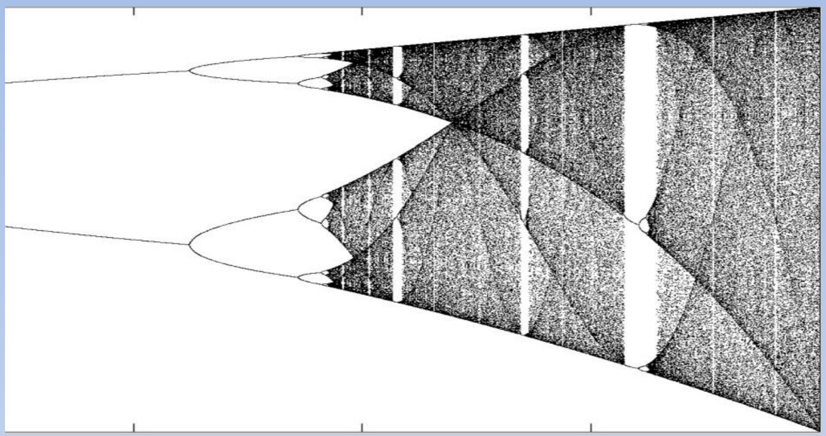
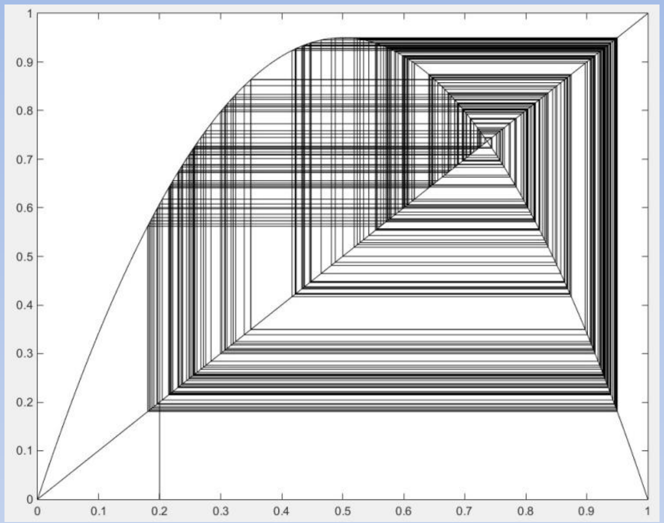
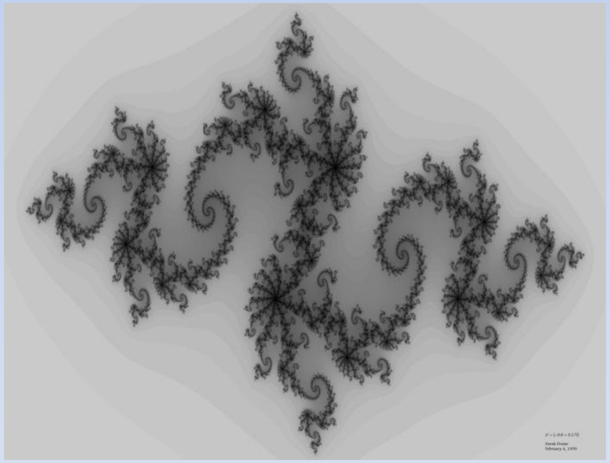
Complicated dynamics from simple functions……

Math Outside the Box, Oct 18 2016







Randall Pyke
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This presentation: www.sfu.ca/~rpyke → Presentations → Dynamics

Discrete dynamical systems: Variables evolve in time in discrete time steps.

Some examples:

- Temperature on Burnaby Mountain each day at noon
- Density of traffic on Highway #1 eastward at Gaglardi overpass each hour
- Total amount of sunshine (minutes) each day at Stanley Park
- Number of salmon running Wilson creek each day
- Closing price of a stock each day
- Population of bees each spring (e.g. March 1) in a bee farm in Richmond
- Number of people infected with Zika virus each month in Florida

An influential paper by biologist Robert May….

Nature Vol. 261 June 10 1976 **459**

review article

Simple mathematical models with very complicated dynamics

Robert M. May*

First-order difference equations arise in many contexts in the biological, economic and social sciences. Such equations, even though simple and deterministic, can exhibit a surprising array of dynamical behaviour, from stable points, to a bifurcating hierarchy of stable cycles, to apparently random fluctuations. There are consequently many fascinating problems, some concerned with delicate mathematical aspects of the fine structure of the trajectories, and some concerned with the practical implications and applications. This is an interpretive review of them.

THERE are many situations, in many disciplines, which can be described, at least to a crude first approximation, by a simple first-order difference equation. Studies of the dynamical properties of such models usually consist of finding constant

Fourth, there is a very brief review of the literature pertaining to the way this spectrum of behaviour—stable points, stable cycles, chaos—can arise in second or higher order difference equations (that is, two or more dimensions; two or more

Deterministic vs random (stochastic) dynamical systems

A *random* (or *stochastic*) *dynamical system* is an evolving system that has random laws governing it's evolution and/or random initial conditions. Here's an example of a random discrete dynamical system $\{y_0, y_1, y_2, \ldots\}$;

• choose $y_0 \in \mathsf{Z}$ (this is a deterministic initial condition, not a random one)

• to determine y_1 roll a die and add that number to y_0. This gives y_1.

