LEMMA 2.1.9. *Let Γ_1, Γ_2 be plane graphs such that each edge is a simple polygonal arc. Then the union of the point sets of Γ_1 and Γ_2 is the point set of a plane graph Γ_3, and Γ_3 is uniquely determined up to homeomorphism. If both Γ_1 and Γ_2 are 2-connected and have at least two points in common, then also Γ_3 is 2-connected.*

PROOF. For $i = 1, 2$, let Γ'_i denote the plane graph which is a subdivision of Γ_i such that each edge of Γ'_i is a straight line segment. Let Γ''_i be the subdivision of Γ'_i such that a point p on an edge e of Γ'_i is a vertex of Γ''_i if either p is a vertex of Γ'_1 or Γ'_2, or p is on an edge of Γ'_{3-i} that crosses e. Then the usual union of graphs Γ''_1 and Γ''_2 is a graph that can play the role of Γ_3. (Note that Γ_3 has no loops or multiple edges since its edges are straight line segments.)

It is obvious that Γ_3 is uniquely determined up to a homeomorphism. The last statement about 2-connectivity is left as an easy exercise. □

Lemma 2.1.9 does not hold for general plane graphs since two arcs can intersect infinitely often.

We shall use the notation $\Gamma_3 = \Gamma_1 \sqcup \Gamma_2$ to denote the graph Γ_3 arising from Γ_1 and Γ_2 as described in the proof of Lemma 2.1.9. Note that the "union" of plane graphs defined above is associative.

LEMMA 2.1.10 (Thomassen [**Th92a**]). *Let $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ $(k \geq 2)$ be 2-connected polygonal arc embedded plane graphs such that, for $i = 2, 3, \dots, k-1$, the graph Γ_i has at least two points in common with each of Γ_{i-1} and Γ_{i+1} and no point in common with any other Γ_j, $|j-i| \geq 2$. Then any point which is in the outer face of each of $\Gamma_1 \sqcup \Gamma_2$, $\Gamma_2 \sqcup \Gamma_3$, $\dots$, $\Gamma_{k-1} \sqcup \Gamma_k$ is also in the outer face of $\Gamma_1 \sqcup \Gamma_2 \sqcup \cdots \sqcup \Gamma_k$.*

PROOF. Suppose p is a point in a bounded face of $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_k$. By Lemma 2.1.9, Γ is 2-connected, and by Proposition 2.1.5, there is a cycle C in Γ such that $p \in \mathrm{int}(C)$. Choose C such that C is in $\Gamma_i \sqcup \Gamma_{i+1} \sqcup \cdots \sqcup \Gamma_j$ and such that $j - i$ is minimum. We will show that $j - i \leq 1$. So assume that $j - i \geq 2$. Among all cycles in $\Gamma_i \sqcup \cdots \sqcup \Gamma_j$ having p in the interior we assume that C is chosen in such a way that the number of edges in C and not in Γ_{j-1} is minimum. Since C intersects both $\Gamma_j \backslash \Gamma_{j-1}$ and $\Gamma_{j-2} \backslash \Gamma_{j-1}$, C has at least two disjoint maximal segments P_1 and P_2 in Γ_{j-1}. Since Γ_{j-1} is connected, it contains a path from P_1 to P_2. Let P_3 be a shortest path in Γ_{j-1} from P_1 to $C - V(P_1)$. Then $C \cup P_3$ has three cycles two of which have p in the interior. The one that contains P_3 has fewer edges not in Γ_{j-1} than C. This contradicts the minimality of C. □

THEOREM 2.1.11. *If P is a simple arc in the plane, then $\mathbb{R}^2 \backslash P$ is arcwise connected.*

PROOF (from [**Th92a**]). Let p, q be two points of $\mathbb{R}^2 \backslash P$, and let d be a positive number such that each of p, q has distance $> 3d$ from P.

