ADAPTIVE CARDINAL UTILITY[1]

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ABSTRACT

Recent research in neuroscience inspires a foundation for a cardinal utility function that is adaptive and perhaps hedonic. We model adaptation as arising from a limited capacity to make fine distinctions, where the utility functions adapt in real time. We give the noisy nature of choice a central role. The mechanism adapts to an arbitrary distribution in a way that is approximately optimal, in terms of minimizing the probability of error. The model predicts the so-called "hedonic treadmill" and sheds light on national happiness measures.

Keywords: Utility, cardinal, adaptation, choice, neuroscience, bounded rationality. JEL Codes: A12, D11.

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1. Introduction

Jeremy Bentham is famous for, among other things, the dictum "the greatest happiness for the greatest number". The happiness that Bentham described was hedonic—derived from pleasure, that is—cardinal, and capable of being summed across individuals to obtain a basic welfare criterion. Conventional welfare economics remains needful of some degree of cardinality. However, in the context of individual decisions, of consumer theory, in particular, economics has completely repudiated any need for cardinality, on the basis of "Occam's Razor," a theme that culminates in the theory of revealed preference.

What does modern neuroscience establish about the hedonic or cardinal nature of utility, as used in economic decisions? What is relevant are three aspects of an economic choice—utility in the revealed preference behavioral sense; neural firing rates in the brain that capture the attractiveness of each option as it is considered as a potential selection; and the hedonic neural reward of actually obtaining an option. Although neuroscience has made significant progress, which is briefly outlined in the next subsection, there remain gaps in the account that rigorously and precisely ties revealed preference to hedonic reward.

There are nevertheless fundamental well-established properties of all neural systems that are relevant to economic decisions. In the first place, neural systems are generally adaptive, and can react rapidly to changes in the environment. Adaptation is a characteristic of any form of perception. For example, if the variance of visual stimuli increases, the sensitivity of relevant visual neurons are reduced.

Two additional interrelated properties of neural systems are imprecision of perception and noisiness. If two sounds are close enough in loudness, for example, an individual may be unable to perceive which is louder.[2] Imprecision provides a clear rationale for adaptation. That is, if the ability to make sharp distinctions is limited, the ability should be shifted to where it is most likely to be needed.

The noisiness of neural systems is pervasive. The firing of a particular neuron is random, but stochasticity does not vanish in larger systems. Choices that are finally made by animals or humans are noisy. Given a particular precisely specified binary choice, an individual will sometimes choose one thing, sometimes another.

We build a model of adaptive economic choice based on these key features of neural systems. The model readily accounts for a number of behavioral anomalies. For example, it accounts directly for a preference for rising reward schedules. Although the model has predictive power, it is not intended in a normative sense. The adaptation that is induced makes the normative interpretation of happiness surveys difficult, for example.

1.1. **Psychology and Neuroscience Background.** A useful framework for considering the background literature involves distinguishing between 'liking', 'wanting' and learning. (See Berridge, Robinson, and Aldredge, 2009, for example.) In the first place, liking refers to the hedonic experience of final consumption. Secondly, wanting refers to the processes that incentivize an individual to seek such consumption experiences. Although liking and wanting clearly ought to be closely related in a well-functioning organism, they can be shown to be independent—it is possible, that is, to want something one does not like, or to like something one does not want. The third relevant process is learning—that is, how an individual can come to predict outcomes. Again, although learning and wanting ought to be closely related, they are dissociable. From the point of view of modelling choice, liking and wanting are the central phenomena.

[2]The study of the limits to perceptual precision is a major concern of psychology. Foley and Matlin (2009) is a textbook treatment.

Two papers by neuroscientists that are apparently motivated by economic theory are Lak, Stauffer, and Schultz (2014) and Stauffer, Lak and Schultz (2014). These focus on dopamine neuron firing rates, and fall directly under the rubric of learning. That is, a well-established empirical regularity is that dopamine neurons fire in response to "reward prediction error"—unexpected reward, or one that is larger than expected. This is not directly the anticipated experience of consuming each option in a choice situation, or the experience of actual consumption. Nevertheless, the fit with economic theory is notable. That is, dopamine neuron firing rates, as induced by reward prediction error, and relevant in learning, directly reflect the economic utility of a gamble, and are predictive of stochastic choice. Stauffer, Lak and Schultz (2014) show dopamine activity reflects the behaviorally established von Neumann Morgenstern utility of a gamble. Lak, Stauffer, and Schultz (2014) show that this analysis can be extended to multi-attribute bundles.

Similarly, Glimcher (2004) is a systematic account of neuroeconomics by a pioneer, demonstrating how economics can motivate neuroscience. For example, he describes evidence that neurons in the "lateral intraparietal" (LIP) area of the brain encode prior probabilities, posterior probabilities, and value, as are needed to generate expected utility. Finally, there is a large literature on neural adaptation, in an economic choice context, in particular. For example, Louie, Glimcher, and Webb (2015) is one of several papers arguing for a a particular form of adaptation—divisive normalization. A simple ratio formula resolves various apparent choice anomalies, such as a dependence on irrelevant alternatives.

1.2. **Sketch of the Paper.** We first consider a stylized model of how a neuronal network makes a binary economic choice. The options have been drawn iid according to a cdf F, say, and each is processed by such a network. The mapping from actual value to the value used in the choice has two components. The first component is a deterministic step function which generates adaptation by means of endogenously setting the location of the steps, the thresholds. The second component is noise, so the model is then essentially one of quantized random utility.

Our key new contribution is to model rapid and automatic adaptation, and to discuss how such adaptation is reflected in economic choices. Such automatic adaptation might also characterize neural processes for perception, for example. When an outcome arrives between two thresholds, the thresholds move closer together, each by a given increment. An irreducible Markov chain with a unique invariant distribution describes the dynamics of the thresholds. As the increment is made smaller, the invariant distribution puts full weight on the thresholds being at the quantiles of the distribution F. Thus, the thresholds adapt to the distribution. Crucially, such adaptation is approximately optimal.

We next derive some formal properties of the model in Section 3. We show it generates an observable cardinal utility function that reflects F, under the probability of error criterion. The cardinality derives from the probabilities arising in noisy choice. This utility can be empirically identified from noisy choice data, as can the noise, so adaptation has observable choice consequences. It would be of substantial interest to investigate how the cardinal utility here relates to neural firing rates—both those reflecting the contemplation of an option and those deriving from its consumption.

A difficulty in identifying cardinal utility from noisy choice is the endogeneity of utility—set at F in the simplest case. That is, if the sequence of observations from F were not typical, utility would shift away from F. Indeed, there is a fundamental trade-off between speed of adjustment and the precision of adjustment.

Our model accounts immediately for the "hedonic treadmill"—that is, the reversion of average utility to its original level despite a vast shift in the distribution of rewards. The adaptation of utility, as reflected in the hedonic treadmill, suggests a reconsideration of the use of national happiness measures, as has been recently advocated. The discussion suggests the importance of distinguishing between positive and normative views. A positive interpretation is all that is needed here and this interpretation already raises issues. A normative interpretation of a criterion that is adaptive raises still further issues. For example, doubling all rewards leads to complete hedonic reversion, which seems questionable as a normative property. It is noted, however, how this depends on the probability of error criterion used here. Under the expected fitness criterion, average utility may depend on the distribution F. This may offer positive insights into surveys of national happiness, but is hardly a firm normative basis for them.

1.3. **A Few Related Papers in Economics.** There are a handful of related papers in economics. Robson (2001) provides the basis on which the current paper is constructed. There are thresholds, but no noise. In order to minimize the probability of error, the thresholds should be at the quantiles of the distribution F, but there is no process for ensuring this allocation.

Netzer (2009) investigates the Robson model using the expected fitness criterion instead of the probability of error. He shows that, whereas the Robson approach generates a limiting density of thresholds given by f, where f is the pdf associated with F, the expected fitness approach generates a limiting density of thresholds proportional to $f^{2/3}$. This again puts fewer thresholds where f is low, but to a less dramatic extent. Intuitively, the expected fitness criterion is more concerned with low f than is the probability of error criterion, since, although the probability of error is small if f is small, the size of the error is large.

Rayo and Becker (2007) address the issue of adaptation using an alternative model of bounded rationality. Individuals cannot maximize expected utility precisely, but all choices that come within some band are considered equivalent. The problem is to construct the optimal utility function. Under simplifying and limiting assumptions, optimal utility is a step function, jumping from 0 to 1 at maximized expected income, so concentrating all incentives at the point of greatest interest, and adapting to the distribution.

Woodford (2012) adopts an approach based on informational transfer and rational inattention. Rather than the constraint being the number of thresholds, the constraint is the informational transmission capacity of the channel. Adaptation arises to use the available resources to make the most accurate distinctions possible and the key predictions arise from this adaptation.

All of these previous papers describe how adaptation would be advantageous and make predictions presuming this has occurred; the contribution of the present paper is to show *how*—to provide an explicit real-time low-rationality adjustment mechanism.

2. The Model

Consider the following abstract view of how a decision is orchestrated in the brain. We will concentrate, for simplicity, on the case of two options. Each option provides a stimulus $y \in [0,1]$. This is interpreted as a sensory cue—a visual one, for example. This stimulus is processed by a neural network in the brain, generating firing of decision neurons given by $z = \tilde{h}(y) \in [0,1]$. These neurons anticipate the hedonic consequences of potential consumption of each of the options. This function $\tilde{h}$ involves noise, and is necessarily inaccurate. The inaccuracy reflects a limited ability to make fine perceptual distinctions. This, in turn, produces a benefit from adaptation.