• to obtain the next number in the sequence, y_{i+1}, roll a die and add the number turning up to y_i.

A feature of this example (as for all random systems) is that the future evolution is *unpredictable*; it cannot be predicted exactly (although one can calculate the *probabilities* that the future states may assume). Also, every time you create this sequence *even using the same initial condition* y_0, you typically get *different* evolutions; it is not determined by the initial conditions.

Deterministic vs random (stochastic) dynamical systems

A *deterministic dynamical system* is one where the dynamical laws are deterministic (i.e., they are specified completely and unequivocally, not randomly). For example, the discrete dynamical system $\{x_0, x_1, x_2, \ldots\}$ defined by;

- choose $x_0 \in \mathsf{R}$
- $x_1 = 3x_0 + 2$
- $x_2 = 3x_1 + 2$
- continue in this manner; $x_{i+1} = 3x_i + 2$

In a deterministic dynamical system the evolution is, at least in principle, completely predictable. That is, given the initial condition one could predict *exactly* what the future evolution is. Also note that creating this sequence with the same initial condition x_0 will *always* result in the same evolution $x_1, x_2, \ldots$. We can express this fact mathematically as $x_n = g(n, x_0)$ for some (perhaps not explicitly available) function g; x_n is uniquely determined by n and x_0 alone.

Mathematically modelling (deterministic) discrete dynamical systems.

Dynamics determined by a function $f : \mathbf{R} \rightarrow \mathbf{R}$ (one dimensional)

Initial point (data) x_0 Subsequent points;

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f(x_1) \\ x_3 &= f(x_2) \end{aligned}$$

(iteration)

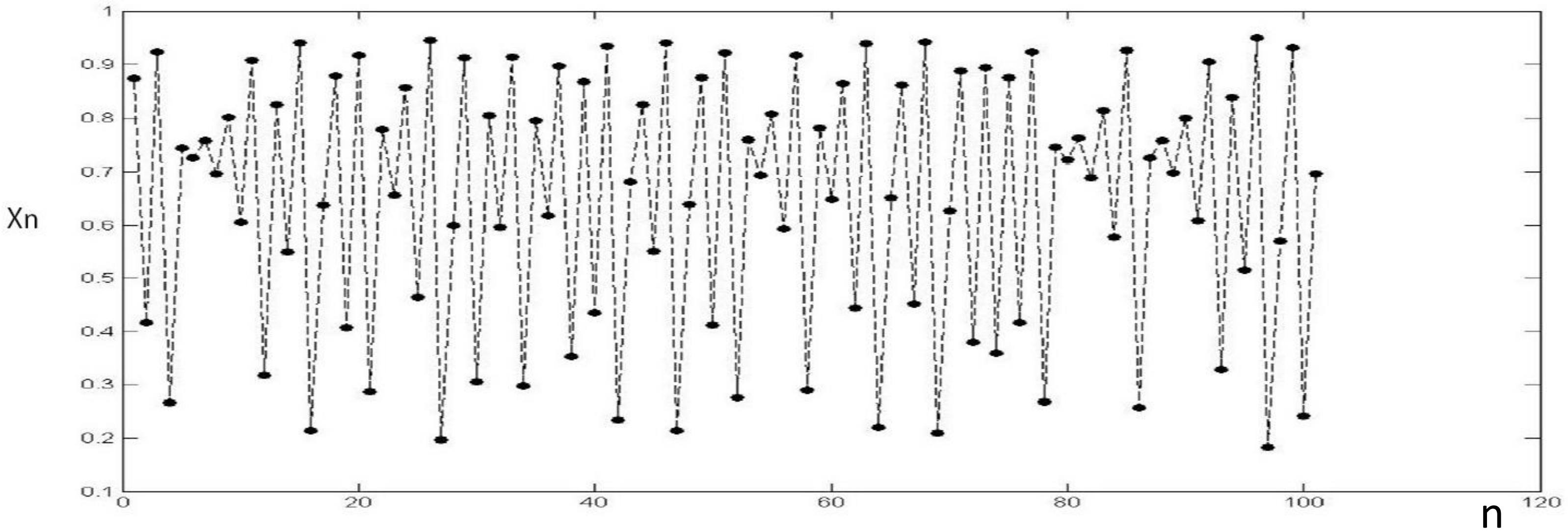
$$\begin{aligned} f^2 &\equiv f \circ f \\ f^3 &\equiv f \circ f \circ f \end{aligned}$$

Example:

