Self-similarity of the Cantor Set

Let $\mathcal{C}$ denote the Cantor middle-thirds set. We define the interval $A_{a_1a_2\cdots a_n}$ to be the interval that contains all numbers whose ternary expansion begins with $a_1a_2\cdots a_n$;

$$\begin{aligned} A_{a_1a_2\cdots a_n} &= \{x \in [0,1] \mid [x]_3 = .a_1a_2\cdots a_n a_{n+1}a_{n+2}\cdots\} \\ &= \{x \in [0,1] \mid [x]_3 = .a_1a_2\cdots a_n\,\vec{a}\,\} \end{aligned}$$

where $\vec{a}$ is an arbitrary (infinitely long) sequence of $0^{\prime s}$, $1^{\prime s}$, and $2^{\prime s}$. For example,

$$\begin{aligned} A_0 &= [0,\ 1/3] \\ A_1 &= [1/3,\ 2/3] \\ A_2 &= [2/3,\ 1] \\ A_{00} &= [0,\ 1/9] \\ A_{22} &= [8/9,\ 1] \\ A_{022} &= [8/27,\ 9/27] \end{aligned}$$

Then we define the sets $\mathcal{C}_{a_1a_2\cdots a_n}$ to be those parts of the Cantor set that lie in $A_{a_1a_2\cdots a_n}$;

$$\begin{aligned} \mathcal{C}_{a_1a_2\cdots a_n} &= \mathcal{C} \cap A_{a_1a_2\cdots a_n} \\ &= \{x \in \mathcal{C} \mid [x]_3 = .a_1a_2\cdots a_n\,\vec{c}\,\} \end{aligned}$$

where $\vec{c}$ is an arbitrary sequence of $0^{\prime s}$ and $2^{\prime s}$ (note that for $\mathcal{C}_{a_1a_2\cdots a_n}$, $a_i \in \{0,2\}$).

Let $3^m\,\mathcal{C}_{a_1a_2\cdots a_n} = \{x \,|\, x = 3^m y \text{ for some } y \in \mathcal{C}_{a_1a_2\cdots a_n}\}$. That is, $3^m\,\mathcal{C}_{a_1a_2\cdots a_n}$ are the numbers in $\mathcal{C}_{a_1a_2\cdots a_n}$ multiplied by 3^m. Now recall that $[3^m x]_3$ is the ternary expansion of x shifted to the left by m places (if m is positive). Therefore, $3^n\,\mathcal{C}_{a_1a_2\cdots a_n} = \{x \mid [x]_3 = a_1a_2\cdots a_n\,.\,\vec{c}\,\}$. Now if we let $z = a_1 3^{n-1} + a_2 3^{n-2} + \cdots + a_{n-1}3 + a_n$ (so that $[z]_3 = a_1a_2\cdots a_n$), then

$$3^n\,\mathcal{C}_{a_1a_2\cdots a_n} - z = \{x \mid [x]_3 = .\vec{c}\,\}$$

where $\vec{c}$ is an arbitrary sequence of $0^{\prime s}$ and $2^{\prime s}$ (this is because if x is a number such that $[x]_3 = .a_1a_2\cdots a_n\,\vec{c}$, then $[3^n x - z]_3 = .\vec{c}\,$).

In other words, the set of numbers $3^n\,\mathcal{C}_{a_1a_2\cdots a_n} - z$ are precisely those numbers in $[0,1]$ whose ternary expansion contains only $0^{\prime s}$ and $2^{\prime s}$. That is, $3^n\,\mathcal{C}_{a_1a_2\cdots a_n} - z$ <u>is</u> the Cantor set. This demonstrates the self-similarity of the Cantor set (it is similar to pieces of itself after magnifying and shifting the pieces). For example,

$$\begin{aligned} 3\,\mathcal{C}_2 - 2 &= \mathcal{C} \\ 9\,\mathcal{C}_{22} - 8 &= \mathcal{C} \\ 27\,\mathcal{C}_{022} - 8 &= \mathcal{C} \end{aligned}$$

(recall the sets A_2, A_{22}, and A_{022} given above). You can also see this by looking at Figure 2.5, page 68, in the text.