Suppose, in particular, that $\tilde{h}(y) = h(y) + \tilde{d}\delta$. The function h : $[0,1] \to \{0, \delta, 2\delta, 3\delta, ..., N\delta = 1\}$, is a non-decreasing step function characterized by thresholds x_n, at which a jump is made from one level $(n-1)\delta$ to the next higher level $n\delta$, for $n = 1, ..., N$. We have $0 \leq x_1 \leq .... \leq x_N \leq 1$ where we set $x_0 = 0$ and $x_{N+1} = 1$. Adaptation is captured within the deterministic component function h by means of shifts in the thresholds $x_n, n = 1, ..., N$.

The random variable $\tilde{d}$ that represents noise has a symmetric distribution on $\{-D, ..., D\}$. If $n \in \{D, ...N-D\}$, away from the ends, that is, then $\tilde{d} = 0$ with probability π_0, and $\tilde{d} = d$ or $-d$ with probability π_d, for $d = 1, ..., D$. We assume that $2D < N$, so that such thresholds exist. Of course, $\pi_0 + 2\sum_1^D \pi_d = 1$. In addition, it assumed that these probabilities form a decreasing convex sequence so that

$$\pi_0 - \pi_1 > \pi_1 - \pi_2 > .... > \pi_{D-1} - \pi_D > \pi_D > 0. \tag{2.1}$$

These noise probabilities need to be specified differently near the ends, if $n < D$ or if $n > N - D$, that is, since otherwise the distribution specified above would go beyond the bounds. The probabilities of outcomes of $\tilde{d}$ that would put $\tilde{h}(y)$ beyond the ends are re-assigned to the originating outcome.[3] That is, if $n < D$—

$$Pr\{\tilde{d} = m\} = \pi_m, m = 1, ..., D$$

$$Pr\{\tilde{d} = -m\} = \pi_m, m = 1, ..., n$$

$$Pr\{\tilde{d} = 0\} = \pi_0 + \pi_{n+1} + ... + \pi_D.$$

An analogous treatment applies if $n > N - D$—

$$Pr\{\tilde{d} = -m\} = \pi_m, m = 1, ..., D$$

$$Pr\{\tilde{d} = m\} = \pi_m, m = 1, ..., N-n$$

$$Pr\{\tilde{d} = 0\} = \pi_0 + \pi_{N-n+1} + ... + \pi_D.$$

This issue might be finessed by allowing the function $\tilde{h}$ to go outside $[0, 1]$. However, the bounded range of neural activity, which is assumed to limit the range of h should apply equally to limit the range of $\tilde{h}$.

[3]This treatment has the desirable technical feature that the matrix governing the evolution of an approximating differential equation system is symmetric and negative definite.

There are two options, y_1 and y_2, say, generating neural activity in decision neurons $\tilde{h}(y^i), i = 1,2$. If $\tilde{h}(y^i) > \tilde{h}(y^j)$, i is chosen, as is clearly optimal. That is, the probability that $y^i > y^j$ is then necessarily greater than $1/2$. If $\tilde{h}(y^1) = \tilde{h}(y^2)$, each option is chosen with probability $1/2$.

It is without much loss of generality to suppose that y represents fitness. That is, if y instead represents money, for example, which generates fitness according to an increasing concave function, only minor notational changes need to be made. The $y^i, i = 1,2$, are assumed to be independent, distributed according to the same continuous cumulative distribution function, F, with probability density function, f. Although the realizations are random *ex ante*, they are realized prior to choice. The distribution F nevertheless plays an important role because the thresholds are assumed to be set in the light only of F rather than the realizations.

The simplest special case of this model is when there is no noise, so that $\pi_0 = 1$ (as in Robson, 2001). Now errors arise only when both $y_i, i = 1,2$ lie in the same interval $[x_n, x_{n+1})$. Minimizing the probability of error implies that that the thresholds should be equally spaced in terms of probabilities; should be then at the quantiles of the distribution.[4] The thresholds should adapt to the distribution F. The present approach, involving explicit noise, smooths over the just noticeable differences. The thresholds are retained purely as a

[4]An attractive alternative criterion is maximization of expected fitness or, equivalently, minimization of the expected fitness loss. It can be shown, however, that the analysis is more complex in this case. Each threshold now ought to be at the *mean* of the distribution conditional on being between the two neighboring thresholds. It is then not possible to use purely qualitative observations to estimate the mean, as was true for the median. Nevertheless, approximately optimal rules can be derived that use the distances to the two neighboring thresholds. Analyzing the expected fitness criterion in the presence of noise is cumbersome so the present paper plumps for investigating the implications of noise under the simpler probability of error criterion.

technical device to render the adjustment process tractable, since it involves adjusting a finite number of parameters.[5]

In this model with noise, it remains true that the thresholds should be denser where f is higher. Concentrating the the thresholds like this concentrates the distribution of each noisy signal $\tilde{h}(y_i)$ around the true value $h(y^i)$ and so reduces the probability of error. In the limit when the number of thresholds tends to infinity, and the noise distribution is scaled in proportion, the adjustment process becomes slow. However, the limit yields basically a standard continuous model of noisy utility, with the important novelty that utility is endogenous.

The key contribution here is then to address the questions: How could the thresholds here adjust to a novel distribution, F? What implications are there of the adaptation process? We now formulate and address these questions.

In modelling how the thresholds adjust, we make the the simplifying assumption that there is a single stream of outcomes, represented as y, and abstract from the choices made. Alternatively, we could interpret the analysis as supposing that y^1 and y^2 arrive in alternate periods, with the system adapting to each of them, and a choice between them being made in every even period.

Suppose then, for simplicity, are confined to a finite grid $\mathscr{G} = \{0, \epsilon, 2\epsilon, ..., (G-1)\epsilon, 1\}$, for an integer G such that $G\epsilon = 1$. The thresholds are time dependent, given as $x_n^t \in \mathscr{G}$, where $0 \leq x_1^t \leq .... \leq x_N^t \leq 1$, at time $t = 1, 2, ...$ Consider the rule of thumb for adjusting the thresholds—

$$(2.2) \quad x_n^{t+1} = \begin{cases} x_n^t + \epsilon \text{ with probability } \xi \text{ if } h(y) + \tilde{d}\delta = n\delta \\ x_n^t - \epsilon \text{ with probability } \xi \text{ if } h(y) + \tilde{d}\delta = (n-1)\delta \\ x_n^t \text{ otherwise} \end{cases},$$

[5]The threshold model without noise has the conceptual drawback that, despite being a model of imperfect discrimination, the edges of intervals are sharply discerned. The model with explicit noise suffers less from the criticism in the sense that errors depend largely on the noise.

for $n = 1, ..., N$ where $\xi \in (0, 1)$.

3. Results

3.1. **Convergence.** For general N, we have

Theorem 3.1. *Consider the model with or without noise, so that $D \geq 0$. In the limit as $\epsilon \to 0$, the invariant joint distribution of the thresholds x_n^t converges to a point mass at the vector x_n^*, where $F(x_n^*) = \frac{n}{N+1}, n = 1, ..., N$.*[6]

Proof. See Appendix.

The basic property that must hold in the limit as $\epsilon \to 0$ is that all signals $\tilde{h}(y) = h(y) + \tilde{r}\delta$ are equally likely—each threshold must be equally likely to move left or right, in the limit, under the rule of thumb. The stated result that $F(x_n^*) = n/(N+1)$, for $n = 1, ..., N$, then follows, given the treatment of the noise near the ends.

The basic property implies that the placement of the thresholds maximizes the rate of Shannon information transfer, neglecting the differential meaning of signals near the extremes, as in Laughlin (1981). This argument, which is for a single channel and an abstract criterion, contrasts with that for the present binary choice and minimization of the probability of error.

The property that $F(x_n^*) = n/(N+1)$, for $n = 1, ..., N$ is that utility, U, say, adapts to the distribution, F. Indeed, in the limit as $N \to \infty$, $U = F$. If F has a typical unimodal shape, utility has an S-shape similar to that in Kahneman-Tversky. This suggests an application to choice under uncertainty.[7]

Intuition for Proof.

Step 1. It can be shown that an ordinary differential equation system (ODE) approximates the Markov chain, when ϵ is small. This

[6] For any particular $\epsilon > 0$, the invariant distribution has full support; only in the limit does this invariant distribution converge to the x_n^*.

[7] See Netzer (2009), for example.

approximation is obtained by letting $\epsilon \to 0$ over a fixed interval of time, but also offsetting this by increasing the number of repetitions. This compensating increase in the rate of repetition does not affect the invariant distribution, of course. The differential equation system is obtained by shrinking the small interval of time. A complication is that the increments in the position of the thresholds are not independent. Nevertheless, each increment has a bounded effect on subsequent increments and the full effect of the strong law of large numbers can be shown to still apply.

Step 2. A key step is then to show the ODE system is globally asymptotically stable. This follows from the application of a Lyapunov function. The system converges to the desired vector $x_n^*, n = 1, ..., N$, as $t \to \infty$, from any initial vector. This follows because the matrix governing the $F(x_n)$ is symmetric, Metzlerian and irreducible. Further it has a diagonal that is always weakly dominant and sometimes is strictly dominant. It follows that the matrix is negative definite, applying results from McKenzie (1959).

Step 3. Suppose, by way of establishing a contradiction, that some limit of the invariant distributions of the Markov chain does not put full weight on x_n^*, $n = 1, ..., N$. The ODE system must lower the expected value of the Lyapunov function, using this limit of the invariant distributions. Hence the Markov chain also lowers the expected value of the Lyapunov function, using the invariant distribution of that Markov chain as the initial distribution, whenever ϵ is small enough. This yields the desired contradiction.

The rule of thumb illustrates that there exist low rationality mechanisms that can generate fast complete adaptation to an arbitrary distribution. This process is intended to reflect the automatic process that might take place within neurons. The thresholds are a device that renders the analysis of adaptation more tractable. They still allow the step function h to approximate an arbitrary increasing continuous function.