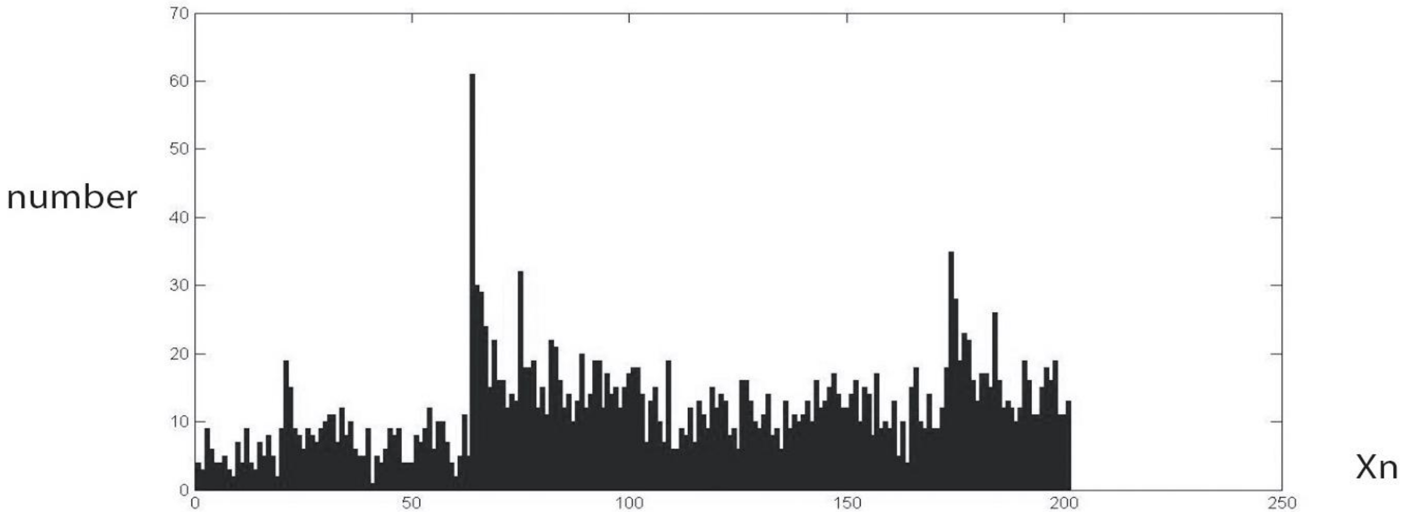
$$\begin{aligned} f(x) &= x^2 - 1; \\ x_0 &= -2 \\ x_1 &= f(x_0) = f(-2) = 3 \\ x_2 &= f(x_1) = f(f(x_0)) = f^2(x_0) = f(3) = 8 \\ x_3 &= f(x_2) = f(f(x_1)) = f(f(f(x_0))) = f^3(x_0) = f(8) = 63 \\ &\vdots \quad \vdots \quad \vdots \end{aligned}$$

Visualizing orbits $\{x_0,\ x_1,\ x_2,\ \ldots\},\quad x_i = f(x_{i-1}) = f^{\,i}(x_0)$

Time series



Histogram



Types of orbits: Periodic orbits, aperiodic orbits

Period 1 $\{-2,\ -2,\ -2,\ -2,\ \ldots\};\qquad f(-2) = -2\quad \text{(fixed point)}$

Period 2 $\{\ 2,\ -1,\ \ 2,\ -1,\ \ 2,\ -1,\ \ 2,\ -1,\ \ldots\};\ \ f(2)=-1,\ f(-1)=2$

Period 3 $\{3,\ -2,\ \ 4,\ \ 3,\ -2,\ \ 4,\ \ 3,\ -2,\ \ 4,\ \ 3,\ \ldots\}$

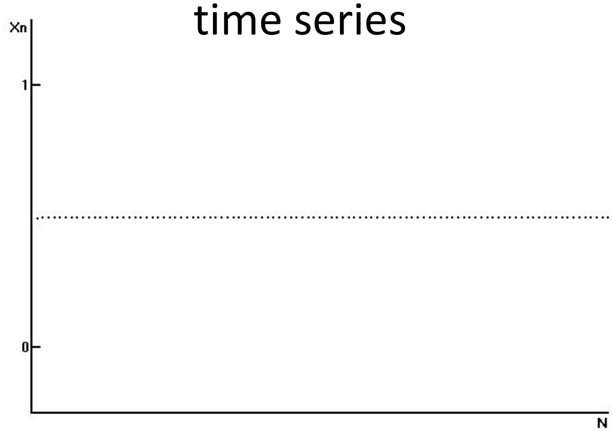
Aperiodic $\{-1.2,\ 2,\ 3.1,\ 5.4,\ -7.3,\ 11,\ 13.5,\ -1.8,\ 5.5,\ 19.2,\ -12,\ 23.5,\ \cdots\}$

