$Q$ and such that all vertices of $B_k$ are on the outer face boundary. The reader may check the example of Figure 2.6 where $B_k$ consists of the black vertices $7, 8, b, c, d$ and $U = \{7, a, d\}$.

By the definition of an st-ordering, the subgraph of $G$ induced by $\{v_{k+1}, \ldots, v_n\}$ is connected. Therefore, if there is no embedding of $Q$ with $U$ on the outer face boundary, the graph $G$ is not planar.

## 2.8. Circle packing representations

Let $G$ be a plane graph. A *circle packing* (abbreviated CP) of $G$ is a set of circles $\{C_v \mid v \in V(G)\}$ in the plane such that:

(i) The interiors of the circles $C_v$, $v \in V(G)$, are pairwise disjoint open discs.
(ii) $C_u$ and $C_v$ $(u, v \in V(G))$ intersect if and only if $uv \in E(G)$.
(iii) By putting vertices $v \in V(G)$ in the centers of the corresponding circles $C_v$ and embed every edge $uv \in E(G)$ as a straight line segment joining $u$ and $v$ through $C_u \cap C_v$, we get a plane representation of $G$ which is equivalent to $G$ (where "equivalent" is defined after Theorem 2.6.7).

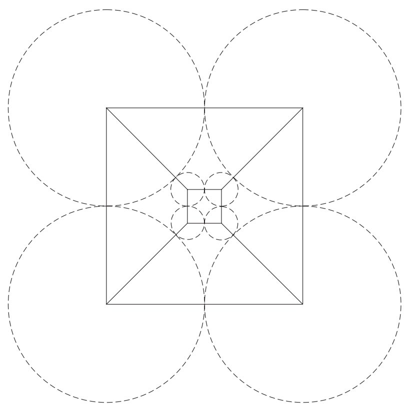

FIGURE 2.7. A CP representation of the 3-cube

Given a CP of $G$, the straight line representation of $G$ defined in (iii) is said to be a *circle packing representation* of $G$ (or just a *circle packing* of $G$). A circle packing representation of the graph of the 3-cube is shown in Figure 2.7.

Jackson and Ringel [**JR84, Ri85**] conjectured that every plane graph admits a circle packing representation. They used the term "coin representation" and the problem has also become known as *Ringel's coin problem.* However, as Sachs [**Sa94**] points out in his survey on this problem, it was solved already by Koebe [**Ko36**] who obtained the existence of circle packing representations as a corollary of a general theorem on conformal mapping of "contact domains". Section 2.9 shows that the relation between circle packings of graphs and conformal mappings is very strong.

In this section we present the result of Brightwell and Scheinerman [**BS93**] that every 3-connected planar graph and its dual have simultaneous straight line embeddings in the plane such that only dual pairs of edges intersect and every such pair is perpendicular. This proves an old conjecture of Tutte [**Tu63**]. This result is a corollary of the Primal-Dual Circle Packing Theorem 2.8.8. Another by-product of this result is Steinitz' Theorem [**St22**] which characterizes the graphs of the convex 3-dimensional polyhedra as the 3-connected planar graphs.

It is convenient to consider circle packings in the *extended plane* (the plane together with a point $\infty$ which we call *infinity*), and a circle packing may contain a special circle, denoted by $C_\omega$, which behaves differently. Instead of (i), we require that none of the circles intersects the exterior of $C_\omega$. We call $C_\omega$ a circle *centered at infinity*. To get the corresponding CP representation in (iii), each edge from a vertex $v$ to the vertex of $C_\omega$ is represented by the half-line from the center of $C_v$ through $C_v \cap C_\omega$ (towards infinity). See Figure 2.8 for an example of a CP representation with a circle centered at infinity.

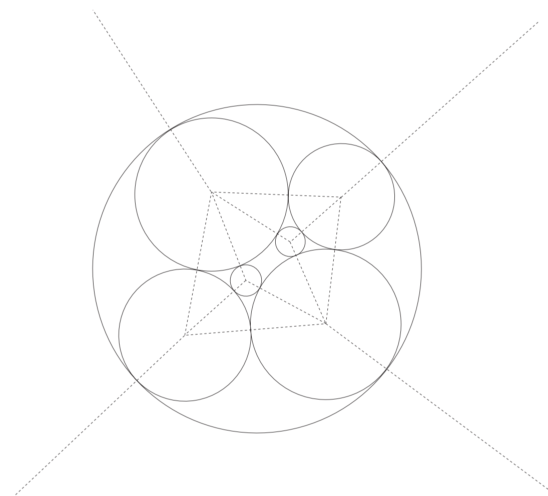

FIGURE 2.8. A CP with a circle centered at infinity

Let us view $\mathbb{R}^2$ as the complex plane $\mathbb{C}$ and the extended plane as $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. Suppose that we have a transformation $w : \mathbb{C} \to \mathbb{C}$. It is well-known and easy to see that $w$ preserves circles (i.e., for every circle $C$ in the plane, $w(C)$ is a circle) if and only if $w$ can be expressed in the form:

$$w(z) = az + b \qquad \text{or} \qquad w(z) = a\overline{z} + b$$

where $a \neq 0$ and $b$ are complex numbers. In the extended plane $\mathbb{C}^*$ there are more general maps which preserve circles if we think of a line as a circle through $\infty$. Consider transformations $w : \mathbb{C}^* \to \mathbb{C}^*$ of the following form:

$$w(z) = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0,$$

where $w(\infty) = a/c$ if $c \neq 0$ and $w(\infty) = \infty$ if $c = 0$. Also, $w(-d/c) = \infty$. These maps are called *fractional linear transformations* or *Möbius transformations*. It is well known (and easy to see) that every fractional linear transformation maps circles and lines to circles and lines in $\mathbb{C}^*$ (lines in $\mathbb{C}^*$ correspond to usual lines in the plane together with the point $\infty$). Every circle that does not contain the point $z = -d/c$ is mapped by $w$ onto a circle. Therefore every fractional linear transformation maps a CP onto another CP if $z = -d/c$ does not lie on any of the circles in the CP. If $z = -d/c$ is the center of a circle in a CP, then the transformed CP has that circle centered at infinity. An immediate corollary is the following:

LEMMA 2.8.1. *If a graph $G$ has a circle packing representation and $v$ is a vertex of $G$, then there is a CP representation of $G$ such that the circle corresponding to $v$ is centered at infinity.*

Let $G$ be a connected plane graph and $G^*$ its geometric dual. A *primal-dual CP* (abbreviated PDCP) of $G$ is a pair of simultaneous circle packings (in the extended plane) of $G$ and of $G^*$, respectively, such that for any dual pair of edges $e = uv \in E(G)$ and $e^* = u^*v^* \in E(G^*)$, the circles $C_u$ and $C_v$ corresponding to $e$ touch at the same point as the circles $C_{u^*}, C_{v^*}$ corresponding to $e^*$, and the line through the centers of $C_u$ and $C_v$ is perpendicular to the line through the centers of $C_{u^*}$ and $C_{v^*}$. We assume that the circle in a PDCP of $G$ and $G^*$ corresponding to the unbounded face of $G$ is centered at infinity.