If the distribution is known to come from a parametric class, then the rule of thumb described above will be slower than the full Bayesian approach. If the cdf is known to normal, for example, but with an unknown mean and variance, then all thresholds should shift in response to an observation, to reflect the updated mean and variance.

The rule of thumb is fully non-parametric, capable of adaptation to an arbitrary cdf. It is, loosely speaking, completely agnostic about the implications of a local observation for the estimated distribution far away. Consider the model with no noise, for simplicity, where there are three pdf's $f_i, i = 1, 2, 3$, each arising with probability $1/3$. f_1 is 2 on $[0, 1/3)$ and $1/2$ on $[1/3, 1]$. f_2 is $1/2$ on $[0, 1/3)$ and on $(2/3, 1]$ but 2 on $[1/3, 2/3]$. f_3 is 2 on $(2/3, 1]$ and $1/2$ on $[0, 2/3]$. By symmetry, the probability of each of the three intervals is $1/3$, so $x_1 = 1/3$ and $x_2 = 2/3$ is optimal. How would an observation in $[0, 1/3)$ change the estimated distribution? The posterior probability of f_1 is now $2/3$ and that of f_2 and f_3 are now each $1/6$. It follows that x_1 should move to the left, to move towards the median of the distribution conditional on being between 0 and $2/3$, but x_2 is still at the median between $1/3$ and 1, because f_2 and f_3 remain equally likely.

If the number of thresholds N were increased, it would be compelling to move not just the thresholds that were immediate neighbors of an observation, but also those somewhat further away, presumably with a declining probability. If this were not done, adaptation would be slowed to a standstill with increasing N. Once this is done, the limiting continuous model could also incorporate adaptation.

3.2. **Approximate Optimality.** We focus here on the the criterion of the overall probability of error, $\bar{P}$, say. Errors arises if $y^i < y^j$, so y^j should be chosen, but $\tilde{h}(y^i) > \tilde{h}(y^j)$, so y^i is actually chosen. If, that is, $\tilde{h}(y^i) = n_i\delta + \tilde{d}^i\delta$, then an error arises if $\tilde{d}^i - \tilde{d}^j > n^j - n^i > 0$, that is, if the noise favors the worse option sufficiently. There is also an error with probability $1/2$ whenever $\tilde{h}(y^i) = \tilde{h}(y^j)$. The criterion $\bar{P}$

is then the expectation of all such errors, making due allowance for the treatment of noise near the ends.

If $D = 0$, it is immediate that the equilibrium is exactly optimal, for all N, in the sense of minimizing $\bar{P}$. In this case, the only errors have probability $1/2$ and arise whenever $h(y^i) = h(y^j)$—that is, if $n^i = n^j$. See Robson (2001).[8]

If $D > 0$, however, the end effects imply that the equal spacing of the thresholds in probability cannot be exactly optimal, as is shown by the following—

Example 3.1. Suppose $N = 2$ and $D = 1$. This example shows how the treatment at the ends ensures that the thresholds lie at the quantiles and, at the same time, that such equality cannot be optimal. More generally, this example illustrates the arguments (in the Appendix) concerning the equilibrium, as in Theorem 3.1, and concerning optimality, as in Theorem 3.2 below. Define $\Delta_n = F(x_{n+1}) - F(x_n) \geq 0, n = 0, 1, 2$, where $\sum_{n=0}^{2} \Delta_n = 1$. Consider first the equilibrium. It follows that

$$\Pr\{\tilde{h}(y) = 0\} = (\pi_0 + \pi_1)\Delta_0 + \pi_1 \Delta_1.$$

To see this, note that, if $h(y) = 0$, which has probability Δ_0, there is a probability of $(\pi_0 + \pi_1)$ that $\tilde{h}(y) = 0$, given how the noise distribution is modified at the ends. If $h(y) = 1$, on the other hand, which has probability Δ_1, there is probability π_1 that $\tilde{h}(y) = 0$.

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[8] It is of interest to reconsider the model when the draws are not independent. Suppose, more generally, that the two draws have a cdf $F(y^1, y^2)$, with pdf $f(y^1, y^2)$, say, which are symmetric in y^1, y^2, for simplicity. Consider the case with one threshold at x, say. The probability of error is then

$$\bar{P} = (1/2) \iint_{y^1, y^2 \leq x} f(y^1, y^2) dy^1 dy^2 + (1/2) \iint_{y^1, y^2 \geq x} f(y^1, y^2) dy^1 dy^2.$$

Differentiating with respect to x and using symmetry yields the first-order condition for minimizing $\bar{P}$—

$$\int_{y \leq x} f(x, y) dy = \int_{y \geq x} f(x, y) dy.$$

That is, the optimal x is such that the probability of y^2 exceeding x conditional on $y^1 = x$ has to equal the probability of y^2 falling short of x conditional on $y^1 = x$. The need to condition on each x makes this problem more demanding in terms of data. Since the algorithm faces a harder problem, convergence will be slower. I thank Jakub Steiner for this observation.

It follows similarly that

$$\Pr\{\tilde{h}(y) = 1\} = \pi_0\Delta_1 + \pi_1\Delta_0 + \pi_1\Delta_2,$$

and that

$$\Pr\{\tilde{h}(y) = 2\} = (\pi_0 + \pi_1)\Delta_2 + \pi_1\Delta_1.$$

The equilibrium is defined by the conditions that

$$\frac{dx_n}{dt} = \xi[\Pr\{\tilde{h}(y) = n\delta\} - \Pr\{\tilde{h}(y) = (n-1)\delta\}] = 0, n = 1, 2.$$

It follows from simple algebra that $\Delta_n = 1/3, n = 0, 1, 2$.

The overall probability of error, on the other hand, is

$$\bar{P} = (1/2)\{(\Delta_0)^2 + (\Delta_1)^2 + (\Delta_2)^2\} +$$

$$\{\Delta_0\Delta_1 + \Delta_1\Delta_2\}\{(\pi_0 + \pi_1)\pi_1 + \pi_0\pi_1 + 2(\pi_1)^2\} + \Delta_0\Delta_2(\pi_1)^2.$$

Consider the middle term in the above expression, which is the most complex. Suppose that $h(y^1) = 0$ and $h(y^2) = \delta$, which has probability $\Delta_0\Delta_1$. Conditional on this event, there is probability $\pi_0 + \pi_1$ that $\tilde{h}(y^1) = 0$ and probability π_1 that $\tilde{h}(y^2) = 0$, in which case there is then probability $1/2$ of an error. Still conditional on this event, there is probability π_0 that $\tilde{h}(y^2) = \delta$ and probability π_1 that $\tilde{h}(y^1) = \delta$, in which case there is a probability $1/2$ of an error. Finally, still conditional on $h(y^1) = 0$ and $h(y^2) = \delta$, there is a probability $(\pi_1)^2$ that $\tilde{h}(y^1) = \delta$ and $\tilde{h}(y^2) = 0$, in which case an error has probability 1. There is a factor of 2 because the roles of y^1 and y^2 can be interchanged and a similar argument applies based on $\Delta_1\Delta_2$. This accounts for the middle term in the expression for $\bar{P}$.

The first-order conditions for minimizing $\bar{P}$ subject to $\sum_{n=0}^{2} \Delta_n = 1$ are then

$$\frac{\partial \bar{P}}{\partial \Delta_0} = \Delta_0 + \Delta_1 T + \Delta_2(\pi_1)^2 = \lambda$$

$$\frac{\partial \bar{P}}{\partial \Delta_1} = \Delta_1 + (\Delta_0 + \Delta_2)T = \lambda$$

and

$$\frac{\partial \bar{P}}{\partial \Delta_2} = \Delta_2 + \Delta_1 T + \Delta_0(\pi_1)^2 = \lambda$$

where λ is a Lagrange multiplier and $T = \{2\pi_0\pi_1 + 3(\pi_1)^2\}$.

However, if $\Delta_n = 1/3, n = 0, 1, 2$ it is immediate that

$$\frac{\partial \bar{P}}{\partial \Delta_0} = \frac{\partial \bar{P}}{\partial \Delta_2} < \frac{\partial \bar{P}}{\partial \Delta_1}$$

and hence that $\Delta_0 = \Delta_2 = \Delta_1$ is not optimal.

Nevertheless the relative importance of the end effects vanishes as N increases, with D fixed, so the equilibrium is approximately optimal—

Theorem 3.2. *Consider the model with noise, so that $D > 0$, and assume that $N > 4D$. There exists $\bar{\pi}_0 \in (0, 1)$ such that, whenever $\pi_0 > \bar{\pi}_0$, then— i) The equilibrium that $F(x_n^*) = n/(N+1)$, for $n = 1, ..., N$ is "conditionally optimal". That is, the choice of $\Delta_{2D} = ...\Delta_{N-2D} = \frac{1}{N+1}$ is optimal* conditional *on fixing $\Delta_0 = ...\Delta_{2D-1} = \Delta_{N-2D+1} = ... = \Delta_N = \frac{1}{N+1}$. ii) The equilibrium is approximately optimal* unconditionally. *That is, suppose the optimal probability of error is given by $P^*(N)$, where all $\Delta_n, n = 0, ..., N$ are freely chosen, and the equilibrium probability of error is $\bar{P}(N)$. It follows that* $\lim_{N\to\infty} NP^*(N) = \lim_{N\to\infty} N\bar{P}(N) > 0$.

Proof. See Appendix.

Essentially, the result ii) is that $\bar{P}(N)$ and $P^*(N)$ have the same leading term if these expressions are expanded in powers of $1/N$.

