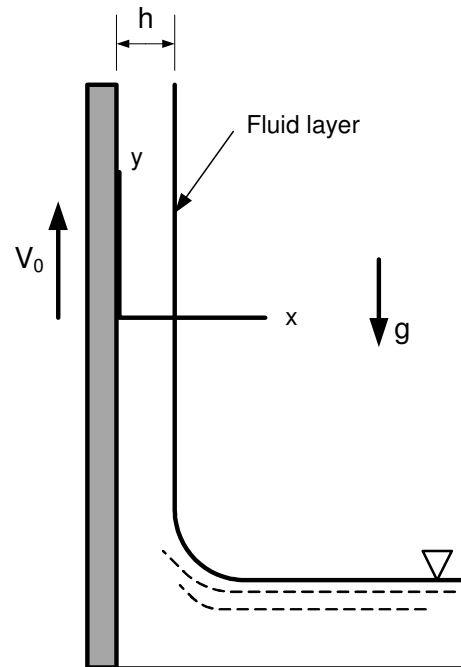


ENSC 283 Week # 10, Tutorial # 6 – Plane Couette Flow

Problem 1: A wide moving belt passes through a container of a viscous liquid. The belt moves vertically upward with a constant velocity V_0 as illustrated in the figure. Because of viscous forces the belt picks up a film of fluid of thickness h . Gravity tends to make the fluid drain down the belt. Assume that the flow is laminar, steady, and fully developed. Use the Navier-Stokes equations to determine an expression for the average velocity of the fluid film as it is dragged up the belt.



Solution

Step 1: Write out what you are required to solve for (this is so you don't forget to answer everything the question is asking for)

Find:

- V , the average velocity of the fluid film

Step 2: Calculations

Since the flow is assumed to be uniform, the only velocity component is in the y direction (the v component) so that $u = w = 0$. It follows from the continuity equation that $\partial v / \partial y = 0$, and for steady flow $\partial v / \partial t = 0$, so that $v = v(x)$. Under these conditions the Navier-Stokes equations for the x and the z direction (perpendicular to the paper) simply reduce to

$$\frac{\partial p}{\partial x} = 0 \quad \frac{\partial p}{\partial z} = 0 \quad (\text{Eq1})$$

This result indicates that the pressure does not vary over a horizontal plane, and because the pressure on the surface of the film ($x = h$) is atmospheric (or zero gage pressure). The Navier-Stokes equation in the y direction thus reduces to

$$0 = -\rho g + \mu \frac{d^2 v}{dx^2} \quad (\text{Eq2})$$

or

$$\frac{d^2 v}{dx^2} = \frac{\gamma}{\mu} \quad (\text{Eq3})$$

Integration of Eq3 yields

$$\frac{dv}{dx} = \frac{\gamma}{\mu} x + c_1 \quad (\text{Eq4})$$

On the film surface ($x = h$) we assume the shearing stress is zero, since the drag of air on the film is negligible. The shearing stress at the free surface (or any interior parallel surface) is designated as τ_{xy} where

$$\tau_{xy} = \mu \left(\frac{dv}{dx} \right) \quad (\text{Eq5})$$

Thus, if $\tau_{xy} = 0$ at $x = h$, it follows from Eq4 that

$$c_1 = -\frac{\gamma h}{\mu} \quad (\text{Eq6})$$

A second integration of Eq4 gives the velocity distribution in the film as

$$v = \frac{\gamma}{2\mu} x^2 - \frac{\gamma h}{\mu} x + c_2 \quad (\text{Eq7})$$

At the belt ($x = 0$) the fluid velocity must match the belt velocity V_0 , so that

$$c_2 = V_0 \quad (\text{Eq8})$$

and the velocity distribution is therefore

$$v = \frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + V_0 \quad (\text{Eq9})$$

With the velocity distribution known we can determine the flowrate per unit width, q , from the relationship

$$q = \int_0^h v \, dx = \int_0^h \left(\frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + V_0 \right) dx \quad (\text{Eq10})$$

and thus

$$q = V_0 h - \frac{\gamma h^3}{3\mu} \quad (\text{Eq11})$$

The average velocity V (where $q = Vh$) is therefore

$$V = V_0 - \frac{\gamma h^2}{3\mu} \quad (\text{Eq12})$$

Comment

Eq9 can be written in dimensionless form as

$$\frac{v}{V_0} = c \left(\frac{x}{h} \right)^2 - 2c \left(\frac{x}{h} \right) + 1 \quad (\text{Eq13})$$

where $c = \gamma h^2 / 2\mu V_0$. This velocity profile is shown in the following figure. Note that even though the belt is moving upward, for $c > 1$ (e.g. for fluids with small enough viscosity or with a small enough belt speed) there are portions of the fluid that flow downward (as indicated by $v/V_0 < 0$).

It is interesting to note from this result that there will be a net upward flow of liquid (positive V) only if $V_0 > \gamma h^2 / 3\mu$. It takes a relatively large belt speed to lift a small viscosity fluid.