Our aim is to show that every 3-connected plane graph $G$ admits a PDCP. This immediately yields simultaneous straight line representations of $G$ and its dual graph (with the vertex of $G^*$ corresponding to the unbounded face of $G$ at infinity) such that every pair of dual edges are perpendicular (cf. Theorem 2.8.10).

We need some auxiliary results. We assume in this section that $G$ is a 2-connected plane graph. We define the *vertex-face graph*[2] of $G$, $\Gamma = \Gamma(G)$, as the plane graph obtained as follows. The vertices of $\Gamma(G)$ correspond to vertices and faces of $G$, i.e., $V(\Gamma) = V(G) \cup V(G^*)$. The vertices of $G^*$ are obtained by selecting a point in each face of $G$. The edges in $\Gamma$ correspond to vertex-face incidence in $G$ and are embedded in the plane so that only their endpoints are in $G \cup G^*$. Then $\Gamma$ is bipartite and all

[2] The vertex-face graph is also known in the literature as the *radial graph*.

faces of $\Gamma$ are bounded by 4-cycles. The vertex-face graph of the graph of the 3-prism is represented in Figure 2.9. White vertices correspond to vertices of the 3-prism and black vertices to its faces.

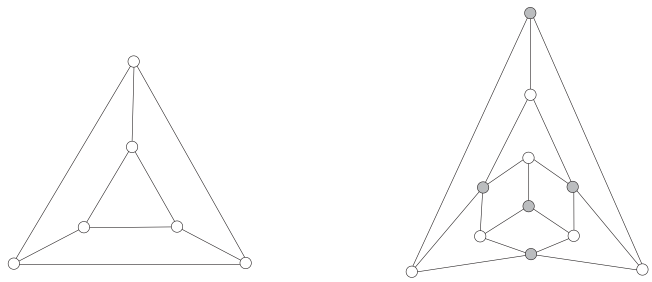

FIGURE 2.9. The 3-prism and its vertex-face graph

LEMMA 2.8.2. *Let $G$ be a 2-connected plane graph with at least 4 vertices and let $\Gamma$ be its vertex-face graph. Then the following assertions are equivalent:*

(a) *$G$ is 3-connected.*
(b) *Every 4-cycle in $\Gamma$ is facial.*
(c) *For every proper subset $S \subset V(\Gamma)$ that contains at least 5 vertices of $\Gamma$ we have*[3]

$$2|S| - |E(\Gamma(S))| \geq 5\,. \tag{2.3}$$

PROOF. If a 4-cycle in $\Gamma$ is nonfacial, then any two opposite vertices of the cycle separate $G$ or $G^*$. Hence (a) $\Rightarrow$ (b). If $G$ has a separating set $\{x, y\}$, then $\{x, y\}$ together with two faces whose boundaries intersect distinct components of $G - \{x, y\}$ form a nonfacial 4-cycle in $\Gamma$. Hence (b) $\Rightarrow$ (a).

By Proposition 2.1.6, $2|S| - |E(\Gamma(S))| \geq 4$ with equality if and only if $\Gamma(S)$ is a quadrangulation. So, if we have equality, one of the facial 4-cycles in $\Gamma(S)$ is nonfacial in $\Gamma$. Conversely, if $\Gamma$ has a nonfacial 4-cycle $C$, we obtain equality by letting $S$ be $V(C)$ together with the vertices in the interior or the exterior of $C$. This shows that (b) is equivalent to (c). $\square$

In the sequel small Greek letters (e.g., $\nu$ or $\tau$) will be used for points in $\mathbb{C}^*$ which are vertices of $\Gamma$. We assume that the vertex of $\Gamma$ corresponding to the unbounded face of $G$ is the point at infinity and we denote it by $\omega$. The vertex-deleted subgraph $\Gamma - \omega$ will be denoted by $\Gamma'$. Given a PDCP $\{C_\nu \mid \nu \in V(\Gamma)\}$ of $G$, we will denote by $r_\nu$ the radius of $C_\nu$, $\nu \in V(\Gamma')$. It will be assumed that the circle $C_\omega$ of $\omega$ is centered at infinity and that $r_\omega$ is a negative number equal, in absolute value, to the radius of $C_\omega$.

[3] Recall that $\Gamma(S)$ is the subgraph of $\Gamma$ induced by $S$.

LEMMA 2.8.3. *Let $r_\nu$, $\nu \in V(\Gamma)$, be the radii of a PDCP of $G$. If $\nu \in V(\Gamma')$ and $\nu\omega \notin E(\Gamma)$, then*

$$\sum_{\substack{\tau \\ \nu\tau\in E(\Gamma)}} \operatorname{arctg} \frac{r_\tau}{r_\nu} = \pi. \tag{2.4}$$

*Let $v_1, \dots, v_k$ be the vertices of $\Gamma$ such that $v_i\omega \in E(\Gamma)$, $i = 1, \dots, k$, and let $\alpha_i = 2\sum_\tau \operatorname{arctg}(r_\tau/r_{v_i})$ where the sum is over all neighbors $\tau$ of $v_i$ in $\Gamma'$. Then*

$$0 < \alpha_i < \pi \quad (1 \le i \le k) \qquad \textit{and} \qquad \sum_{i=1}^{k} \alpha_i = (k-2)\pi. \tag{2.5}$$

PROOF. We assume that $\nu$ is a vertex of $G$. (The proof is similar if $\nu$ is in $G^*$.) If $\nu$ is not a neighbor of $\omega$ (in $\Gamma$), then $C_\nu$ is the inscribed circle of the polygon in $G^*$ whose vertices are the neighbors (in $\Gamma$) of $\nu$. In Figure 2.10, one of these neighbors is shown. The angle $\alpha$ in Figure 2.10 equals $\operatorname{arctg}(r_\tau/r_\nu)$. This implies (2.4) since twice the sum of all such angles is equal to $2\pi$.