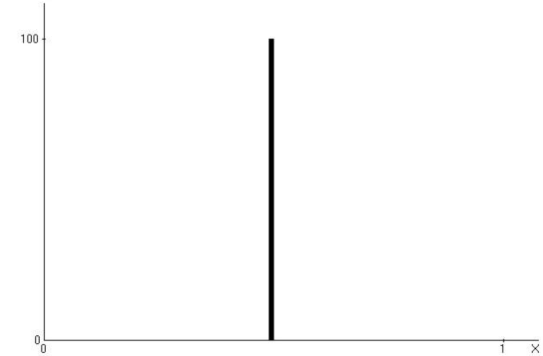
Visualizing orbits

time series

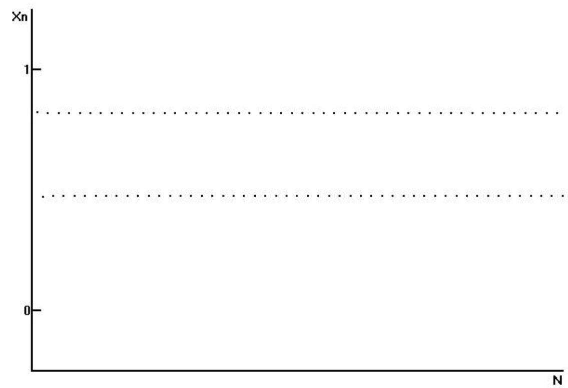
histogram

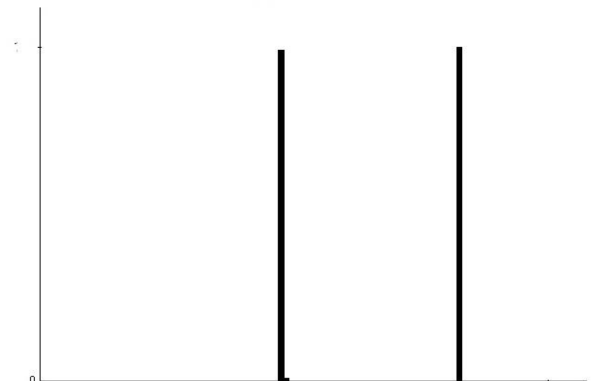
Period 1



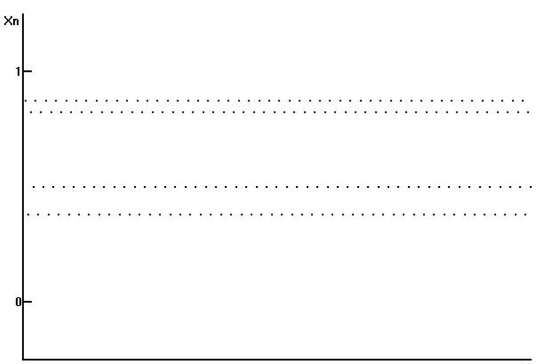


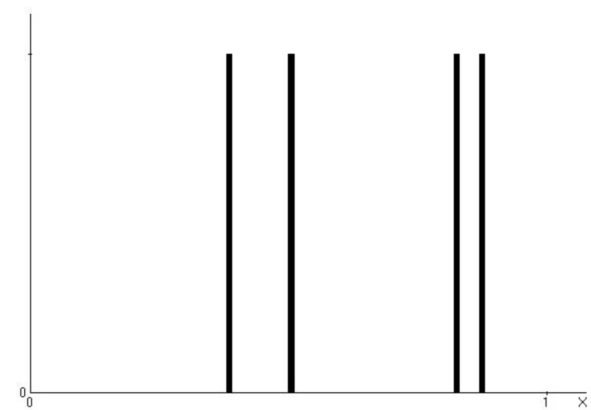
Period 2



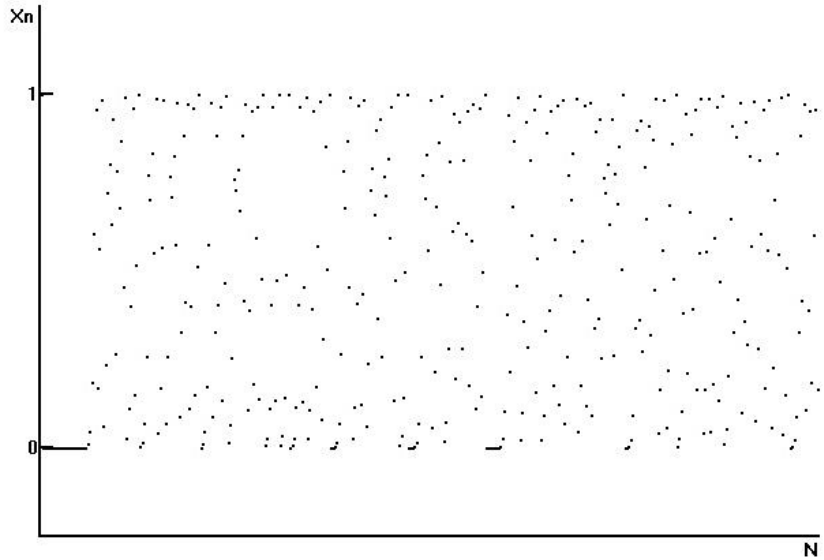


Period 3





Aperiodic orbit
(“ergodic” orbit)