A different treatment at the ends would generally lead to unequal $\Delta_n, n = 0, ..., N$. There is no treatment that guarantees full optimality and, in any case, it is seems conceptually problematic to explicitly design noise. The treatment of ends here dramatizes how utility adapts to the distribution. But adaptation arises under any treatment, since the Δ_n are always independent how F depends on x. Furthermore, the conclusion in Theorem 3.2 is robust to the end treatment. That is, a different treatment at the ends would lead a different precise pattern of $\Delta_n, n = 0, ..., N$ but these would tend to equality away from the ends, as $N \to \infty$, and the approximate optimality result would still be valid.[9]

4. Cardinal Utility

The adaptive utility underlying choices here is cardinal, and can be inferred from data, in principle. This identification is facilitated by the way the separable error $\tilde{r}$ specified in terms of utility. We investigate this in this section.

The thresholds serve merely as a technical device that facilitates discussion of adaptation. Apart from this issue, the model can be readily formulated as continuous. It simplifies the discussion to consider the continuous case. Consider the limit of the model in which $N \to \infty$ but where D/N tends to a positive constant.[10] We have essentially random utility, where y^i is chosen if and only $U(y^i) + \tilde{r^i} > U(y^j) + \tilde{r^j}$. Suppose $\tilde{d}$ has cdf Π with support $[-\bar{D}, \bar{D}]$.

Suppose the data are pairs (y_1, y_2) with associated observed probabilities that y_1 is chosen,

$$J(y_1, y_2) = \int_{-\bar{D}}^{\bar{D}} \Pi(r + U(y_1) - U(y_2))\pi(r)dr. \tag{4.1}$$

Hence

$$J_1(y, y) = U'(y) \int_{-\bar{R}}^{\bar{R}} (\pi(r))^2 dr.$$

This identifies U, up to an additive and a positive multiplicative constant. The multiplicative constant cannot be identified in the sense that scaling utility and the noise together produces indistinguishable data. Without loss of generality, then,

$$U(y) = \frac{\int_0^y J_1(z, z)dz}{\int_0^1 J_1(z, z)dz}, \text{ so that } U(1) = 1.$$

[9] Further remarks on this issue are in Footnote 23 in the Appendix.

[10] The conditions on the π_d as in Eq (2.1) needed for Theorem 3.1 can be extended so they hold throughout the approximation.

Debreu (1958) discusses the representation of stochastic choice by cardinal utility, in such situation. In the present notation, $J(y^1, y^2)$ is represented by a utility function U when $J(y^1, y^2) > 1/2$ if and only $U(y^1) > U(y^2)$, as is clearly true here.[11] The novelty here is explicit endogeneity of the underlying utility.

Given U, the noise cdf Π can then also be obtained from the data. The function J reflects the difference between two draws from the noise distribution Π. Given the symmetry of this distribution about 0, J is also the cdf of the sum of two draws from Π, or the two fold convolution of F with itself given as Π^*, say. The Laplace transform of Π^* is the square of the Laplace transform of F. That is, $\mathscr{L}(\Pi^*) = (\mathscr{L}(\Pi))^2$, where $\mathscr{L}$ is the Laplace transform. The Laplace transform has a unique inverse so it follows that $\Pi = \mathscr{L}^{-1}(\sqrt{\mathscr{L}(\Pi^{*2})})$.[12]

We have thus shown that—

Lemma 4.1. *If the probability of choosing y^1 over y^2 is as in Eq (4.1), then both utility and the noise distribution can be derived from $J(y^1, y^2)$. That is, $U(y) = \frac{\int_0^y J_1(z,z)dz}{\int_0^1 J_1(z,z)dz}$ and $\Pi = \mathscr{L}^{-1}(\sqrt{\mathscr{L}(\Pi^{*2})})$ as discussed above.*

The cardinality of utility found here derives its cardinality from probability, although in a different sense than is true for von Neumann Morgenstern utility. It is not derived from neural firing rates. Nonetheless, it would be fascinating to investigate how such cardinal utility relates to the neural firing rates observed in a choice situation and to the cardinal hedonic reward from actual consumption of a chosen option.

The foregoing argument presupposes that the y_i are drawn iid from a given fixed F, when indeed the prediction is that U is F, in this probability of error case. Reverting to finite threshold model, an

[11] Note that $J(y, y) = \int_{-D}^{D} \Pi d\Pi = 1/2$ and that $J_1(y^1, y^2) > 0$. Roberts (1971) discusses general probabilistic choice functions.

[12] In more detail, the Laplace transform of a cdf Π is given by $\mathscr{L}(\Pi)(s) = \int_0^\infty e^{-st} d\Pi(t)$, where $s > 0$, as in Feller (1971, Ch. XIII). Solving $\mathscr{L}(\Pi^*)(s) = (\mathscr{L}(\Pi))^2(s)$ for $\mathscr{L}(\Pi)(s)$ yields two roots, but the negative root can be discarded.

econometric complication arises if the underlying h function adapts rapidly to F, since the mechanism will then be unduly affected by a run of odd outcomes that are unlikely under F. This issue cannot be solved satisfactorily by simply making adaptation slow, since adaptation should be optimally be fast if F changes frequently. Adaptation should be faster for penny-ante poker than for a move to the West Coast.

In the model, there is a tradeoff between speed and accuracy of adjustment which is controlled by the parameter ε—when ε is large, adaptation is rapid, but imprecise; if ε is small, adaptation is slow but precise. This property seems bound to be independent of the precise formulation here.

5. Happiness in Economics

The present explicitly dynamic adaptive model generates the "hedonic treadmill", as perhaps its most direct implication. That is, if the distribution of rewards shifts up, there will be corresponding surge in average utility, but this will be only temporary, with average utility reverting eventually to the baseline, as shown in Figure 1.

A widely cited paper on the "hedonic treadmill" is Schkade and Kahneman (1998), who argue that the utility used in making a decision to move to California, for example, disagrees with the utility actually experienced after such a move. This experienced utility exhibits a hedonic treadmill effect.[13]

The explanation the current model provides for the hedonic treadmill can be applied to the criterion of average national happiness, as has been recently suggested as an alternative to GNP. And no

[13] Schkade and Kahneman also argue that decision utility is "wrong" and that such decisions ought to be made of the basis of correctly anticipated experienced utility. Robson and Samuelson (2011) use an alternative model due to Rayo and Becker (2007) to analyze this, generating distinct utilities, but finding no error.

doubt GNP has defects—it is incomplete, in particular. Clark *et al.* (2018) go beyond the present *positive* approach to hedonic utility to embrace a *normative* view. Adaptation raises various issues for the positive view that is adopted here, but it is more problematic as a normative approach. If the distribution of outcomes is scaled up, for example, average utility is eventually unaffected. This does not seem a desirable normative property of a welfare criterion.

In the probability of error case, there is *complete* adaptation, in general, because $\int F(x)dF(x) = 1/2$ for all cdf's F. This argument holds for the limiting model, whether or not noise is assumed to remain in the limit. This is because the mechanism described here generates purely *relative* valuations. This was done in the interests of simplicity to highlight what is novel here, although it seems *a priori* that some element of absolute valuation should remain.[14] Nevertheless the issues raised here with complete adaptation will still arise, but to a lesser extent, given some relative adaptation.

Further, although we have largely confined attention here to the probability of error criterion, to facilitate the analysis of adjustment dynamics in the presence of noise, it is worth pointing out that the expected fitness criterion differs on this score. That is, if the criterion is expected fitness and there is no noise, it follows that $U'(x) = kf(x)^{2/3}$ where k is such that $U(1) = 1$ (as in Netzer, 2009). Although it is still true that $\int U(x)dF(x)$ is fully invariant to rescaling the cdf F, it depends on the scale-invariant properties of the distribution. Indeed, average utility can be made arbitrarily close to 1—the absolute maximum value—by taking distributions converging to a point mass at 1 in a particular fashion.

The problem is to maximize $\int_0^1 U(x)f(x)dx$ subject to $F'(x) = f(x)$ $U'(x) = kf(x)^{2/3}$, where $F(0) = U(0) = 0$ and $F(1) = 1$, and where k is such that $U(1) = 1$.

[14] Conceivably, an effectively absolute quality to valuations could be introduced if the distribution F were forced to retain a full range of possibilities.

This can be translated to the equivalent problem

$$\max_f \frac{V(1)}{W(1)}$$

subject to

$$F'(x) = f(x); W'(x) = f(x)^{2/3}; V'(x) = W(x)f(x),$$

where $F(0) = W(0) = V(0) = 0$, and $F(1) = 1$. This "Problem of Mayer" is awkward in the sense that, although first-order necessary conditions are straightforward, it is difficult to obtain local or global sufficient conditions. (See Hestenes, 1966, Ch. 7.) This issue can be finessed here since it can be shown directly that an unbeatable payoff can be obtained in a limiting sense. It must be that $V(1)/W(1) \leq 1$, since $V(1)$ is the expectation of $W(x)$ which has maximum value 1. Moreover—

Lemma 5.1. *There is a sequence of f_n, tending to a point mass at* 1, *such that $V_n(1)/W_n(1) \to 1$.*

Proof. See Appendix.

Surveys of happiness vary to a remarkably small extent with such obvious factors as income. Although Denmark is much richer than Bhutan, for example, it is only somewhat happier. This approximate constancy may reflect adaptation. It would be interesting to explain the variation that is left in terms of such a scale-invariant effect of the distribution.

6. CONCLUSIONS

A key motivation here was to develop a model based on neuroscientific evidence about how decisions are orchestrated in the brain. There is evidence that economic decisions are made by an adaptive neural mechanism that may have hedonic underpinnings.

We present a simple model where utility shifts in response to changing circumstances. This adaptation acts to reduce the error caused by a limited ability to make fine distinctions, and is evolutionarily optimal.