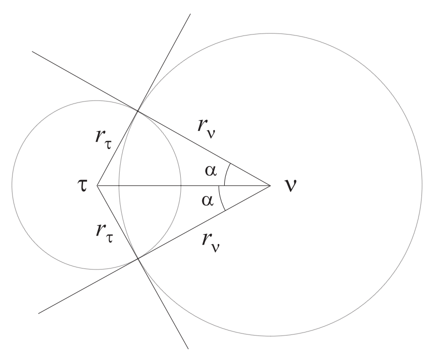


FIGURE 2.10. The angle of $\tau$ at $\nu$

If $v_i$ is adjacent to $\omega$, then $\alpha_i$ is the angle at $v_i$ of the outer facial cycle of $G$. Since $C_\omega$ is inscribed in that polygon, (2.5) holds. □

Suppose that we have simultaneous CP representations of $G$ and $G^* - \omega$ such that for each edge $\nu\tau \in E(\Gamma')$, the circles $C_\nu$ and $C_\tau$ cross at the right angle. Then we say that we have a *weak PDCP*. We shall show that the existence of positive numbers $r_\nu$, $\nu \in V(\Gamma')$, satisfying (2.4) and (2.5) is sufficient for the existence of a weak PDCP. Suppose now that such "radii" $r_\nu$, $\nu \in V(\Gamma)$, exist. For each face $\Phi = \nu x \tau y$ of $\Gamma$, the radii uniquely determine the shape and the size of a quadrangle $Q(\Phi)$ corresponding to

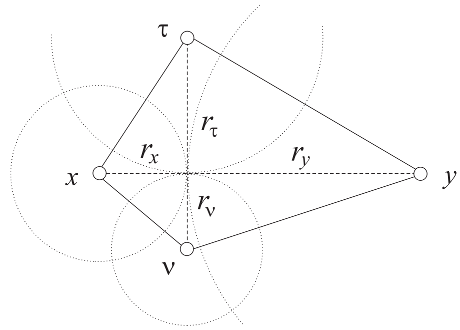


FIGURE 2.11. The quadrilateral $Q(\nu x \tau y)$

$\Phi$ in a possible PDCP representation of $G$ with given radii. Suppose that $\nu, x, y \neq \omega$. Then the angle of $Q(\Phi)$ at $\nu$ is equal to

$$\alpha(\Phi, \nu) = \operatorname{arctg} \frac{r_x}{r_\nu} + \operatorname{arctg} \frac{r_y}{r_\nu} \tag{2.6}$$

and the length of the side $\nu x$ is equal to $\sqrt{r_\nu^2 + r_x^2}$. See Figure 2.11. If $\tau = \omega$, then $Q(\Phi)$ is a quadrangle with two bounded and two unbounded sides but the triangle $\nu xy$ is uniquely determined. By the angle condition (2.4), the quadrangles $Q(\Phi)$ fit together around each vertex $\nu \neq \omega$.

Next we prove that a locally plane representation of $\Gamma$ as obtained above by pasting together the quadrilaterals $Q(\Phi)$ determines a tiling of the entire plane. For this we need some auxiliary results on graphs drawn in the plane which may be of independent interest.

First we prove a special case of an elementary fact about covering spaces.

LEMMA 2.8.4. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a covering map, i.e., $f$ is continuous, onto, and for each $p \in \mathbb{R}^2$ there exist open neighborhoods $U$ and $W$ of $p$ and $f(p)$, respectively, such that the restriction of $f$ to $U$ is a homeomorphism of $U$ onto $W$. Suppose further that the set of points $q$ such that $|f^{-1}(f(q))| > 1$ is bounded. Then $f$ is a homeomorphism.*

PROOF. The set $S$ of points $q$ such that $|f^{-1}(q)| \geq 2$ is clearly open. Also, the set of points $q$ such that $|f^{-1}(q)| = 1$ is easily proved to be open. This is possible only if $S$ is empty. □

The conclusion of Lemma 2.8.4 can be derived even without the assumption that $\{q \in \mathbb{R}^2 \mid |f^{-1}(f(q))| > 1\}$ is bounded (since $\mathbb{R}^2$ is simply connected, see e.g. Massey [**Ma67**]).

PROPOSITION 2.8.5. *Let $G$ be a 2-connected plane graph with polygonal edges. Let $H$ be a drawing of $G$ in the plane (possibly with edge* *crossings) such that all edges of $H$ are polygonal arcs. Suppose further that:*

(i) *For each vertex $x$ of $G$, the edges incident with $x$ in $H$ are pairwise noncrossing and leave $x$ in the same clockwise order as in $G$.*
(ii) *Each facial cycle in $G$ corresponds to a simple closed curve in $H$.*
(iii) *If $C$ is a facial cycle bounding a bounded face in $G$, and $e$ is an edge of $G$ leaving $C$, then the first segment of $e$ is in the exterior of $C$ in $H$.*

*Then $H$ is a plane representation of $G$, i.e., $H$ has no edge crossings.*

Proof. Let $f : G \to H$ be an isomorphism. We extend $f$ to a continuous map of the point set of $G$ onto the point set of $H$ such that $f$ is 1-1 on each edge of $G$. Next we extend $f$ to a continuous map from $\mathbb{R}^2 \to \mathbb{R}^2$ using the Jordan-Schönflies Theorem as follows. Let $C_0$ denote the outer cycle of $G$. For every facial cycle $C \neq C_0$ of $G$ we use (ii) and the Jordan-Schönflies Theorem to extend $f$ to $\mathrm{int}(C)$ such that the restriction of the new map (which we also call $f$) to $\overline{\mathrm{int}}(C)$ is a homeomorphism onto $\overline{\mathrm{int}}(f(C))$. If $p$ is a point in $\mathrm{int}(C_0)$, then by (i) and (iii), $f(p)$ is an interior point in the compact set $f(\overline{\mathrm{int}}(C_0))$. Hence the boundary of $f(\overline{\mathrm{int}}(C_0))$ is a subset of $f(C_0)$. This implies that either $f(\overline{\mathrm{int}}(C_0)) = \overline{\mathrm{int}}(f(C_0))$ or $f(\overline{\mathrm{int}}(C_0)) = \overline{\mathrm{ext}}(f(C_0))$. As $f(\overline{\mathrm{int}}(C_0))$ is compact, the former equality holds. By the Jordan-Schönflies Theorem, $f$ can be extended to $\mathbb{R}^2$ such that the restriction to $\overline{\mathrm{ext}}(C_0)$ is a homeomorphism onto $\overline{\mathrm{ext}}(f(C_0))$. By Lemma 2.8.4, $f$ is a homeomorphism of $\mathbb{R}^2$ onto $\mathbb{R}^2$. In particular, $H$ is a plane representation of $G$. □

