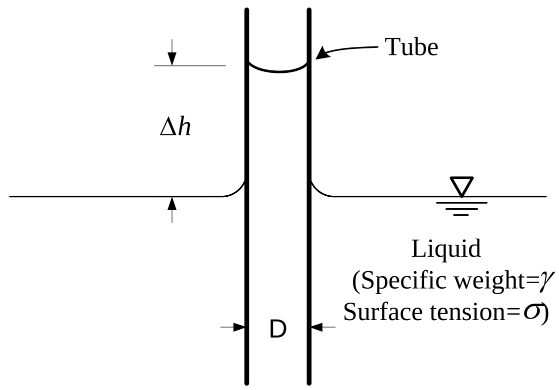
ENSC 283 Week # 13, Tutorial # 9 – Capillary Effect: Use of Dimensional Matrix

Problem: When a small tube is dipped into a pool of liquid, surface tension causes a meniscus to form at the free surface, which is elevated or depressed depending on the contact angle at the liquid-solid-gas interface. Experiments indicate that the magnitude of this capillary effect, Δh, is a function of the tube diameter, D, liquid specific weight, γ, and surface tension, σ. Determine the number of independent Π parameters that can be formed and obtain a set.



__Solution__

Step 1: Write out what you are required to solve for (this is so you don't forget to answer everything the question is asking for)

Find:

- Number of independent Π parameters
- One set of Π parameters

Step 2: Calculations

1) Write the function Δh and count variables:

$\Delta h = f(D, \gamma, \sigma)$ there are four variables (n=4)

2) Choose primary dimensions (use both $\{M, L, T\}$and $\{F, L, T\}$ dimensions to illustrate the problem in determining j).

3) (a) $\{M, L, T\}$

Δh	D	γ	σ
$\{L\}$	$\{L\}$	$\left\{\frac{M}{L^2T^2}\right\}$	$\left\{\frac{M}{T^2}\right\}$

$r = 3$ primary dimensions

(b) $\{F, L, T\}$

Δh	D	γ	σ
$\{L\}$	$\{L\}$	$\left\{\frac{F}{L^3}\right\}$	$\left\{\frac{F}{L}\right\}$

$r = 2$ primary dimensions

Thus for each set of primary dimensions we ask, "Is j (number of repeating parameters) equal to r?"
Initially guess $j = r$ and try to find power product values of pi groups. For $\{M, L, T\}$ system we cannot find all power values, i.e. the power values are dependent to each other, but for $\{F, L, T\}$ system all power values can be found. Thus:

$\{M, L, T\}$

$j = 2$
$j \neq r$

4) $j = 2$. Choose D, γ as repeating parameters.

5) $n - j = 2$ dimensionless groups will result.

$\Pi_1 = D^a\gamma^b\Delta h$ and

$$(L)^a\left(\frac{M}{L^2t^2}\right)^b(L) = M^0L^0t^0$$

$$\left.\begin{aligned} M&: \quad b + 0 = 0 \\ L&: \quad a - 2b + 1 = 0 \\ t&: \quad -2b + 0 = 0 \end{aligned}\right\} \quad \begin{aligned} b &= 0 \\ a &= -1 \end{aligned}$$

Therefore, $\Pi_1 = \frac{\Delta h}{D}$

$\Pi_2 = D^c\gamma^d\sigma$ and

$$(L)^c\left(\frac{M}{L^2t^2}\right)^d\left(\frac{M}{t^2}\right) = M^0L^0t^0$$

$\{F, L, T\}$

$j = 2$
$j = r$

4) $j = 2$. Choose D, γ as repeating parameters.

5) $n - j = 2$ dimensionless groups will result.

$\Pi_1 = D^e\gamma^f\Delta h$ and

$$(L)^e\left(\frac{F}{L^3}\right)^f(L) = F^0L^0t^0$$

$$\left.\begin{aligned} F&: \quad f = 0 \\ L&: \quad e - 3f + 1 = 0 \end{aligned}\right\} \quad e = -1$$

Therefore, $\Pi_1 = \frac{\Delta h}{D}$

$\Pi_2 = D^g\gamma^h\sigma$ and

$$(L)^g\left(\frac{F}{L^3}\right)^h\left(\frac{F}{L}\right) = F^0L^0t^0$$

$$M: \quad d + 1 = 0, \quad L: \quad c - 2d = 0, \quad t: \quad -2d - 2 = 0 \quad \Rightarrow \quad d = -1, \quad c = -2$$

Therefore, $\Pi_2 = \frac{\sigma}{D^2\gamma}$

$$F: \quad h + 1 = 0, \quad L: \quad g - 3h - 1 = 0 \quad \Rightarrow \quad h = -1, \quad g = -2$$

Therefore, $\Pi_2 = \frac{\sigma}{D^2\gamma}$

6) Check, using *F*, *L*, *t* dimensions

$$\Pi_1 = \frac{\Delta h}{D} \quad and \quad \frac{L}{L} = 1$$

$$\Pi_2 = \frac{\sigma}{D^2\gamma} \quad and \quad \frac{F}{L}\frac{1}{L^2}\frac{L^3}{F} = 1$$

Check, using *M*, *L*, *t* dimensions

$$\Pi_1 = \frac{\Delta h}{D} \quad and \quad \frac{L}{L} = 1$$

$$\Pi_2 = \frac{\sigma}{D^2\gamma} \quad and \quad \frac{M}{t^2}\frac{1}{L^2}\frac{L^2t^2}{M} = 1$$

Therefore, both systems of dimensions yield the same dimensionless Π parameters. The predicted functional relationship is

$$\Pi_1 = f(\Pi_2) \text{ or } \frac{\Delta h}{D} = f\left(\frac{\sigma}{D^2\gamma}\right)$$

Notes:

1) This result is reasonable on physical grounds. The fluid is static; we would not expect time to be an important dimension.
2) The analytical relation for this problem is $\Delta h = 4\sigma\cos\theta/(\rho g D)$, θ is the contact angle. Hence $\Delta h/D$ is directly proportional to $\sigma/D^2\gamma$.