This model sheds light on the hedonic treadmill, and related phenomena in economics. A related model, that considers choice between *gambles*, promises to illuminate Kahneman and Tversky (1979), and the substantial risk-aversion and risk-preference observed in experiments, to an extent that is inconsistent with attitudes in contexts with higher stakes (Rabin, 2000).

7. Appendix—Proofs

Proof of Theorem 3.1.

Suppose that the (finite) Markov chain described by Equation (2.2) is represented by the matrix A_G.[15] This is defined on the state space $S_G = \mathscr{G}^N$, describing the position of the N thresholds, so A_G is a $|S_G|$ by $|S_G|$ matrix. It is irreducible, so that there exists a integer P such that $(A_G)^P$ has only strictly positive entries.[16] Finally, define the overall state space as $S = [0,1]^N$.

Consider an initial state $x_G^t \in S_G$ where $0 \leq x_{1,G}^t \leq x_{2,G}^t \leq ... \leq x_{N,G}^t \leq 1$. Let $x_G^t = (x_{1,G}^t, ...x_{N,G}^t)$. Consider then the random variable $x_G^{t+\Delta}$ that represents the state of the chain at $t + \Delta$, where $\Delta > 0$ is assumed divisible by ϵ, and is fixed, for the moment. Suppose there are R iterations of the chain, where $R = \Delta/\epsilon = G\Delta$. These iterations arise at times $t + r\epsilon$ for $r = 1, ..., R$. Suppose the process is constant between iterations, so it is defined for all $t' \in [t, t + \Delta]$.

We consider the limit as $R \to \infty$ (so it is always implicit that $G \to \infty$ as well). Taking this limit implies $\epsilon \to 0$, but also speeds up the

[15] See Mathematical Society of Japan (1987, 260, p. 963), for example.

[16] Consider any initial configuration, x^0, say, and any desired final configuration, x^T, say. The simplest argument to this effect relies on $\xi < 1$. First move x_1^0 to x_1^T by means of outcomes just to the right or left, as required, that do not affect any other thresholds. This might entail x_1 crossing the position of other thresholds, but temporarily suspend the usual convention of renumbering the thresholds, if so. This will take at most G periods. Then move x_2^0 to x_2^T in an analogous way. And so on. There is a finite time, GN, such that the probability of all this is positive.

process in a compensatory way, making the limit non-trivial. This speeding up is only a technical device and has no effect on the invariant distribution, in particular.

We adopt the notational simplification that

$$H_n(x) = \xi \Pr\{h(y) + \tilde{d}\delta = n\delta\}, n = 0, ..., N. \tag{7.1}$$

Given that $N > 2D$, it follows that, if $n \geq D$ and $n \leq N - D$, so the end effects do not arise, then

$$\Pr\{h(y) + \tilde{d}\delta = n\delta\} = \pi_0\Delta_n + \sum_{s=1}^{D} \pi_s\Delta_{n+s} + \sum_{s=1}^{D} \pi_s\Delta_{n-s}, \tag{7.2}$$

where $\Delta_n = \Pr\{h(y) = n\delta\} = F(x_{n+1}) - F(x_n), n = 0, ...N.$

However, if, for example $n < D$, then

(7.3)

$$\Pr\{h(y) + \tilde{d}\delta = n\delta\} = \left(\pi_0 + \sum_{s=n+1}^{D} \pi_s\right)\Delta_n + \sum_{s=1}^{D} \pi_s\Delta_{n+s} + \sum_{s=1}^{n} \pi_s\Delta_{n-s}.$$

An analogous expression holds if $n > N - D$.

We have then that $x_{n,G}^{t+\Delta} = x_{n,G}^{t} + \sum_{r=1}^{R} \epsilon_r$ where

$$\epsilon_r = \begin{cases} \epsilon \text{ with probability } H_n(x_G^{t+r\epsilon}) \\ -\epsilon \text{ with probability } H_{n-1}(x_G^{t+r\epsilon}) \\ 0 \text{ otherwise} \end{cases}.$$

It follows that

$$\frac{x_{n,G}^{t+\Delta} - x_{n,G}^{t}}{\Delta} = \frac{\sum_{r=1}^{R} \epsilon_r/\epsilon}{R}.$$

We will apply a version of the strong law of large numbers to this expression to obtain the limiting ODE system. A complication is that the increments ϵ^r are not independent because previous draws of ϵ^r $r < s$ affect the current value of $x_G^{t+s\epsilon}$ and hence the distribution of ϵ^s. However, the strong law still holds because the effect of an earlier outcome falling in $[x_n, x_{n+1})$ is to reduce the probability of a subsequent outcome doing so.

We consider a limit where the system starts at an arbitrary $\bar{x} \in S$. To allow for this, let $\bar{x}_G$ be any of the points in S_G that are closest to $\bar{x}$. Of course, $\bar{x}_G \to \bar{x}$, as $R, G \to \infty$.

Lemma 7.1. *Define $p_r = H_n(x_G^{t+r\epsilon}) - H_{n-1}(x_G^{t+r\epsilon})$ as the expected value of ϵ^r/ϵ. It follows that*

$$\frac{\sum_{r=1}^{R}(\epsilon^r/\epsilon - p_r)}{R} \to 0,$$

with probability 1 for $r = 1, ..., R$ as $R \to \infty$.

Proof. We apply Theorem 1 of Etemadi (1983). Define $w_r = \epsilon_r^+/\epsilon = 1$, if an outcome lies in $[x_n, x_{n+1})$, $w_r = \epsilon_r^+/\epsilon = 0$, otherwise. It follows that $E(w_r) = p_r^+ \leq 1$, say, satisfying (a) for Theorem 1. To consider the nature of the correlations consider draws r and s where $r < s$. Define $p_s^+(1) = \Pr\{w_s = 1 | w_r = 1\}$ and $p_s^+(0) = \Pr\{w_s = 1 | w_r = 0\}$. It follows that

$$E(w_r w_s) = E(w_r w_s | w_r = 1)\Pr\{w_r = 1\} + E(w_r w_s | w_r = 0)\Pr\{w_r = 0\},$$

so that

$$E(w_r w_s) = p_r^+ p_s^+(1) \leq p_r^+ p_s^+,$$

satisfying (b) of Theorem 1. That is, the effect of $\epsilon_r^+/\epsilon = 1$ is to increase x_n, decrease x_{n+1}, and hence to reduce p_s^+, so that there is non-positive covariance between the increments. Finally, $\operatorname{var}(w_r) = (1 - p_r^+)p_r^+ \leq 1/4$ so that $\sum_{r=1}^{\infty} \operatorname{var}(w_r)/r^2 < \infty$, satisfying (c) of Theorem 1. Hence all the conditions of Theorem 1 of Etemadi are satisfied.

Hence

$$\frac{\sum_{r=1}^{R} \epsilon_r^+/\epsilon - p_r^+}{R} \to 0,$$

with probability 1, for $r = 1, ..., R$ as $R \to \infty$.

A similar analysis clearly holds for ϵ_r^-/ϵ, where $\epsilon_r^-/\epsilon = 1$, if an outcome lies in $[x_{n-1}, x_n)$, and $\epsilon_r^-/\epsilon = 0$, otherwise. Let $p_r^- = \Pr\{\epsilon_r^-/\epsilon = 1\}$ and $p_r = p_r^+ - p_r^-$. Since $\epsilon_r = \epsilon_r^+ - \epsilon_r^-$, the result follows.[17]

[17] Etemadi implicitly assumes that the distributions of the initial random variables do not change as further variables are added. Although these distributions do change here, inspection of his proof

□

Any realized trajectory of the entire system of $x_G^{t+r\epsilon}$ must be continuous, so all the Riemann integrals exist, and it follows that, where x^s is any realized trajectory over time s,

$$\frac{x_{n,G}^{t+\Delta} - \bar{x}_{n,G}^{t}}{\Delta} \to \frac{\int_{s=t}^{t+\Delta} \left[H_n(x^s) - H_{n-1}(x^s)\right] ds}{\Delta}, n = 1, ..., N,$$

with probability 1, as $R \to \infty$. Taking the limit as $\Delta \to 0$ yields—

$$\frac{dx_n}{dt} = H_n(x) - H_{n-1}(x), n = 1, ...N.^{18} \tag{7.4}$$

---

shows it remains valid. An alternative perhaps more general proof relies on the property that all the covariances tend to zero as $R \to \infty$. This also makes clear that it is irrelevant that the distribution of initial random variables changes with R. To see that the weak law follows under this approach, redefine the binary random variables $w_r = \epsilon_r^+/\epsilon - p_r^+$, where $p_r^+ = \Pr\{w_r = 1\}$, for $r = 1, ..., R$. Of course, $E(w_r) = 0$ and it follows readily that $E((w_r)^2) = p_r^+(1 - p_r^+) \leq 1/4$. The w_r are not independent. Consider w_r and w_s, for example, where $r < s$. The problem is that the outcome of w_r affects p_s^+. Define $p_s^+(1) = \Pr\{w_s = 1|w_r = 1\}$ and $p_s^+(0) = \Pr\{w_s = 1|w_r = 0\}$. It follows that

$$E(w_r w_s) = E(w_r w_s|w_s = 1)\Pr\{w_s = 1\} + E(w_r w_s|w_s = 0)\Pr\{w_s = 0\}.$$

This expression reduces to

$$E(w_r w_s) = p_r^+(1 - p_r^+)(p_s^+(1) - p_s^+(0))(1 - 2(p_s^+(0) + p_s^+(1))$$

Hence $|E(w_r w_s)| \leq (3/4)|p_s^+(1) - p_s^+(0)|$. The effect of w_r on p_s^+ arises from the effect of a shift in x_n of at most Δ/R in absolute value on $H_n(x)$. Since $H_n(x)$ is continuously differentiable, there exists K such that $|p_s^+(1) - p_s^+(0)| \leq K/R$. Let $S_R = \sum_{r=0}^{R} w_r$. Chebyshev's inequality (Feller, 1971, p. 151) applied to the random variable S_R/R is then

$$\Pr\left\{\frac{|S_R|}{R} \geq \delta\right\} \leq \frac{E(|S_R|^2)}{R^2\delta^2}.$$

Since $E(|S_R|^2) \leq \frac{R}{4} + \frac{R^2 3K}{4R}$ it follows that

$$\Pr\left\{\frac{|S_R|}{R} \geq \delta\right\} \leq \frac{1+3K}{4R\delta^2},$$

for all $\delta > 0$. Hence

$$\Pr\left\{\frac{|\sum_{r=1}^{R} \epsilon_r^+/\epsilon - p_r^+|}{R} \geq \delta\right\} \to 0,$$

for $r = 1, ..., R$ as $R \to \infty$. This is the weak law of large numbers here for the redefined w_r^+. This argument applies to the analogous w_r^-, so the weak law holds for the original $w_r = \epsilon_r/\epsilon$.