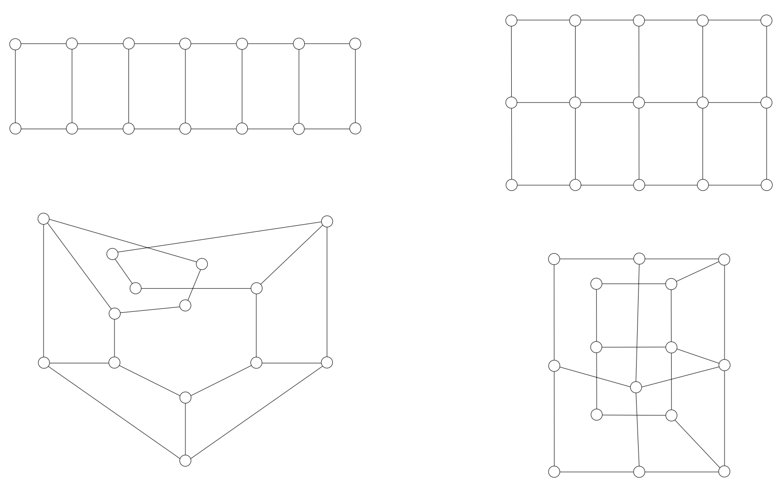

Figure 2.12. Locally bad drawings

Figure 2.12 shows that just a slight weakening of conditions (i)–(iii) of Proposition 2.8.5 does not result in the conclusion of that proposition.

LEMMA 2.8.6. *Let $G$ be a 3-connected plane graph and $\Gamma = \Gamma(G)$ its vertex-face graph. If there are positive numbers $r_\nu$, $\nu \in V(\Gamma')$, such that (2.4)–(2.5) are satisfied, then there exists a weak PDCP of $G, G^*$ with radii $r_\nu$, $\nu \in V(\Gamma')$ and with the same local clockwise orientations as in $G$, $G^*$.*

PROOF. For every vertex $\nu \in V(\Gamma')$, we shall determine a point $\bar{\nu}$ in $\mathbb{R}^2$ such that the circles $C_\nu$ with center $\bar{\nu}$ and radius $r_\nu$ ($\nu \in V(\Gamma')$) form the desired weak PDCP. We start with a vertex $\tau_0$ of $\Gamma'$ and draw any edge $\tau_0\tau_1$, its length being equal to $\sqrt{r_{\tau_0}^2 + r_{\tau_1}^2}$ (cf. Figure 2.10). Then the position of each neighbor of $\tau_0$ is uniquely determined. So, having drawn $\bar{\tau}_0$ and $\bar{\tau}_1$ we consider, for each vertex $\tau$ in $\Gamma'$, a path from $\tau_0$ to $\tau$ and thus get a position for $\bar{\tau}$. Clearly, the position for $\tau_0$ does not change if we walk from $\tau_0$ to $\tau_0$ along a facial 4-cycle or, more generally, along any cycle $C$ (as can be proved by induction on the number of facial 4-cycles in $\overline{\mathrm{int}}(C)$), or along any closed walk from $\tau_0$ to $\tau_0$. Hence, we get the same point $\bar{\tau}$ if we consider another path from $\tau_0$ to $\tau$. Also, we get the same drawing of $\Gamma'$ if we start with any other edge instead of $\tau_0\tau_1$ and repeat the construction. It follows that (2.4) is satisfied at every vertex. It only remains to show that the drawing of $\Gamma'$ has no edge crossings. But this follows from Proposition 2.8.5. (Note that we need (2.5) in order to show that the outer facial cycle of $\Gamma'$ satisfies (ii).) □

Lemma 2.8.7 below shows that for any 3-connected plane graph $G$, there exist positive real numbers associated with the vertices of $\Gamma = \Gamma(G)$ such that (2.4)–(2.5) are satisfied.

LEMMA 2.8.7. *Let $G$ be a 3-connected plane graph with outer cycle $C = v_1v_2\dots v_k$. Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be real numbers such that $0 < \alpha_i < \pi$ ($i = 1, 2, \dots, k$) and $\alpha_1 + \dots + \alpha_k = (k-2)\pi$. Then there are positive numbers $r_\nu$, $\nu \in V(\Gamma')$, such that (2.4) holds for $\nu \neq v_1, \dots, v_k$ and, for each $i = 1, \dots, k$,*

$$2 \sum_{v_i\tau \in E(\Gamma')} \operatorname{arctg} \frac{r_\tau}{r_{v_i}} = \alpha_i \tag{2.7}$$

*where the summation is taken over all neighbors $\tau$ of $v_i$ in $\Gamma'$. The numbers $r_\nu$, $\nu \in V(\Gamma')$, are unique up to a multiplicative constant.*

PROOF. Suppose that we have a list of positive numbers, $r = (r_\nu \mid \nu \in V(\Gamma'))$. For each $\nu \in V(\Gamma') \setminus \{v_1, \dots, v_k\}$ we define

$$\vartheta_\nu = \vartheta_\nu(r) = \sum_{\nu\tau \in E(\Gamma)} \operatorname{arctg} \frac{r_\tau}{r_\nu} - \pi .$$

Also, we define for $i = 1, \dots, k$

$$\vartheta_{v_i} = \vartheta_{v_i}(r) = \sum_{v_i\tau\in E(\Gamma')} \operatorname{arctg}\frac{r_\tau}{r_{v_i}} \; - \; \tfrac{1}{2}\alpha_i\,.$$

The number

$$\mu(r) = \sum_{\nu\in V(\Gamma')} \vartheta_\nu^2 \tag{2.8}$$

is a measure for how far $r$ is from a solution. To prove the theorem, it suffices to see that there are positive numbers $r = (r_\nu)$ such that $\mu(r) = 0$.