We shall join p, q by a simple polygonal arc in $\mathbb{R}^2 \backslash P$. Since P is the image of a continuous (and hence uniformly continuous) map, we can partition P into segments $P_1, P_2, \ldots, P_k$ such that P_i joins p_i and p_{i+1} for $i = 1, 2, \ldots, k$ and such that each point on P_i has distance less than d from p_i ($i = 1, 2, \ldots, k$). Let d' be a positive number smaller than the minimal distance between P_i and P_j, $1 \leq i$, $i + 2 \leq j \leq k$, and $(i, j) \neq (1, k)$. Note that $d' \leq d$. For each $i = 1, 2, \ldots, k$, we partition P_i into segments $P_{i,1}, P_{i,2}, \ldots, P_{i,k_i}$ such that $P_{i,j}$ joins a point $p_{i,j}$ with $p_{i,j+1}$ for $j = 1, 2, \ldots, k_i - 1$ and such that each point on $P_{i,j}$ has distance less than $d'/4$ to $p_{i,j}$. Let Γ_i be the graph which is the union of the boundaries of the squares that consist of horizontal and vertical line segments of length $d'/2$ and have a point $p_{i,j}$ ($1 \leq j \leq k_i$) as the center. Then the graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ satisfy the assumptions of Lemma 2.1.10. Hence both of p and q are in the outer face of $\Gamma_1 \sqcup \ldots \sqcup \Gamma_k$ (because they are outside the disc of radius $3d$ and with center p_{i+1}, while $\Gamma_i \sqcup \Gamma_{i+1}$ is inside that disc, $i = 1, 2, \ldots, k - 1$), and P does not intersect that face. Therefore p and q can be joined by a simple polygonal arc disjoint from P. □

If C is a closed subset of the plane, and Ω is a region in $\mathbb{R}^2 \backslash C$, then a point p in C is *accessible* from Ω if for some (and hence each) point q in Ω, there is a polygonal arc from q to p having only p in common with C. If C is a simple closed curve, then p need not be accessible from Ω. However, if P is any segment of C containing p, then Theorem 2.1.11 implies that $(\mathbb{R}^2 \backslash C) \cup P$ contains a simple polygonal arc P' from q to a region of $\mathbb{R}^2 \backslash C$ distinct from Ω. Then P' intersects C in a point on P. Since P can be chosen to be arbitrarily small, we conclude that the points on C accessible from Ω are dense on C. We also get:

Proposition 2.1.12. *If C is a simple closed curve in the plane, then $\mathbb{R}^2 \backslash C$ has at most two regions.*

Proof. Assume (*reductio ad absurdum*) that q_1, q_2, q_3 are points in distinct regions $\Omega_1, \Omega_2, \Omega_3$ of $\mathbb{R}^2 \backslash C$. Let p_1, p_2, p_3 be distinct points on C and let D_i be a disc around p_i ($i = 1, 2, 3$) such that D_1, D_2, D_3 are pairwise disjoint and contain none of q_1, q_2, q_3. By the remark following Theorem 2.1.11, in Ω_i there is a simple polygonal arc $P_{i,j}$ from q_i to D_j for $i, j = 1, 2, 3$. We may assume that $P_{i,j} \cap P_{i,j'} = \{q_i\}$ for $j \neq j'$, and $P_{i,j} \cap P_{i',j'} = \emptyset$ when $i \neq i'$. We can now extend (by adding three segments of C) the union of the curves $P_{i,j}$ ($i, j = 1, 2, 3$) to a plane graph with vertex set $\{q_1, q_2, q_3, p_1, p_2, p_3\}$ isomorphic to $K_{3,3}$. This contradicts Corollary 2.1.7. □

Propositions 2.1.8 and 2.1.12 constitute what is usually called the *Jordan Curve Theorem*.

The Jordan Curve Theorem. *Any simple closed curve C in the plane divides the plane into exactly two arcwise connected components. Both of these regions have C as the boundary.*

The Jordan Curve Theorem is named after Camille Jordan. Apparently, the first correct proof was given by Veblen in 1905 [**Ve05**]. This result is a special case of the Jordan-Schönflies Theorem which we prove in the next section.

Discrete versions of the Jordan Curve Theorem have been considered by Little [**Li88**], Stahl [**St83**], and Vince [**Vi89**].

2.2. The Jordan-Schönflies Theorem

Now that we have proved the Jordan Curve Theorem we can extend some of the previous results. For example, Corollary 2.1.4 easily extends to the case where C is any simple closed curve and P is a simple arc in $\overline{\text{int}}(C)$ such that only its ends lie on C. Also Lemma 2.1.9 remains valid if Γ_1 and Γ_2 are plane graphs consisting of a simple closed curve C (which is the outer cycle in both Γ_1 and Γ_2) and polygonal curves in $\overline{\text{int}}(C)$. (Lemma 2.1.9 would not be valid if Γ_1 and Γ_2 had distinct outer cycles, or if the interior edges are not polygonal arcs.)

If C and C' are simple closed curves, and Γ and Γ' are 2-connected plane graphs whose exterior faces are bounded by C and C', respectively, then Γ and Γ' are said to be *plane-isomorphic* if there is an isomorphism γ of Γ to Γ' which maps C onto C' such that a cycle in Γ bounds a face of Γ if and only if the image of the cycle is a face boundary in Γ'. The isomorphism γ is said to be a *plane-isomorphism* of Γ and Γ'.

