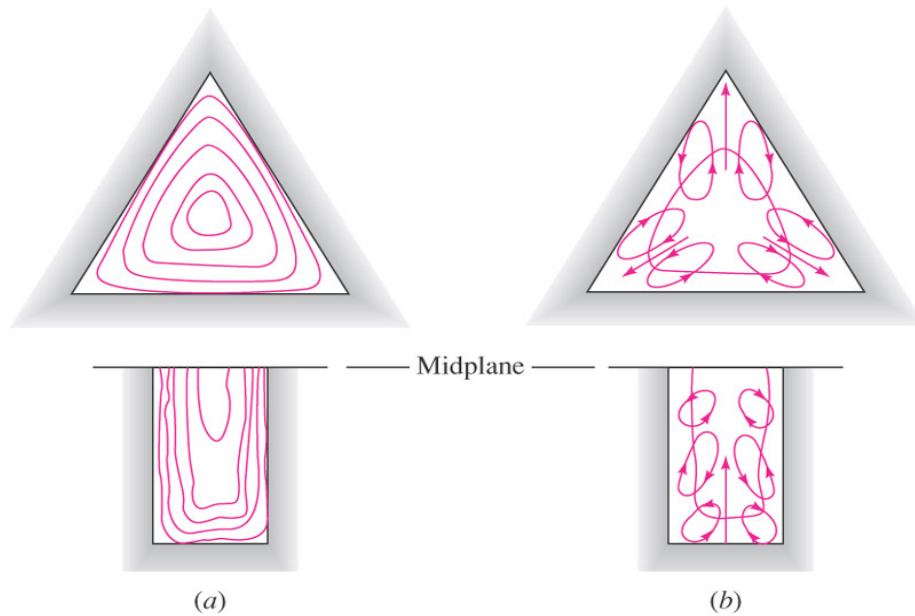

Fig. 12: Flow in non-circular ducts.

For noncircular ducts, the above analyses are correct, however, the cross-sectional area A does not equal to πR^2 and the cross-sectional perimeter wetted by the shear stress, $\wp$, is not equal to $2\pi R$. For this reason a non-circular duct is said to have a hydraulic diameter:

$$D_h = \frac{4 \times area}{wetted\ perimeter} = \frac{4A}{\wp}$$

We use the hydraulic diameter, as a length scale for non-circular ducts, and use the analyses derived for the circular pipes.

Minor losses in pipe systems

For any pipe systems, in addition to the Moody-type friction loss, computed for the length of pipe, there are additional minor losses, including: pipe entrance or exit, sudden expansion or contraction, bends, elbows, tees, other fittings, valves (open or partially closed), and gradual expansions or contractions.

The flow patterns in most of the above-mentioned geometries are quite complex, see Fig. 13, and thus the losses commonly measured experimentally and correlated with the pipe flow parameters.

The measured minor loss is usually given as a ratio of the head loss through the device to the velocity head $V^2/(2g)$ of the associated piping system:

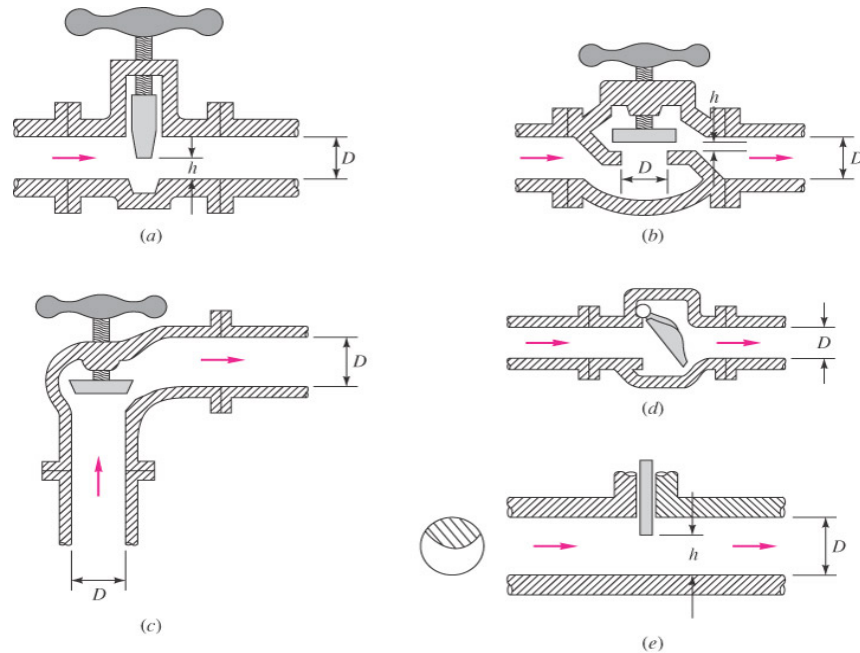$$Loss\ coefficient\ K = \frac{h_m}{V^2/(2g)} = \frac{\Delta p}{\frac{1}{2}\rho V^2}$$

Fig. 13: Typical commercial valve geometries a) gate valve, b) globe valve, c)angle valve, d)swing-check valve, e) disk type gate valve.

A single pipe system may have many minor losses. Since all are correlated with $V^2/(2g)$, they can be summed into a single total system loss if the pipe has constant diameter:

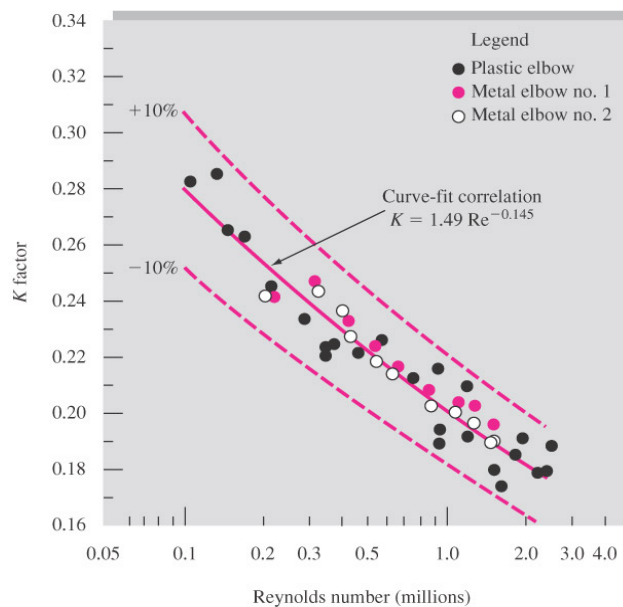$$\Delta h_{tot} = h_f + \sum h_m = \frac{V^2}{2g}\left(f\frac{L}{d} + \sum K\right)$$

Fig. 14: K factor for 90 degree elbows.

K is a non-dimensional factor, like f, and is often correlated with the raw size of the pipe, tables and figures are available in the textbook for various minor changes, see Fig. 14 for example. Note that we must sum the losses separately if the pipe size changes so that V^2 changes. The length L is the total length of the pipe axis.