We claim that

$$\sum_{\nu\in V(\Gamma')} \vartheta_\nu = 0\,. \tag{2.9}$$

To see this, we expand the sum of $\vartheta_\nu$ as:

$$\begin{aligned}\sum_{\nu\in V(\Gamma')} \vartheta_\nu \;\; &= \sum_{\nu\tau\in E(\Gamma')} \Big(\operatorname{arctg}\frac{r_\tau}{r_\nu} + \operatorname{arctg}\frac{r_\nu}{r_\tau}\Big)\\ &\quad -\pi(|V(\Gamma')| - k) - \frac{1}{2}\sum_{i=1}^{k}\alpha_i\,. \end{aligned}\tag{2.10}$$

Since $\Gamma$ is a quadrangulation, we have by Proposition 2.1.6(b)

$$2|V(\Gamma')| = |E(\Gamma)| + 2 = |E(\Gamma')| + k + 2\,. \tag{2.11}$$

Since $\operatorname{arctg}(x) + \operatorname{arctg}(1/x) = \pi/2$ for every $x > 0$, (2.10) and (2.11) clearly imply (2.9).

Let $S$ be a proper nonempty subset of $V(\Gamma')$. Denote by $t$ the number of vertices among $v_1, \dots, v_k$ that are contained in $S$. By applying (2.3) on the set $S \cup \{\omega\} \subset V(\Gamma)$, we see that

$$2|S| - |E(\Gamma(S))| \ge t + 3 \tag{2.12}$$

if $|S| \ge 4$. It is easy to see that (2.12) holds also in cases when $|S| \in \{2, 3\}$ and $t = 0$. If $t > 0$ and $|S| \in \{2, 3\}$, then we have:

$$2|S| - |E(\Gamma(S))| \ge t + 2\,. \tag{2.13}$$

Let $\mathcal{Q}$ be the set of all sequences $r = (r_\nu \mid \nu \in V(\Gamma'))$ (of radii candidates) such that $0 < r_\nu \le 1$ and $r_\nu = 1$ if $\vartheta_\nu > 0$ ($\nu \in V(\Gamma')$). Moreover, we require that $r_\nu = 1$ for some $\nu \in V(\Gamma')$. Clearly, $\mathcal{Q}$ is nonempty since the sequence with $r_\nu = 1$ for each $\nu \in V(\Gamma')$ belongs to $\mathcal{Q}$.

Let $m = \inf\{\mu(r) \mid r \in \mathcal{Q}\}$. We claim that the infimum is attained, i.e., $m = \mu(r)$ for some $r \in \mathcal{Q}$. Let $r^{(1)}, r^{(2)}, r^{(3)}, \dots$ be a sequence in $\mathcal{Q}$ such that $\mu(r^{(i)}) \to m$ as $i \to \infty$. By standard arguments, there is a subsequence such that for each $\nu \in V(\Gamma')$ the corresponding numbers $r_\nu^{(i)}$ converge. We may assume that this holds for the sequence $r^{(1)}, r^{(2)}, r^{(3)}, \dots$.

Let $S \subseteq V(\Gamma')$ be the set of vertices $\nu$ for which $\lim_{i\to\infty} r_\nu^{(i)} \neq 0$. Suppose that $S \neq V(\Gamma')$. By a calculation similar to that in (2.10) we get

$$\begin{aligned}\sum_{\nu\in S} \vartheta_\nu(r^{(i)}) &= \frac{\pi}{2}|E(\Gamma(S))| - \pi(|S| - t) \\ &\quad -\frac{1}{2}\sum_{v_j\in S}\alpha_j + \sum_{\nu,\tau}\operatorname{arctg}\frac{r_\tau^{(i)}}{r_\nu^{(i)}} \end{aligned} \tag{2.14}$$

where the last sum is taken over all edges $\nu\tau \in E(\Gamma')$ such that $\nu \in S$ and $\tau \notin S$. By definition of $S$, this latter sum tends to 0 as $i \to \infty$. Therefore $\sum_{\nu\in S}\vartheta_\nu(r^{(i)})$ tends to

$$-\frac{\pi}{2}\big(2|S| - |E(\Gamma(S))| - t - 2\big) + \frac{1}{2}\sum_{v_j\in S}(\pi - \alpha_j) - \pi \tag{2.15}$$

as $i \to \infty$. Since $\pi - \alpha_j > 0$ for $j = 1, \ldots, k$ and $\sum_{j=1}^{k}(\pi - \alpha_j) = 2\pi$, (2.15) implies that $\sum_{\nu\in S}\vartheta_\nu(r^{(i)}) < 0$ if $i$ is large enough and if (2.12) holds. The same is true when we have equality in (2.13) since in that case $t < k$. The remaining case when $|S| = 1$ trivially gives the same conclusion. This result and (2.9) imply that

$$\sum_{\nu\notin S}\vartheta_\nu(r^{(i)}) > 0$$

if $i$ is sufficiently large. But $\vartheta_\nu(r^{(i)}) > 0$ implies that $r_\nu^{(i)} = 1$, a contradiction to the definition of $S$. Hence $S = V(\Gamma')$.

Let $r = \lim_{i\to\infty} r^{(i)}$. Since the functions $\vartheta_\nu$ are continuous, $r \in \mathcal{Q}$. Now we prove that $m = \mu(r) = 0$. Suppose that this is not the case. Let $S'$ be the set of vertices $\nu$ with $\vartheta_\nu(r) < 0$. By (2.9), $S' \neq V(\Gamma')$ and $S' \neq \emptyset$. Let $r'_\nu = r_\nu$ if $\nu \notin S'$ and let $r'_\nu = \alpha r_\nu$ if $\nu \in S'$, where $\alpha < 1$. If $\alpha$ is close enough to 1 (so that no $\vartheta_\nu(r')$, $\nu \in S'$, becomes positive), then $r' \in \mathcal{Q}$. Using (2.9) and the definition of $\vartheta_\nu$, it is easy to see that $\mu(r') < \mu(r)$ if $\alpha$ is close enough to 1. This contradicts the minimality of $\mu(r)$.