time series

histogram

The logistic equation: $f_a(x) = ax(1-x), \quad 0 \leq a \leq 4$

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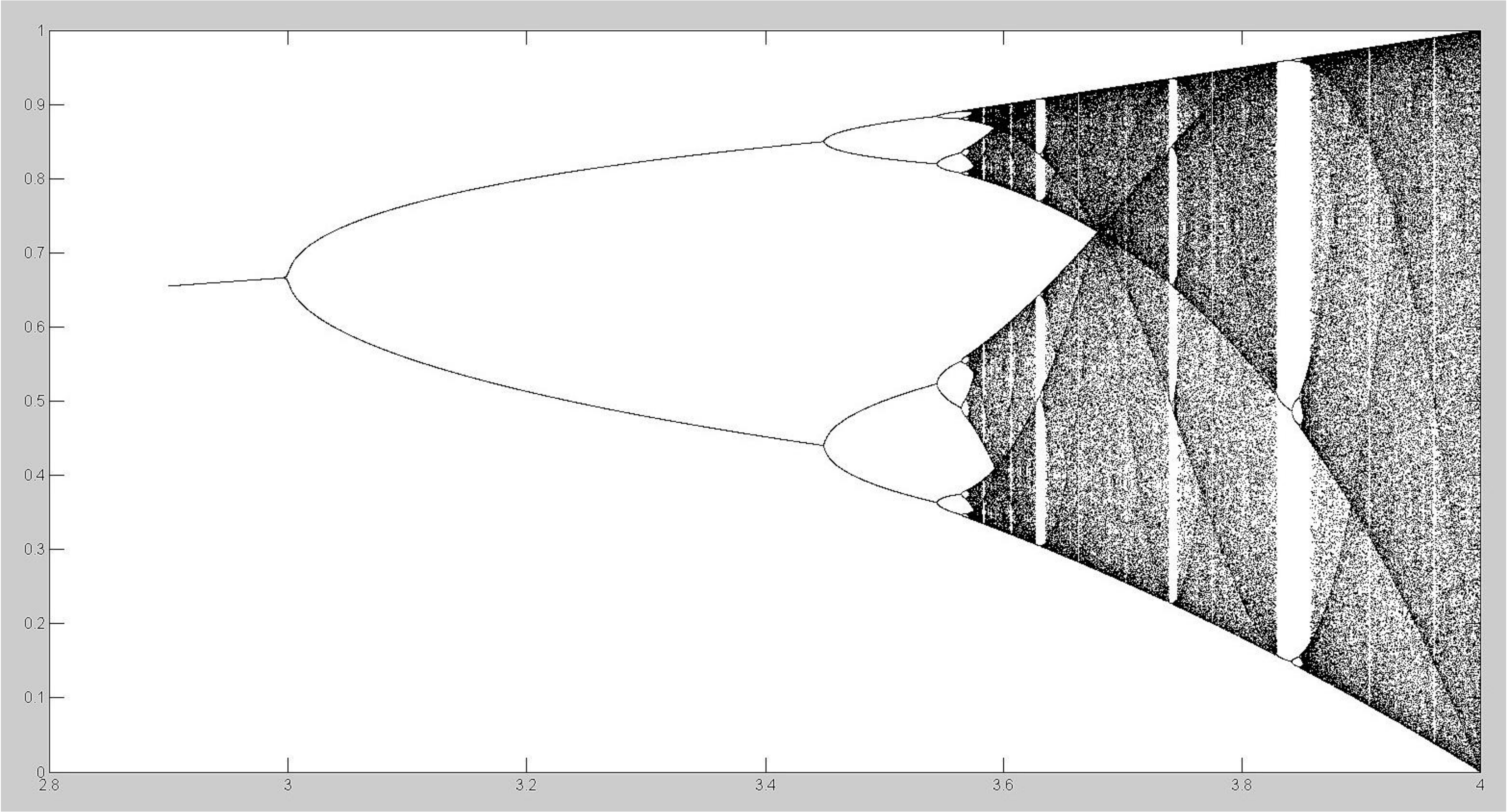
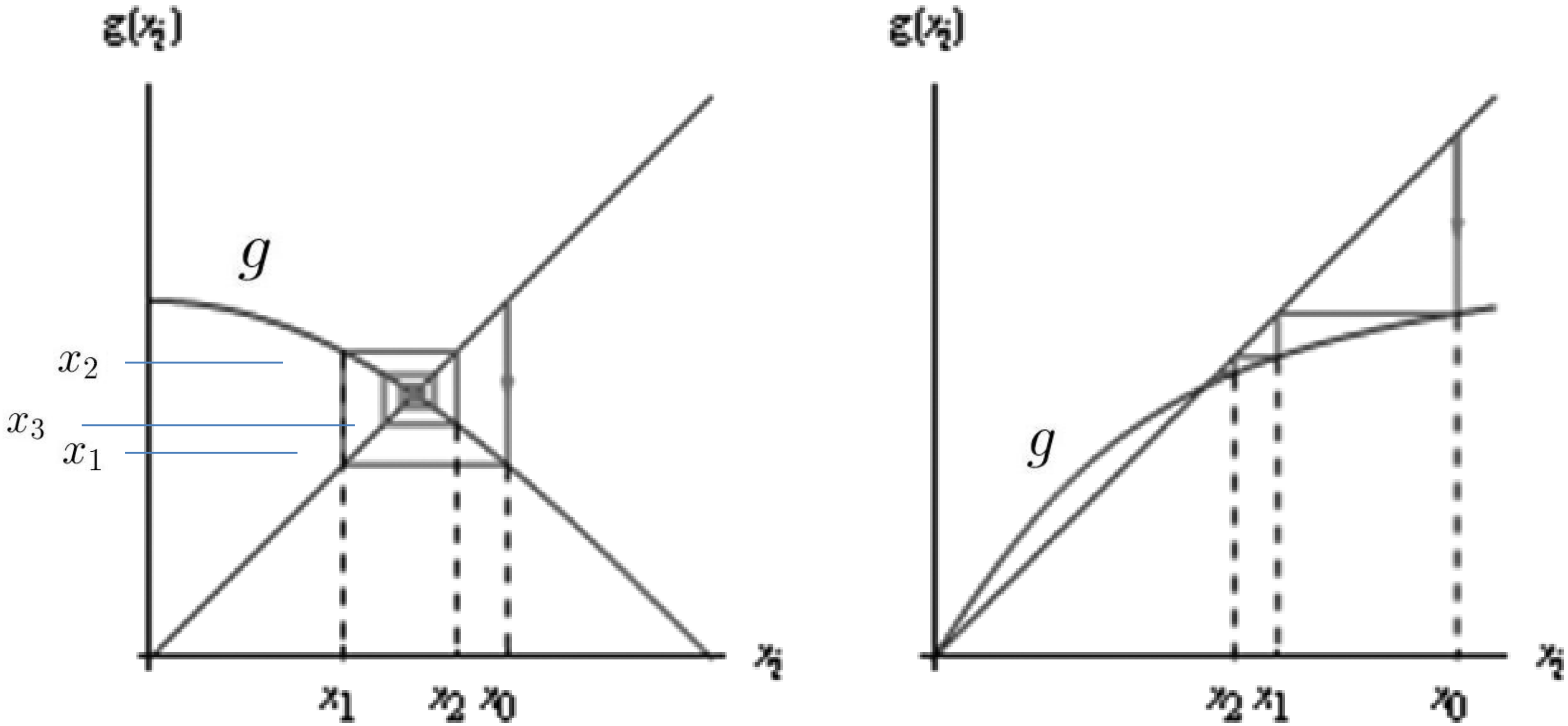
A summary of the dynamics of the logistic equation:



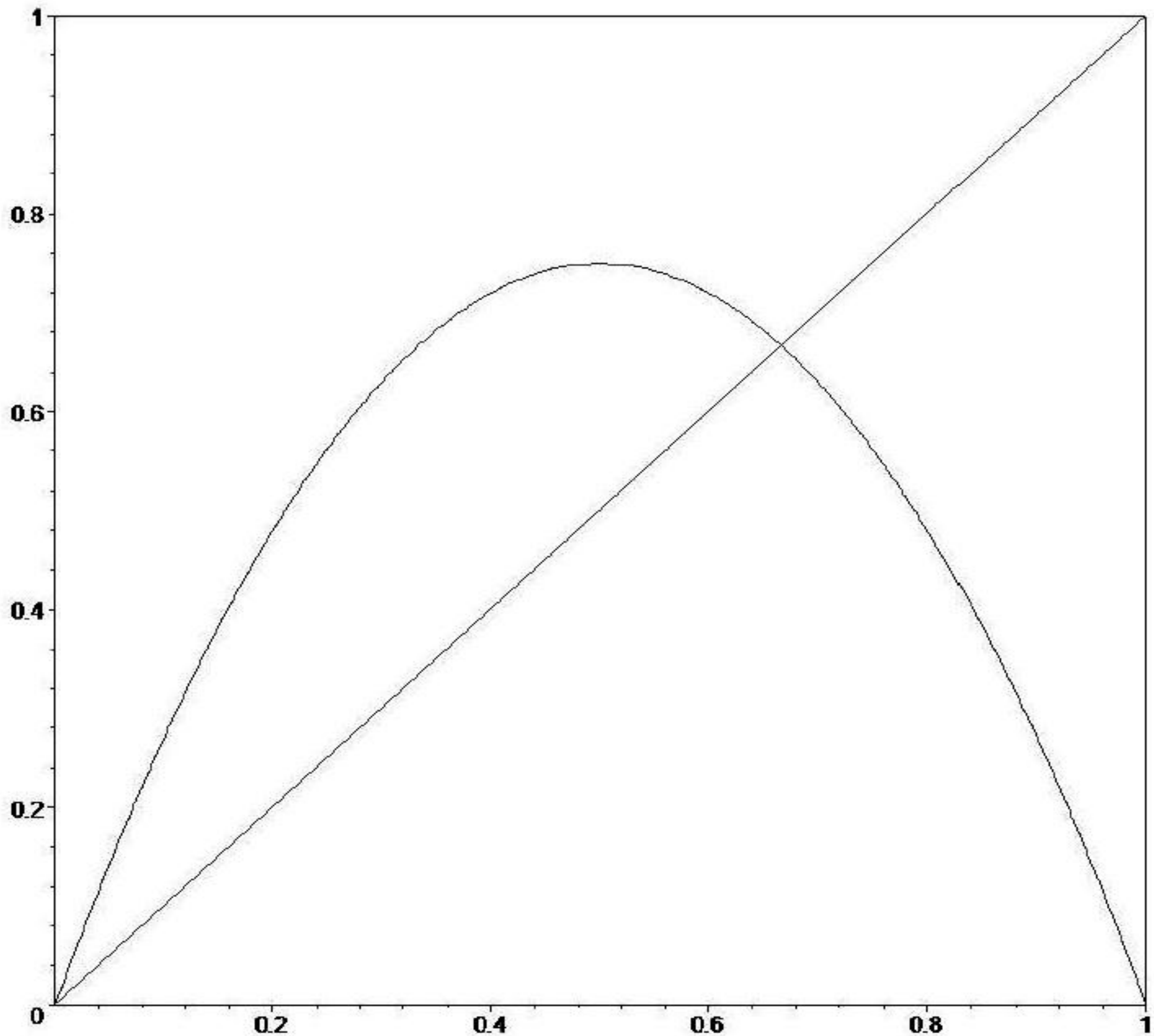
Graphical iteration $x_i = g(x_{i-1})$