[18] This expression is valid even if there are ties so that $x_n = x_{n+1}$, for example. In this case, x_n and x_{n+1} immediately split apart, relying on the convention that $x_n \leq x_{n+1}$.

Existence and uniqueness of the limiting realized path is then a consequence of existence and uniqueness results for ordinary differential equations (ODE's).(See Coddington and Levinson, 1955, Theorem 1.3.1, and the remarks on p. 19, for example. All the $H_n(x), n = 0, ..., N$ are Lipshitz continuous.)

The ODE system given by Eqs (7.1), (7.2), (7.3), and (7.4) can be summarized as

$$\frac{dx}{dt} = \xi C z + \xi b \tag{7.5}$$

where $z_n = F(x_n), n = 1, ..., N$, so that $\Delta_n = z_{n+1} - z_n, n = 0, ..., N$. The $n \times n$ matrix C and the $n \times 1$ vector b are independent of x.

It follows that the elements of C, C_{nm}, are given by

$$\xi C_{nm} = \frac{dH_n}{dz_m} - \frac{dH_{n-1}}{dz_m} = -\frac{dH_n}{d\Delta_m} + \frac{dH_n}{d\Delta_{m-1}} + \frac{dH_{n-1}}{d\Delta_m} - \frac{dH_{n-1}}{d\Delta_{m-1}}. \tag{7.6}$$

Lemma 7.2. *The matrix C is symmetric and negative definite.*

Proof. Consider first $n \geq D+1$ and $n \leq N-D-1$. It is immediate that

$$C_{nn} = -2(\pi_0 - \pi_1) < 0$$

and that

$$C_{n,n+s} = C_{n,n-s} = (\pi_{s-1} - \pi_s) - (\pi_s - \pi_{s+1}) > 0, s = 1, ..., D+1$$

Hence,

$$\sum_{-(D+1)}^{D+1} C_{n,n+s} = -2(\pi_0 - \pi_1) + 2(\pi_0 - \pi_1) - 2(\pi_1 - \pi_2) +$$

$$... + 2(\pi_{D-1} - \pi_D) - 2\pi_D + 2\pi_D = 0,$$

so that C satisfies weak diagonal dominance for rows n where $n \geq D+1$ and $n \leq N-D-1$.

Consider now $n \leq D$. (The case for $n \geq N-D$ is entirely analogous.) Using Eq (7.6) it follows that

$$C_{nn} = -2(\pi_0 - \pi_1) - \pi_n - 2\sum_{n+1}^{D} \pi_s < 0.$$

Adopt the convention throughout that $\pi_d = 0$ if $d > D$ and sums are 0 if the range described is empty. We also have

$$C_{n,n+1} = (\pi_0 - \pi_1) - (\pi_1 - \pi_2) + \sum_{n+1}^{D} \pi_s > 0.$$

For $s = 2, ..., D+1$, it again follows that

$$C_{n,n+s} = (\pi_{s-1} - \pi_s) - (\pi_s - \pi_{s+1}) > 0.$$

Hence

$$\sum_{0}^{D+1} C_{n,n+s} = -(\pi_0 - \pi_1) - \sum_{s=n}^{D} \pi_s < 0.$$

Hence C satisfies strict diagonal dominance in the first row, if $n = 1$, that is. In addition, if $n > 1$ and $n \le D$,

$$C_{n,n-1} = (\pi_0 - \pi_1) - (\pi_1 - \pi_2) + \sum_{s=n}^{D} \pi_s \text{ so } \sum_{s=-1}^{D+1} C_{n,n+s} = -(\pi_1 - \pi_2) < 0.$$

It follows that

$$\sum_{s=-(n-1)}^{D+1} C_{n,n+s} = -(\pi_{n-1} - \pi_n) < 0.$$

Thus C satisfies strict diagonal dominance for rows $1, ..., D$ and, analogously, for rows $N - D, ..., N$.

Altogether, C satisfies weak diagonal dominance for all rows and strict diagonal dominance for some rows.

We now show that C is symmetric. The above argument shows that

$$C_{n,n+s} = C_{n,n-s} = (\pi_{s-1} - \pi_s) - (\pi_s - \pi_{s+1}) > 0, n, n-s, n+s = 1, ..., N, s \ge 2.$$

Hence $C_{n,n+s} = C_{(n+s),n}$, for all $|s| \ge 2$. The only remaining issue is then to show that $C_{n,n+1} = C_{(n+1),n}$ for all $n, n+1 = 1, ..., N$. This is immediate if $n \ge D+1$ and $n \le N - D - 1$. Suppose then $n \le D$. Now $C_{n,n+1} = (\pi_0 - \pi_1) - (\pi_1 - \pi_2) + \sum_{s=n+1}^{D} \pi_s$ and $C_{n,n-1} = (\pi_0 - \pi_1) - (\pi_1 - \pi_2) + \sum_{s=n}^{D} \pi_s$. Hence $C_{(n+1),n} = (\pi_0 - \pi_1) - (\pi_1 - \pi_2) + \sum_{s=n+1}^{D} \pi_s = C_{n,n+1}$.

An analogous argument applies for $n \ge N - D$

In addition—i) C is "Metzlerian" since $C_{mn} \geq 0$ for all $m \neq n$. ii) Given the above observations, it follows that C has a "quasidominant diagonal" iii) It follows that C is irreducible since all elements one off the diagonal on either side are are strictly positive.

It then follows from McKenzie (1959) that C satisfies "negative row diagonal dominance" and so is negative definite and non-singular. (See also Giorgi and Zuccotti, 2015, Theorems 1 and 2.) □

Lemma 7.3. *The ODE system (7.5) is globally asymptotically stable, with $x(t) \to x^*$ for all $x(0)$, where $F(x_n^*) = \frac{n}{N+1}$.*

Proof. We have, from Eq (7.5),

$$\frac{dz}{dt} = E\frac{dx}{dt} = \xi EC(z + C^{-1}b), \tag{7.7}$$

where $E = \text{diag}(F_1', ..., F_N')$. Define then a candidate Lyapounov function

$$V = -(z + C^{-1}b)^T C(z + C^{-1}b).$$

Since C is negative definite, $V \geq 0$ and $V = 0$ if and only if $z = -C^{-1}b = z^*$. Further

$$\frac{dV}{dt} = -2\xi(Cz + b)^T E(Cz + b) \leq 0$$

for all z, since E is positive definite. Further $\frac{dV}{dt} = 0$ if and only if $z = z^*$. That is, V is a Lyapounov function for the ODE system given by Eq (7.7).[19]

Hence if x^* is the unique solution of $F(x_n^*) = z_n^*, n = 1, ..., N$ then $x(t)$ is globally asymptotically stable—that is, $x(t) \to x^*$ for all $x(0)$.

Consider $x_n, n = 1, ..., N$ such that $F(x_n) = \frac{n}{N+1} = z_n, n = 1, ..., N$. It follows that $\Delta_n = F(x_{n+1}) - F(x_n) = \frac{1}{N+1}, n = 0, ..., N$. From Eqs (7.2) and (7.3), $H_n(x) = \frac{\xi}{N+1}, n = 0, ..., N$ and $\frac{dx_n}{dt} = 0, n = 1, ..., N$. Hence it must be that this is the unique rest point so $F(x_n^*) = \frac{n}{N+1}$. □

[19] See Mathematical Society of Japan (1987, 126F, p. 492), for example.

We can now complete the proof of Theorem 3.1. Suppose that Ω_G is the cdf representing the unique invariant distribution of the Markov chain with transition matrix A_G. Extend Ω_G to be defined on the entire space $S = [0,1]^N$. By compactness, it follows that there exists a subsequence of the Ω_G that converges weakly to a cdf Ω defined on S (Billingsley, 1968, Chapter 1). That is, $\Omega_G \Rightarrow \Omega$ as $G \to \infty$. We will show that Ω puts full weight on the singleton x^*. Once this is shown, it follows that the full sequence must also converge to this Ω.

Suppose, then by way of contradiction, that Ω does not put full weight on x^*, so that $\int V(\bar{x})d\Omega(\bar{x}) > 0$.