Suppose now that there are distinct solutions $r$ and $r'$ such that $\max\{r_\nu \mid \nu \in V(\Gamma')\} = \max\{r'_\nu \mid \nu \in V(\Gamma')\} = 1$. Then $\vartheta_\nu(r) = \vartheta_\nu(r') = 0$ for all $\nu \in V(\Gamma')$. Assume without loss of generality that the set $S = \{\nu \mid r_\nu > r'_\nu\}$ is nonempty. Clearly, $S \neq V(\Gamma')$. From (2.14) applied to $r$ and $r'$, respectively, we get

$$0 = \sum_{\nu\in S}\vartheta_\nu(r) - \sum_{\nu\in S}\vartheta_\nu(r') = \sum_{\nu,\tau}(\operatorname{arctg}\frac{r_\tau}{r_\nu} - \operatorname{arctg}\frac{r'_\tau}{r'_\nu})$$

where the last sum is taken over all edges $\nu\tau \in E(\Gamma')$ such that $\nu \in S$ and $\tau \notin S$. By definition of $S$, the latter sum is negative, a contradiction. The proof is complete. □

We now apply Lemmas 2.8.6 and 2.8.7 to show that every 3-connected planar graph admits a PDCP.

THEOREM 2.8.8 (Brightwell and Scheinerman [**BS93**]). *Let $G$ be a 3-connected plane graph. Then $G$ admits a PDCP representation. The PDCP of $G$ is unique up to fractional linear transformations and reflections in the plane.*

PROOF. Suppose first that the outer cycle of $G$ is a 3-cycle. Using the notation in Lemma 2.8.7 we let $\alpha_1 = \alpha_2 = \alpha_3 = \pi/3$ and then apply Lemma 2.8.7. The list $r = (r_\nu \mid \nu \in V(\Gamma'))$ satisfies (2.4)–(2.5). By Lemma 2.8.6, there is a weak PDCP of $G$ with these radii. In particular, $r_{v_1} = r_{v_2} = r_{v_3}$. This implies that this weak PDCP can be extended to a PDCP by adding the circle $C_\omega$. Because of the uniqueness of the radii in Lemma 2.8.7, the resulting PDCP is unique once the three circles $C_1, C_2, C_3$ corresponding to $v_1, v_2, v_3$, respectively, have been prescribed.

Suppose next that the outer cycle of $G$ has length greater than 3. Then either $G$ or $G^*$ has a facial 3-cycle by the proof of Proposition 2.1.6. Now redraw $G$ and $G^*$ (using a fractional linear transformation), and interchange the roles of $G$ and $G^*$ if necessary, so that the outer cycle of $G$ in the new embedding is a 3-cycle. By the previous paragraph the new embedding of $G, G^*$ has a PDCP representation. The inverse of the applied fractional linear transformation gives rise to a PDCP representation of the original pair $G, G^*$.

For any PDCP representation of $G$, there exists a fractional linear transformation (possibly followed by reflection) taking the PDCP representation into one using the prescribed circles $V_1, C_2, C_3$ in the previous paragraph. That proves uniqueness. □

Mohar [**Mo97a**] proved that, given a 3-connected planar graph $G$ and an $\varepsilon > 0$, one can determine the centers and radii of the PDCP of $G$ with precision $\varepsilon$ in time that is bounded by a polynomial in $|V(G)|$ and $\max\{\log(1/\varepsilon), 1\}$.

An immediate corollary of Theorem 2.8.8 is a result of Koebe [**Ko36**] which was independently discovered by Andreev [**An70a, An70b**] and Thurston [**Th78**].

COROLLARY 2.8.9 (Koebe–Andreev–Thurston). *Every plane graph admits a circle packing representation.*

Koebe (and also Andreev and Thurston) also proved that circle packing representations of planar triangulations are unique up to fractional linear transformations.

In Theorem 2.3.2 we proved that every 3-connected planar graph has a convex representation. Since every PDCP representation is convex, Theorem 2.8.8 yields a much stronger result:

THEOREM 2.8.10 (Brightwell and Scheinerman [**BS93**]). *If $G$ is a planar 3-connected graph, then $G$ and its dual $G^*$ can be embedded in the plane with straight edges and with the outer vertex of $G^*$ at infinity such that they form a geometric dual pair. Both embeddings are convex and each pair of dual edges is perpendicular.*

Theorem 2.8.10 was conjectured by Tutte [**Tu63**] who proved that every planar 3-connected graph $G$ and its dual $G^*$ have simultaneous straight line representations so that only dual pairs of edges cross.

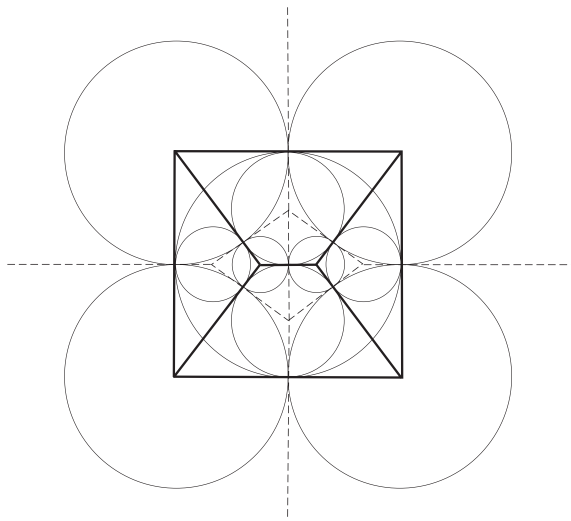

FIGURE 2.13. A PDCP representation of the 3-prism

In Figure 2.13 a PDCP representation of the 3-prism is shown. The edges of the dual graph are represented by broken lines.

Tutte established a necessary and sufficient condition for a 2-connected graph $G$ with a given cycle $C$ to have a convex embedding in the plane such that $C$ bounds the outer face and it is a convex $|V(C)|$-gon. The condition is that $G$ is a subdivision of a 2-connected graph $H$ such that every separating set $\{u, v\}$ of $H$ (if any) is contained in the cycle of $H$ corresponding to $C$. This result was generalized by Thomassen [**Th80a**] to cover also the case where $C$ need not be strictly convex, i.e., it could be a $k$-gon with $k < |V(C)|$. Thomassen [**Th88**] also extended Tutte's result of simultaneous straight line representations of a planar graph and its dual to the 2-connected case. The proof of [**Th80a**] yields a linear time algorithm for convex drawings of planar graphs. See [**CYN84, NC88**] for details.