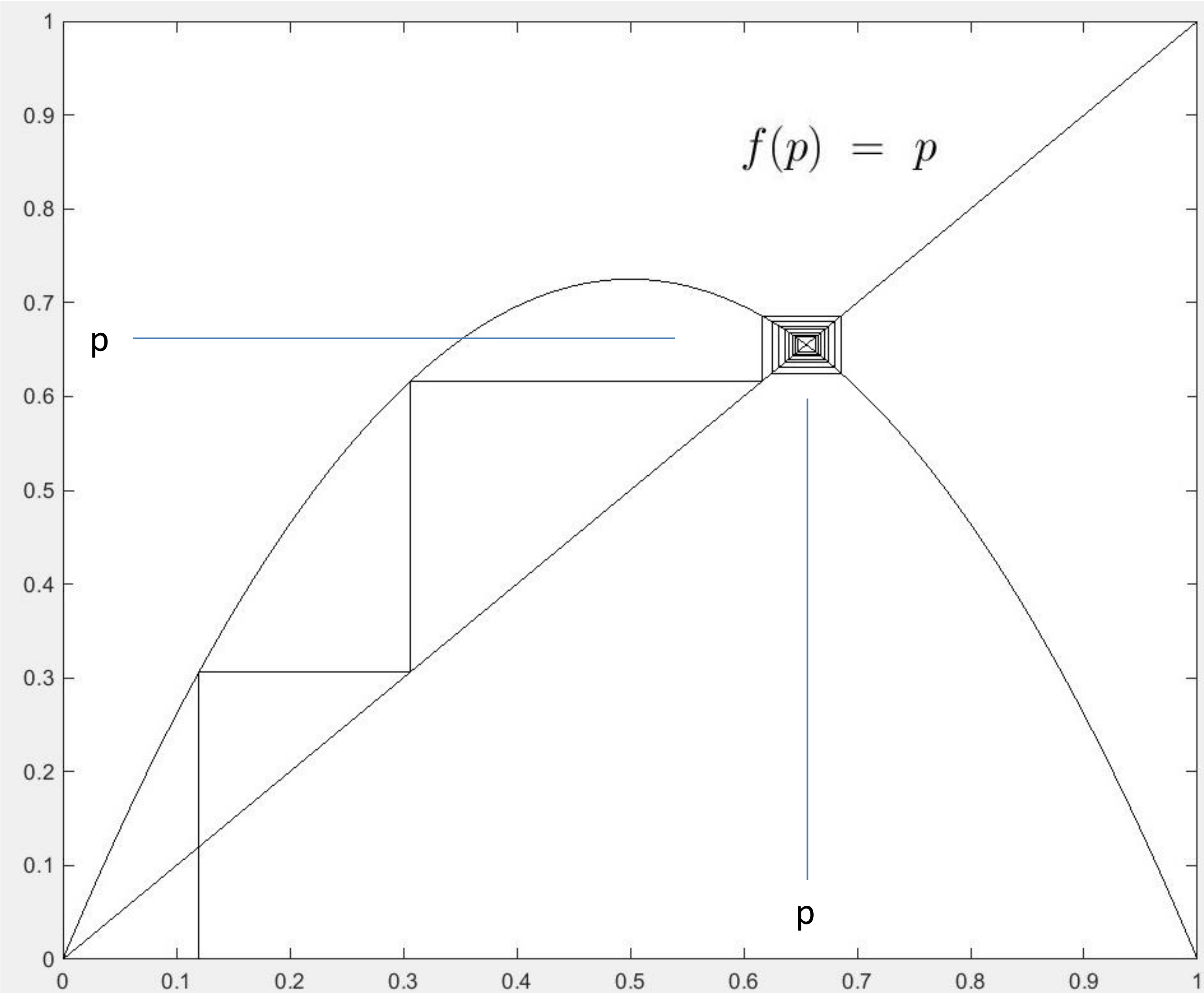
$x_1 = g(x_0) \qquad x_2 = g(x_1) \qquad x_3 = g(x_2)$