Reconsider then the construction that led to the differential equation system that approximates the Markov chain, as described from the beginning of this Appendix. Recall that $\bar{x} \in S$, is arbitrary, where $\bar{x} \neq x^*$ and $0 \leq \bar{x}_1 < \bar{x}_2 < ... < \bar{x}_N \leq 1$. Again, let $\bar{x}_G$ be any of the points in S_G that are closest to $\bar{x}$. Let $x_G^\Delta(\bar{x})$ be the random variable describing the Markov chain at $t = \Delta$ that starts at $\bar{x}_G$ at $t = 0$. Consider now the limit as $G \to \infty$, so that the number of repetitions in the fixed time Δ, given by $R = \Delta/\epsilon$ also tends to infinity. Suppose $x^\Delta(\bar{x})$ is the solution to Eq (7.5), that is, to $\dot{x} = \xi(Cz + b)$, at $t = \Delta$, given it has initial value $\bar{x}$ at $t = 0$.

Given that $\bar{x} \neq x^*$ and $0 \leq \bar{x}_1 < \bar{x}_2 < ... < \bar{x}_N \leq 1$, it follows that $V(x^\Delta(\bar{x})) < V(\bar{x})$, since we showed that $\dot{V}(x) < 0$ on $[0, \Delta]$. By hypothesis, $\int V(\bar{x})d\Omega(\bar{x}) > 0$. It follows that

$$\int V(x^\Delta(\bar{x}))d\Omega(\bar{x}) < \int V(\bar{x})d\Omega(\bar{x}). \tag{7.8}$$

That this inequality holds in the limit implies that it must hold for large enough R and G, as follows.

First, the derivation of the approximating system $\dot{x} = \xi(Cz + b)$ implies, in particular, that

$$EV(x_G^\Delta(\bar{x})) \to V(x^\Delta(\bar{x})) \text{ as } G \to \infty, \tag{7.9}$$

and it can be shown that this convergence is uniform in $\bar{x}$.[20]

It now follows that

$$|\int EV(x_G^\Delta(\bar{x}))d\Omega_G(\bar{x}) - \int V(x^\Delta(\bar{x}))d\Omega(\bar{x})| \leq$$

$$|\int EV(x_G^\Delta(\bar{x}))d\Omega_G(\bar{x}) - \int V(x^\Delta(\bar{x}))d\Omega_G(\bar{x})| +$$

$$|\int V(x^\Delta(\bar{x}))d\Omega_G(\bar{x}) - \int V(x^\Delta(\bar{x}))d\Omega(\bar{x})|.$$

The first term on the right hand side tends to zero, as $G \to \infty$, by the uniform convergence in Equation (7.9). The second term on the right hand side also tends to zero as $G \to \infty$ since $\Omega_G \Rightarrow \Omega$ and the integrand is continuous. Hence

$$\int EV(x_G^\Delta(\bar{x}))d\Omega_G(\bar{x}) \to \int V(x^\Delta(\bar{x}))d\Omega(\bar{x}), \text{ as } G \to \infty. \tag{7.14}$$

Secondly, since $\Omega_G \Rightarrow \Omega$, as $G \to \infty$, and V is continuous, it follows that

$$\int V(\bar{x})d\Omega_G(\bar{x}) \to \int V(\bar{x})d\Omega(\bar{x}). \tag{7.15}$$

[20]If $\delta(G) = \sup_{\bar{x}\in S} |EV(x_G^\Delta(\bar{x})) - V(x^\Delta(\bar{x})|$, then it is enough to show $\delta(G) \to 0$ as $G \to \infty$. If this is not true, there exists a $\delta > 0$ and a sequence of $\bar{x}_G$ such

$$|EV(x_G^\Delta(\bar{x}_G)) - V(x^\Delta(\bar{x}_G)| \geq \delta. \tag{7.10}$$

There must exist a convergent subsequence of $\bar{x}_G$ such that $\bar{x}_G \to \hat{x} \in S$, say. Since Eq (7.9) holds at $\hat{x}$, it follows that, if G is large enough,

$$|EV(x_G^\Delta(\hat{x})) - V(x^\Delta(\hat{x}))| < \delta/3, \tag{7.11}$$

if G is large enough. The ODE system is continuous in the initial state, given that the $H_n(x)$ are Lipshitz (Coddington and Levinson, 1955, Theorem 1.7.1, p. 22) so that,

$$|V(x^\Delta(\bar{x}_G) - V(x^\Delta(\hat{x})| < \delta/3, \tag{7.12}$$

if G is large enough. The Markov chain is continuous in the initial state for the same reason. Also, the jump in the initial state from finding the closest state in S_G to $\bar{x}_G \in S$ is arbitrarily small if G is large. Hence

$$|EV(x_G^\Delta(\bar{x}_G)) - EV(x_G^\Delta(\hat{x}))| < \delta/3, \tag{7.13}$$

if G is large enough. Now Eqs (7.11), (7.12) and (7.13) contradict Eq (7.10).

Altogether, then Equations (7.8), (7.14) and (7.15) imply that, whenever G is sufficiently large

$$\int EV(x_G^{\Delta}(\bar{x}))d\Omega_G(\bar{x}) < \int V(\bar{x})d\Omega_G(\bar{x}), \tag{7.16}$$

which will yield a contradiction, since Ω_G is the invariant distribution.

To show this explicitly, we revert to matrix notation given that the inequality concerns a finite Markov chain. This chain has transition matrix A_G and an invariant distribution Ω_G with finite support. Suppose then that ω_G is the column vector describing the invariant distribution, so that $\omega_G^T = \omega_G^T A_G$, where T denotes the transpose. As before, let $R = \Delta/\epsilon$. We have

$$EV(x_G^{\Delta}(\bar{x}_G)) = \sum_{x \in S_G} e^T(\bar{x}_G)(A_G)^R(x)V(x),$$

where $e^T(\bar{x}_G)$ is the unit row vector that assigns 1 to $\bar{x}_G \in S_G$ and 0 to all other elements of S_G.[21]

It follows that Equation (7.16) is equivalent to

$$\sum_{x \in S_G} \omega_G^T(A_G)^R(x)V(x) < \sum_{x \in S_G} \omega_G^T(x)V(x),$$

which is a contradiction, since $\omega_G^T(A_G)^R = \omega_G^T$. This completes the proof of Theorem 3.1.

Proof of Theorem 3.2.

Fix $\Delta_0 = ... = \Delta_{2D-1} = \Delta_{N-2D+1} = ... = \Delta_N = \frac{1}{N+1}$, as in Theorem 3.2. Consider free choice of the remaining $\Delta_{2D} = ...\Delta_{N-2D}$. The terms in $\bar{P}(N)$ that involve $\Delta_n, n = 2D, ..., N-2D$ are then

$$\frac{1}{2}(\Delta_n)^2 + \{\Delta_{n-1}\Delta_n + \Delta_n\Delta_{n+1}\}Q_1 + ... + \{\Delta_{n-2D}\Delta_n + \Delta_n\Delta_{n+2D}\}Q_{2D}$$

where

$$Q_d = 2\Pr\{d_1 - d_2 > d\} + \Pr\{d_1 - d_2 = d\} > 0, d = 1, .., 2D.$$

[21]That is, $e^T(\bar{x}_G)(A_G)^R(x)$ is the probability that the Markov chain is in state $x \in S_G$ at Δ given that is in state $\bar{x}_G$ at time 0.

It follows then that

(7.17) $$\frac{\partial \bar{P}}{\partial \Delta_n} = \Delta_n + \{\Delta_{n-1} + \Delta_{n+1}\}Q_1 + ... + \{\Delta_{n-2D} + \Delta_{n+2D}\}Q_{2D}$$

and hence that

$$\frac{\partial^2 \bar{P}}{\partial \Delta_{n+s} \partial \Delta_n} = \begin{cases} 1 \text{ if } s = 0 \\ Q_s \text{ if } s = 1, ..., 2D \\ Q_s \text{ if } s = -1, ..., -2D \end{cases}.$$

In general, $\bar{P}(N)$ is strictly convex in $\Delta_n, n = 2D, ..., N - 2D$ if it satisfies strict diagonal dominance everywhere, that is, if $1 > 2\sum_{s=1}^{2D} Q_s$.[22] If this condition holds, strict diagonal dominance will holds near the ends, since the effect is to truncate some of the Q_s terms. Since $2\sum_{s=1}^{2D} Q_s = 0$ if $\pi_0 = 1$ strict convexity holds if $\pi_0 > \bar{\pi}_0$, for some $\bar{\pi}_0 < 1$.[23]

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[22] Eq (2.1) does not imply this condition.

[23] This condition provides an argument for the robustness of the results here to the treatment of noise at the ends. Taking the first difference of the first-order conditions, $\frac{\partial \bar{P}}{\partial \Delta_n} = k, n = 0, ...N$, from Eq (7.17), for some Lagrange multiplier k, yields a difference equation in the Δ_n of order $4D + 1$, without k. (See Mathematical Society of Japan, 1987, 104, p. 380, for example.) This difference equation has a simple characteristic root of 1, corresponding to equal Δ_n as focussed upon here. The additional roots satisfy

$$1 + \sum_{s=1}^{2D} Q_s(\lambda^s + \lambda^{-s}) = 0.$$

These other $4D$ roots occur in complex conjugate pairs and also in reciprocal pairs. (The symmetry of the Q_s accounts for roots arising as reciprocal pairs. If λ is a root going forward, then it must be a root going backwards, so $1/\lambda$ is also a root.) Since the coefficients are real, as is the desired solution, complex conjugate pairs arise with complex conjugate coefficients. A root with modulus different from 1 has to be given vanishing weight as $N \to \infty$, since $\Delta_n \in [0, 1]$, for all n. That is, if $|\lambda| > 1$, then the corresponding solution explodes away from $n = 0$; if $|\lambda| < 1$ it explodes away from $n = N$. This argument holds regardless of the treatment of noise at the ends. The condition that $1 > 2\sum_{s=1}^{2D} Q_s$ guarantees that there cannot be another (complex) root with modulus 1. Indeed, if $\lambda = e^{i\theta}$ is such a root, then

$$1 + 2\sum_{s=1}^{2D} Q_s \cos(s\theta) = 0,$$

where θ is real. Of course $\cos(s\theta) \geq -1$, so

$$1 + \sum_{s=1}^{2D} Q_s \cos(s\theta) \geq 1 - 2\sum_{s=1}^{2D} Q_s > 0,$$

ruling out any such additional root. That is, under the condition that $1 > 2\sum_{s=1}^{2D} Q_s$, it follows that the Δ_n robustly become close to equal away from the ends, as $N \to \infty$.