The inverse of the stereographic projection which maps the extended plane onto the unit sphere in $\mathbb{R}^3$ takes every CP in the extended plane into a *spherical circle packing*, a set of circles on the unit sphere in $\mathbb{R}^3$ that has the same properties as are required for circle packings in the plane. Theorem 2.8.8 implies the following geometric result.

THEOREM 2.8.11. *If $G$ is a 3-connected planar graph, then there is a convex polyhedron $Q$ in $\mathbb{R}^3$ whose graph is isomorphic to $G$ such that all edges of $Q$ are tangent to the unit sphere in $\mathbb{R}^3$.*

PROOF. By Theorem 2.8.8, $G$ has a PDCP. The inverse of the stereographic projection maps the circles of the PDCP to circles $\tilde{C}_\nu$, $\nu \in V(\Gamma(G))$, on the unit sphere in $\mathbb{R}^3$. Denote by $\Pi_\nu$ the plane in $\mathbb{R}^3$ that contains $\tilde{C}_\nu$. It is easy to see that the planes $\Pi_\nu$, $\nu \in V(G^*)$, determine a convex polytope whose graph is isomorphic to $G$ and whose edges are tangent to the unit sphere. □

The construction in the above proof gives, at the same time, a convex polyhedron whose graph is $G^*$ and whose edges are tangent to the unit sphere at the same points as their dual edges, and are perpendicular to their dual edges.

We have proved that part of Theorem 2.8.8 implies Theorem 2.8.11. Also, the converse holds as pointed out by Sachs [**Sa94**]. To see this, let $G$ be a 3-connected planar graph and $Q$ a convex polyhedron whose graph is isomorphic to $G$ and whose edges are tangent to the unit sphere $S^2$ in $\mathbb{R}^3$. Then the faces of $Q$ intersect the unit sphere in circles which determine a spherical CP. Let $\tilde{C}_\tau$ be the circle corresponding to the face $\tau$ of $Q$. The stereographic projection maps these circles into a CP representation of the dual graph $G^*$ of $G$ in the plane. At the same time, we get a CP of $G$ as follows. Let $\nu$ be a vertex of $Q$. The cone with apex $\nu$ that is tangent to the unit sphere $S^2$ has a circle $\tilde{C}_\nu$ in common with $S^2$. All such circles $\tilde{C}_\nu$, $\nu \in V(Q) = V(G)$, determine a circle packing of $G$ on $S^2$. Clearly, these circles intersect the corresponding circles $\tilde{C}_\tau$ in the CP of $G^*$ perpendicularly (or not at all). The stereographic projection therefore gives rise to a PDCP of $G$ and $G^*$ in the plane.

Theorem 2.8.11 was conjectured by Grünbaum and Shephard [**GS87**] (cf. also Schulte [**Sch87**]) and independently by Sachs [**Sa94**] (see also Lehel and Sachs [**LS90**]).

Theorem 2.8.11 implies the difficult part of Steinitz' Theorem [**St22**].

**Steinitz' Theorem.** *A graph $G$ is the graph of a convex polytope in $\mathbb{R}^3$ if and only if it is planar and 3-connected.*

For the definition of the graph (1-skeleton) of a polytope and for the easy part of Steinitz' Theorem, the reader is referred to Grünbaum [**Gr67**] or Brøndsted [**Br83**].

The uniqueness of the PDCP in Theorem 2.8.8 implies that every automorphism[4] of a 3-connected planar graph $G$ induces a geometric symmetry of the corresponding polyhedron $Q$ in Theorem 2.8.11. This implies an extension of Steinitz' Theorem.

THEOREM 2.8.12 (Mani [**Ma71**]). *If $G$ is a 3-connected planar graph, then there is a convex polyhedron $Q$ in $\mathbb{R}^3$ whose graph is isomorphic to $G$ such that every automorphism of $G$ induces a symmetry of $Q$.*

Schramm has obtained the following generalizations of (part of) Theorems 2.8.8 and 2.8.11, respectively.

THEOREM 2.8.13 (Schramm [**Sc96**]). *Let $G$ be a planar graph and $(P_v\,;\, v \in V(G))$ a collection of strictly convex compact subsets of the plane with smooth boundary. Then there are numbers $\alpha_v > 0$ $(v \in V(G))$ and points $\rho_v \in \mathbb{R}^2$ $(v \in V(G))$ such that the sets $Q_v = \alpha_v P_v + \rho_v$ $(v \in V(G))$ have pairwise disjoint interiors, and $Q_u, Q_v$ intersect if and only if $u$ and $v$ are adjacent in $G$.*

THEOREM 2.8.14 (Schramm [**Sc92**]). *Let $S$ be a strictly convex compact set in $\mathbb{R}^3$ with nonempty interior. If $G$ is a 3-connected planar graph, then there is a convex polyhedron $P$ in $\mathbb{R}^3$, whose graph is isomorphic to $G$, all of whose edges are tangent to $S$.*

Colin de Verdière [**CV89, CV91**] and Mohar [**Mo97a**] obtained analogues of the PDCP results for graphs on arbitrary surfaces.

H. Harborth (private communication) has raised the following

PROBLEM 2.8.15. *Does every planar graph have a straight line representation such that all edges have integer length?*

Brightwell and Scheinerman [**BS93**] showed that this cannot in general be achieved by a circle packing representation since otherwise it would be possible to trisect an angle of $\pi/3$ by ruler and compass.