The Jordan-Schönflies Theorem. *If f is a homeomorphism of a simple closed curve C in the plane onto a closed curve C' in the plane, then f can be extended to a homeomorphism of the entire plane.*

PROOF (from [**Th92a**]). Without loss of generality we may assume that C' is a convex polygon. We shall first extend f to a homeomorphism of $\overline{\text{int}}(C)$ to $\overline{\text{int}}(C')$. Let B denote a countable dense set in $\text{int}(C)$ (for example the points with rational coordinates). As mentioned before Proposition 2.1.12, the points on C accessible from $\text{int}(C)$ are dense on C. Therefore, there exists a countable set $A \subseteq C$ which is dense in C consisting of points accessible from $\text{int}(C)$. Let $p_1, p_2, \ldots$ be a sequence of points in $A \cup B$ such that each point in $A \cup B$ occurs infinitely often in this sequence. Let Γ_0 denote any 2-connected graph consisting of C and some simple polygonal curves in $\overline{\text{int}}(C)$. Let Γ_0' be a graph consisting of C' and simple polygonal curves in $\overline{\text{int}}(C')$ such that Γ_0 and Γ_0' are plane-isomorphic (with isomorphism g_0) such that g_0 and f coincide on C. We now extend f to $C \cup V(\Gamma_0)$ such that g_0 and f coincide on $V(\Gamma_0)$. We shall define a sequence of 2-connected graphs $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ and $\Gamma_0', \Gamma_1', \Gamma_2', \ldots$

such that, for each $n \geq 1$, Γ_n is an extension of a subdivision of Γ_{n-1}, Γ'_n is an extension of a subdivision of Γ'_{n-1}, there is a plane-isomorphism g_n of Γ_n onto Γ'_n which coincides with g_{n-1} on $V(\Gamma_{n-1})$, and Γ_n (respectively Γ'_n) consists of C (respectively C') and simple polygonal curves in $\overline{\text{int}}(C)$ (respectively $\overline{\text{int}}(C')$). Also, we shall assume that $\Gamma'_n \backslash C'$ is connected for each n. We then extend f such that f and g_n coincide on $V(\Gamma_n)$.

Suppose that we have already defined $\Gamma_0, \ldots, \Gamma_{n-1}, \Gamma'_0, \ldots, \Gamma'_{n-1}$, and $g_0, \ldots, g_{n-1}$. We shall define Γ_n, Γ'_n and g_n as follows. We consider the point p_n. If $p_n \in A$, then we let P be a simple polygonal curve from p_n to a point q_n of $\Gamma_{n-1} \backslash C$ such that $\Gamma_{n-1} \cap P = \{p_n, q_n\}$. We let Γ_n denote the graph $\Gamma_{n-1} \cup P$. The arc P is drawn in a face of Γ_{n-1}. By Proposition 2.1.5, this face is bounded by a cycle S, say. We add to Γ'_{n-1} a simple polygonal curve P' in the face bounded by $g_{n-1}(S)$ such that P' joins $f(p_n)$ with $g_{n-1}(q_n)$ (if q_n is a vertex of Γ_{n-1}) or a point on $g_{n-1}(a)$ (if a is an edge of Γ_{n-1} containing the point q_n). Then we put $\Gamma'_n = \Gamma'_{n-1} \cup P'$ and we define the plane-isomorphism g_n from Γ_n to Γ'_n in the obvious way. We extend f to $C \cup V(\Gamma_n)$ such that $f(q_n) = g_n(q_n)$.

If $p_n \in B$, we consider the largest square with vertical and horizontal sides, which has p_n as the center and which is in $\overline{\text{int}}(C)$. Inside this square (whose sides we are not going to add to Γ_{n-1} as they may contain infinitely many points of C) we draw a new square with vertical and horizontal sides each of which has distance $< 1/n$ from the sides of the first square. Inside the new square we draw vertical and horizontal lines such that p_n is on both a vertical line and a horizontal line and such that all regions in the square have diameter $< 1/n$. We let H_n be the union of Γ_{n-1} and the new horizontal and vertical straight line segments possibly together with an additional polygonal curve in $\text{int}(C)$ in order to make H_n 2-connected and $H_n \backslash C$ connected. By Proposition 1.4.2, H_n can be obtained from Γ_{n-1} by successively adding paths in faces. We add the corresponding paths to Γ'_{n-1} and obtain a graph H'_n which is plane-isomorphic to H_n. Then we add vertical and horizontal line segments in $\overline{\text{int}}(C')$ to H'_n such that the resulting graph has no (bounded) region of diameter $\geq 1/2n$. If necessary, we displace some of the lines a little such that they intersect C' only in $f(A)$ and such that all bounded regions have diameter $< 1/n$ and such that each of the new lines has only finite intersection with H'_n. This extends H'_n into a graph that we denote by Γ'_n. We add to H_n polygonal curves such that we obtain a graph Γ_n plane-isomorphic to Γ'_n. Then we extend f such that it is defined on $C \cup V(\Gamma_n)$ and coincides with the plane-isomorphism g_n on $V(\Gamma_n)$.