Conditional optimality entails minimizing $\bar{P}$ over choice of $\Delta_n, n = 2D, ..., N-2D$ subject to $\sum_{n=2D}^{N-2D} \Delta_n = \frac{N-4D+1}{N+1}$. The first-order conditions for an interior minimum are satisfied by choice of $\Delta_n = \frac{1}{N+1}, n = 2D, ..., N-2D$, given $\Delta_0 = ...\Delta_{2D-1} = \Delta_{N-2D+1} = ... = \Delta_N = \frac{1}{N+1}$, since, where λ is a Lagrange multiplier,

$$\frac{\partial \bar{P}}{\partial \Delta_n} = \frac{\{1 + 2Q_1 + ... + 2Q_{2D}\}}{N+1} = \lambda, n = 2D, ..., N-2D.$$

Since $\bar{P}(N)$ is convex these first-order conditions ensure $\Delta_n = \frac{1}{N+1}, n = 2D, ..., N-2D$ is the global optimum for the constrained problem, establishing i).

The following expression holds

$$\bar{P}(N) = \frac{1}{2}\left\{(\Delta_{2D})^2 + ... + (\Delta_{N-2D})^2\right\} + Q_1\left\{\Delta_{2D}\Delta_{2D+1} + ... + \Delta_{N-2D-1}\Delta_{N-2D}\right\} +$$

$$... + Q_{2D}\left\{\Delta_{2D}\Delta_{4D} + ... + \Delta_{N-4D}\Delta_{N-2D}\right\} +$$

$O(D^2)$ quadratic terms with at least one Δ_n for $n = 0, ..., 2D,$ or $n = N-2D+1, ..., N$.

It follows immediately that, since $\Delta_n = \frac{1}{N+1}, n = 0, ..., N$, then

$$\lim_{N\to\infty} N\bar{P}(N) = \frac{1}{2} + Q_1 + ... + Q_{2D}.$$

It follows that $P^*(N)$ is given by the same expression as $\bar{P}(N)$, where all Δ_n are freely chosen. Clearly $P^*(N) \leq \bar{P}(N)$, so

$$\liminf_{N\to\infty} NP^*(N) \leq \limsup_{N\to\infty} NP^*(N) \leq \lim_{N\to\infty} N\bar{P}(N).$$

Suppose, by way of establishing a contradiction, that $\liminf_{N\to\infty} NP^*(N) < \lim_{N\to\infty} N\bar{P}(N)$. Hence there exists a sequence of $N \to \infty$ such that $\lim_{N\to\infty} NP^*(N) < \lim_{N\to\infty} N\bar{P}(N)$. This contradicts the conditional optimality of $\bar{P}(N)$. That is, consider a solution to the constrained problem given by $\hat{\Delta}_n = \frac{1}{N+1}, n = 0, ..., 2D-1, N-2D+1, ..., N$ and

$$\hat{\Delta}_n = k\Delta_n^*, n = 2D, ..., N-2D,$$

where k satisfies

$$k\left(1-\sum_{n=0}^{2D-1}\Delta_n^* - \sum_{n=N-2D+1}^{N}\Delta_n^*\right) = 1-\frac{4D}{N+1}$$

.

It is clear that $\Delta_n^* \to 0$ as $N \to \infty$, so that $k \to 1$ and it follows that, where $\hat{P}(N)$ is the probability of error generated by the $\hat{\Delta}_n$,

$$\lim_{N\to\infty} N\hat{P}(N) = \lim_{N\to\infty} NP^*(N) < \lim_{N\to\infty} N\bar{P}(N).$$

Hence the $\hat{\Delta}_n$ provide a superior solution to the constrained problem, a contradiction. Hence

$$\lim_{N\to\infty} NP^*(N) = \lim_{N\to\infty} N\bar{P}(N) > 0.$$

This completes the proof of Theorem 3.2.

Proof of Lemma 5.1.

Choose any strictly increasing and continuous $f : [0,1] \to (0,\infty)$ where $f(x) \to \infty$ as $x \to 1$ and $\int_0^1 f(x)dx = 1-m$ for $m \in (0,1)$. Define $h(\delta)$ as the unique strictly decreasing solution of $\int_0^{1-\delta} f(x)dx + h(\delta)\delta = 1$, and consider the pdf given by f on $[0,1-\delta)$ and the constant $h(\delta)$ on $[1-\delta,1]$. It follows that $W(x) = W(1-\delta) + h^{2/3}(x - 1+\delta)$ if $x > 1-\delta$ so that $V(1) = \int_0^{1-\delta} W(x)f(x)dx + h\int_{1-\delta}^1 W(x)dx$. After some algebra, it follows that $V(1) = V(1-\delta) + h\delta W(1-\delta) + \frac{h^{2/3}\delta^2}{2}$. As $\delta \to 0$, have $h\delta \to m$ so that $h^{2/3}\delta \to 0$. It follows that $W(1-\delta) \to W(1)$. Hence $\frac{V(1)}{W(1)} \geq h\delta\frac{W(1-\delta)}{W(1)} \to m$, as $\delta \to 0$. For any sequence of $f_n \to 0$ such that $m_n \to 1$, it follows that the associated V_n and W_n satisfy

$$\frac{V_n(1)}{W_n(1)} \to 1,$$

since $\frac{V_n(1)}{W_n(1)} \leq 1$.[24]

[24] The necessary conditions for the problem of Mayer described before Lemma 5.1 are as follows (Hestenes , 1966, Theorem 4.1, p. 315). It is necessary to impose a finite upper bound, $\bar{f}$, say, on the density f, since otherwise existence may not hold. Define then the Hamiltonian

$$\mathscr{H} = \psi_V W f + \psi_F f + \bar{\psi}_W f^{2/3},$$

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where ψ_V, ψ_F and ψ_W are the costate variables corresponding to the state variables V, F and W, respectively. Hence $\psi_V' = \psi_F' = 0$ and $\bar{\psi}_W' = -\frac{\partial \mathscr{H}}{\partial W} = -\psi_V f$, so that $\bar{\psi}_W = \psi_W - \psi_V F$, for a constant ψ_W. The objective is to minimize $-\frac{V(1)}{W(1)}$. It is necessary that $\mathscr{H}$ is minimized over $f \in [0, \bar{f}]$ with transversality conditions $\psi_V(1) = \frac{\psi_0}{W(1)}$ and $\bar{\psi}_W(1) = \psi_W - \psi_V = -\frac{\psi_0 V(1)}{W(1)^2}$, for some constant $\psi_0 \geq 0$. It can be shown (eventually) that these necessary conditions admit a unique closed form solution on $[0, 1]$. This solution involves a strictly increasing $f \in (0, \bar{f})$ initially but then a second and final phase where $f = \bar{f}$. If a well-behaved solution exists, this must be it. As the upper bound $\bar{f} \to \infty$, it follows that $f \to 0$ on the initial phase and $\frac{V(1)}{W(1)} \to 1$. The essential features of this construction are used in the argument given above. But that argument is direct, and finesses issues of existence or sufficiency.

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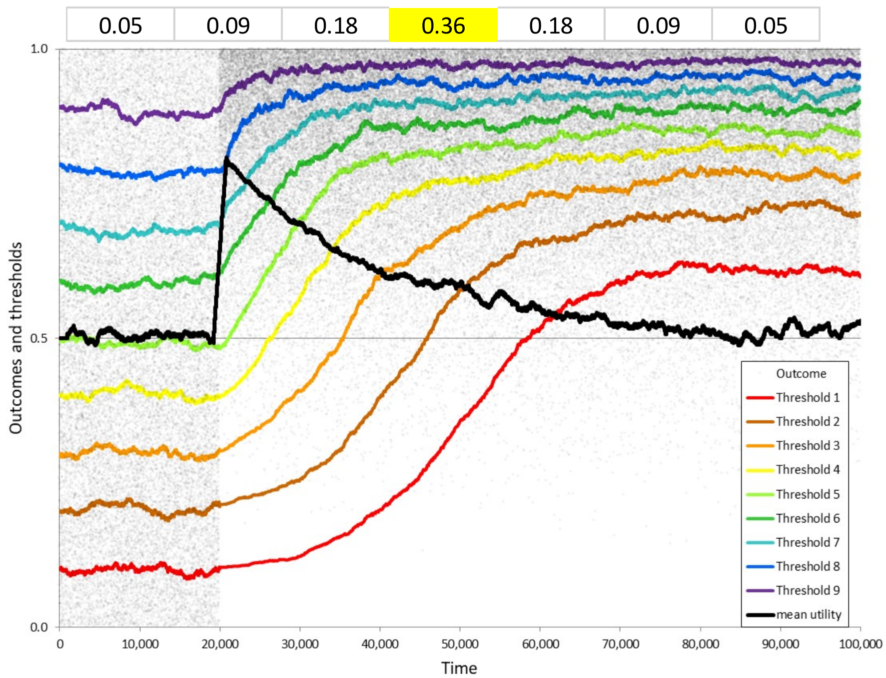

FIGURE 1. Simulation of Hedonic Treadmill.
$F(x) = x$ for $1 - 20,000$ periods; $F(x) = x^5$ for $20,000 - 100,000$.
The probabilities π_d for $d = 0, 1, 2, 3$ are at the top.