## 2.9. The Riemann Mapping Theorem

An open connected set $\Omega \subseteq \mathbb{R}^2$ is *simply connected* if, for every simple closed curve $J \subseteq \Omega$, we have $\text{int}(J) \subseteq \Omega$. Using the proof of the Jordan–Schönflies Theorem presented in Section 2.2, it is not difficult to prove that every open simply connected set $\Omega$ in $\mathbb{R}^2$ is homeomorphic to the open unit disc $\Delta$ in $\mathbb{R}^2$. The Riemann Mapping Theorem says that, if $\Omega$ is bounded, then $\Omega$ is conformally equivalent to $\Delta$, i.e., there exists a homeomorphism $f : \Omega \to \Delta$ which is conformal (analytic). At the International Symposium in Celebration of the Proof of the Bieberbach Conjecture (Purdue University, March 1985), William Thurston conjectured that the conformal mapping of $\Omega$ to the unit disc $\Delta$ can be approximated by manipulating

[4]An *automorphism* of a graph is an isomorphism of the graph onto itself.

hexagonal circle packing configurations in $\Omega$. More precisely, let $p$ and $q$ be fixed distinct points in $\Omega$. Consider the standard hexagonal tiling of the plane into hexagons of diameter $1/n$. Let $G'_n$ be the graph which is the union of those hexagons that are in $\Omega$. Let $G''_n$ be the maximal subgraph of $G'_n$ which contains $p$ and $q$ in its hexagons and which has no two adjacent vertices which separate $G''_n$. For $n$ sufficiently large, $G''_n$ exists and is a subdivision of a 3-connected graph $G_n$. By Theorem 2.8.8, the pair $G_n, G^*_n$ has a PDCP such that the vertex of $G^*_n$ in the unbounded face of $G_n$ corresponds to the unit circle (i.e., the boundary of $\Delta$) centered at infinity. We focus on the circle packing of $G^*_n$. An example is indicated in Figure 2.14. It shows a region $\Omega$ with the hexagonal lattice graph in it and the graph $G^*_n$ without its vertex corresponding to the outer face of $G_n$. A circle packing of $G^*_n$ is shown in Figure 2.15.

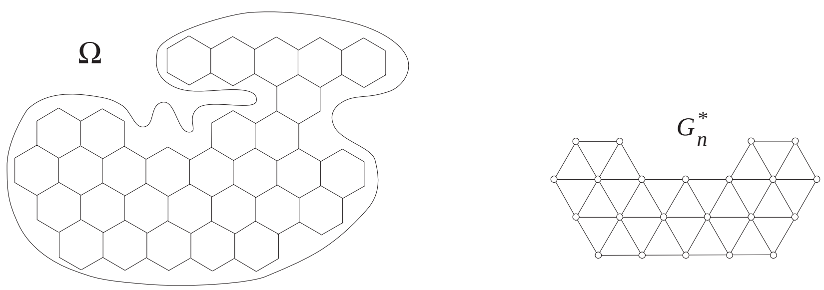


FIGURE 2.14. Discretization of a region $\Omega$

We define a map $f_n$ from the interior of the outer cycle of $G''_n$ into $\Delta$ by mapping all points inside a hexagon to the center of the corresponding circle in $\Delta$. By modifying the circle packing by a Möbius transformation, we may assume that $f_n(p) = 0$, and using a rotation we may assume that $f_n(q)$ is a positive real number. Thurston conjectured that $f_n$ converges to a homeomorphism which is analytic, and this was verified by Rodin and Sullivan [**RS87**].

The Riemann Mapping Theorem has several other proofs but the proof of Rodin and Sullivan based on circle packings is particularly interesting because of its combinatorial and constructive nature. It can be used for computer experiments on conformal mappings, see Dubejko and Stephenson [**DS95b**] and Collins and Stephenson [**CS98p**].

Some other central results in the theory of conformal mappings have been successfully attacked by the use of circle packings, e.g., Schwartz' Lemma (e.g., Beardon and Stephenson [**BS91**], Dubejko and Stephenson [**DS95a**], Rodin [**Ro87, Ro89**]), Koebe uniformization (He and Schramm [**HS93, HS95**]), etc. (Aharonov [**Ah90**]). More references can be found in Stephenson's Cumulative bibliography on circle packings [**St93**].

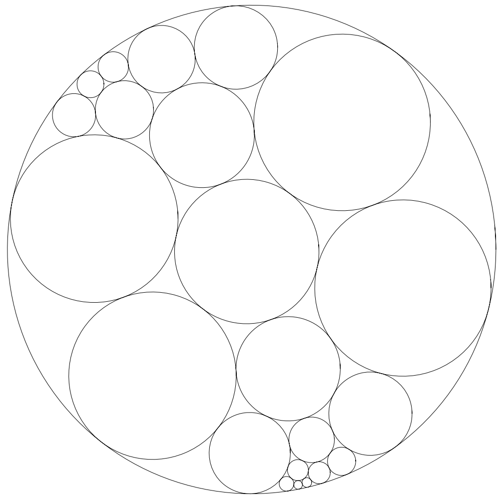

FIGURE 2.15. An approximation to a Riemann mapping

## 2.10. The Jordan Curve Theorem and Kuratowski's Theorem in general topological spaces

We have previously observed that, in Kuratowski's theorem, $K_{3,3}$ is more fundamental in that $K_5$ can be omitted when we restrict Kuratowski's theorem to 3-connected graphs of order at least 6 (cf. Lemma 2.5.5). The fundamental character of $K_{3,3}$ becomes even more clear when we consider more general topological spaces that have the Jordan curve separation property, as demonstrated by the following result that provides a link between the Jordan Curve Theorem and Kuratowski's theorem.

THEOREM 2.10.1 (Thomassen [**Th90a**]). *Let $X$ be an arcwise connected Hausdorff space that cannot be separated by a simple arc. Assume also that $X$ is not homeomorphic to a simple closed curve. Then the following statements are equivalent:*

(a) *Every simple closed curve separates $X$.*
(b) *Every simple closed curve separates $X$ into precisely two arcwise connected components.*
(c) *$K_{3,3}$ cannot be embedded in $X$.*
(d) *Neither $K_{3,3}$ nor $K_5$ can be embedded in $X$.*

PROOF. Clearly (b) $\Rightarrow$ (a) and (d) $\Rightarrow$ (c). It is also easy to prove that (c) $\Rightarrow$ (d). For suppose that $K_5$ is embedded in $X$. Let $v_1, v_2, \dots, v_5$ be the vertices of $K_5$. Let $p_1, p_2$ be points on the edge $v_1v_2$ such that $v_1, p_1, p_2, v_2$ are distinct. Now consider an arc $A$ in $K_5$ which connects $p_1$ and $p_2$ and which contains all of $v_1, v_2, v_3, v_4, v_5$. Since $X \setminus A$ is arcwise