When we extend H'_n into Γ'_n and H_n into Γ_n, we are adding many edges and it is perhaps difficult to visualize what is going on. However, Proposition 1.4.2 tells us that we can look at the extension of H'_n into Γ'_n as the result of a sequence of path additions (each of which is a straight line segment in a face). We then just perform successively the corresponding additions in H_n. Note that we have plenty of freedom for that since the current mapping f is only defined on the current vertex set. The images of the points on the current edges have not been specified yet. In this way we extend f to a 1-1 map defined on $F = C \cup V(\Gamma_0) \cup V(\Gamma_1) \cup \cdots$ whose image is the set $C' \cup V(\Gamma'_0) \cup V(\Gamma'_1) \cup \cdots$. These sets are dense in $\overline{\mathrm{int}}(C)$ and $\overline{\mathrm{int}}(C')$, respectively.

If p is a point in $\mathrm{int}(C)$ on which f is not yet defined, then we consider a sequence $q_1, q_2, \ldots$ converging to p and consisting of points from $V(\Gamma_0) \cup V(\Gamma_1) \cup \cdots$. We shall show that $f(q_1), f(q_2), \ldots$ converges and we let $f(p)$ be the limit. Let d be the distance from p to C, and let p_n be a point of B at distance $< d/3$ from p. Then p is inside the largest square in $\overline{\mathrm{int}}(C)$ having p_n as the center (and also inside what we called the new square if n is sufficiently large). By the construction of Γ_n and Γ'_n it follows that Γ_n has a cycle S such that $p \in \mathrm{int}(S)$ and such that both S and $g_n(S)$ are in discs of radius $< 1/n$. Since f maps $F \cap \mathrm{int}(S)$ into $\mathrm{int}(g_n(S))$ and $F \cap \overline{\mathrm{ext}}(S)$ into $\overline{\mathrm{ext}}(g_n(S))$, it follows in particular, that the sequence $f(q_m), f(q_{m+1}), \ldots$ is in $\mathrm{int}(g_n(S))$ for some m. Since n can be chosen arbitrarily large, $f(q_1), f(q_2), \ldots$ is a Cauchy sequence and hence convergent. It follows that f is well-defined. Moreover, using the above notation, f maps $\mathrm{int}(S)$ into $\mathrm{int}(g_n(S))$. Hence f is continuous in $\mathrm{int}(C)$. Since $V(\Gamma'_0) \cup V(\Gamma'_1) \cup \cdots$ is dense in $\mathrm{int}(C')$, the same argument shows that f maps $\mathrm{int}(C)$ onto $\mathrm{int}(C')$ and that f is 1-1 and that f^{-1} is continuous on $\mathrm{int}(C')$.

It only remains to be shown that f is continuous on C. (Then also f^{-1} is continuous since $\overline{\mathrm{int}}(C)$ is compact.) In order to prove this, it is sufficient to consider a sequence $q_1, q_2, \ldots$ of points in $\mathrm{int}(C)$ converging to q on C and then show that $f(q_1), f(q_2), \ldots$ converges to $f(q)$. Suppose therefore that this is not the case. Since $\overline{\mathrm{int}}(C')$ is compact, we may assume (by considering an appropriate subsequence, if necessary) that $\lim_{n\to\infty} f(q_n) = q' \neq f(q)$. Since f^{-1} is continuous on $\mathrm{int}(C')$, q' is on C'. Since A is dense in C, $f(A)$ is dense in C' and hence each of the two curves on C' from q' to $f(q)$ contain a point $f(q_1)$ and $f(q_2)$, respectively, in $f(A)$. For some n, Γ_n has a path P from q_1 to q_2 having only q_1 and q_2 in common with C. As we have noted at the beginning of this section, P separates $\mathrm{int}(C)$ in two regions. These two regions are mapped on the two distinct regions of $\mathrm{int}(C') \backslash g_n(P)$. Hence we cannot have $\lim_{n\to\infty} f(q_n) = q'$. This contradiction shows that f has the appropriate extension to $\mathrm{int}(C)$.

By similar arguments, f can be extended to $\mathrm{ext}(C)$: Without loss of generality we may assume that $\mathrm{int}(C)$ contains the origin and that both C and C' are in the interior of the quadrangle T with corners $(\pm 1, \pm 1)$. Let L_1, L_2 be the line segments (on lines through the origin) from $(1,1)$ and $(-1,-1)$, respectively, to C. Let p_i be the end of L_i on C, $i = 1, 2$. Let L'_1, L'_2 be simple polygonal arcs from $f(p_1)$ to $(1,1)$ and from $f(p_2)$

to $(-1,-1)$, respectively, such that $L'_1 \cap L'_2 = \emptyset$ and these arcs have only their ends in common with $C \cup T$. It is easy to see that we can extend f to a homeomorphism $C \cup L_1 \cup L_2 \cup T \to C' \cup L'_1 \cup L'_2 \cup T$ so that f is the identity on T. Now we can use the method of the first part of the proof to extend f to a homeomorphism of $\overline{\text{int}}(T)$ (onto itself). Since f is the identity on T, it can be extended to the entire plane such that it is the identity on $\text{ext}(T)$. This determines a required homeomorphism. □

If F is a closed set in the plane, then we say that a point p in F is *curve-accessible* if, for each point $q \notin F$, there is a simple arc from q to p having only p in common with F. The Jordan-Schönflies Theorem implies that every point on a simple closed curve is curve-accessible. Hence, the following is an extension of Proposition 2.1.12.

THEOREM 2.2.1 (Thomassen [**Th92a**]). *If F is a closed set in the plane with at least three curve-accessible points, then $\mathbb{R}^2 \backslash F$ has at most two regions.*

PROOF. If p_1, p_2, p_3 are curve-accessible points in F and q_1, q_2, q_3 belong to distinct regions of $\mathbb{R}^2 \backslash F$, then we get, as in the proof of Proposition 2.1.12, a plane graph isomorphic to $K_{3,3}$ with vertices p_1, p_2, p_3, q_1, q_2, q_3, a contradiction to Corollary 2.1.7. □

In Theorem 2.2.1, "three" cannot be replaced by "two". To see this, we let F be a collection of three or more internally disjoint simple arcs between two fixed points.

Some other consequences of the Jordan–Schönflies Theorem are presented below. First we generalize Corollary 2.1.4.

PROPOSITION 2.2.2. *Let P_1, P_2, P_3 be simple arcs with ends p, q such that $P_i \cap P_j = \{p, q\}$ for $1 \le i < j \le 3$. Then $P_1 \cup P_2 \cup P_3$ has precisely three faces with boundaries $P_1 \cup P_2$, $P_1 \cup P_3$ and $P_2 \cup P_3$, respectively. If the outer face is bounded by $P_1 \cup P_2$, and P'_1, P'_2, P'_3 are simple arcs joining p', q' such that $P'_i \cap P'_j = \{p', q'\}$ for $1 \le i < j \le 3$, and such that $P'_3 \subseteq \overline{\text{int}}(P'_1 \cup P'_2)$, then any homeomorphism f of $P_1 \cup P_2 \cup P_3$ onto $P'_1 \cup P'_2 \cup P'_3$ such that $f(P_i) = P'_i$ $(i = 1, 2, 3)$ can be extended to a homeomorphism of $\mathbb{R}^2$ onto itself.*

PROOF. If $P_1 \subseteq \overline{\text{ext}}(P_2 \cup P_3)$, $P_2 \subseteq \overline{\text{ext}}(P_1 \cup P_3)$, and $P_3 \subseteq \overline{\text{ext}}(P_1 \cup P_2)$, then it is easy to extend $P_1 \cup P_2 \cup P_3$ to a $K_{3,3}$ in the plane, a contradiction. So we may assume that $P_3 \subseteq \overline{\text{int}}(P_1 \cup P_2)$. The first part of the proposition follows easily. To prove the last part, it is sufficient to consider the case where P'_1, P'_2, P'_3 are polygonal arcs. This case is done by using the Jordan-Schönflies Theorem to $\overline{\text{int}}(P_1 \cup P_3)$, $\overline{\text{int}}(P_2 \cup P_3)$, and $\overline{\text{ext}}(P_1 \cup P_2)$, respectively. □

Now we generalize Proposition 2.1.5.