Attractive (stable) fixed points! (nearby points are attracted to the fixed point)

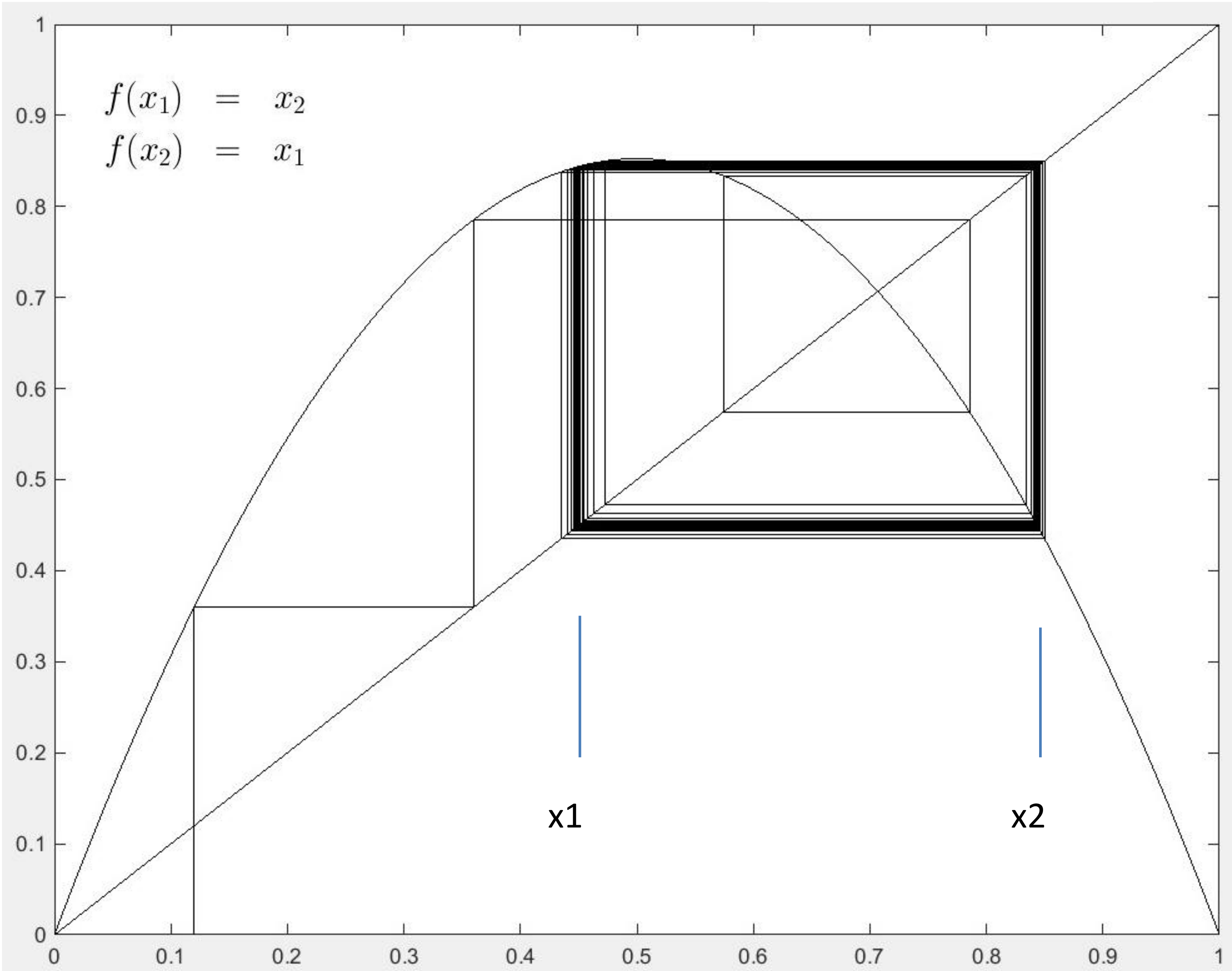
Logistic equation: $f_a(x) = ax(1-x), \quad a \approx 3.2$



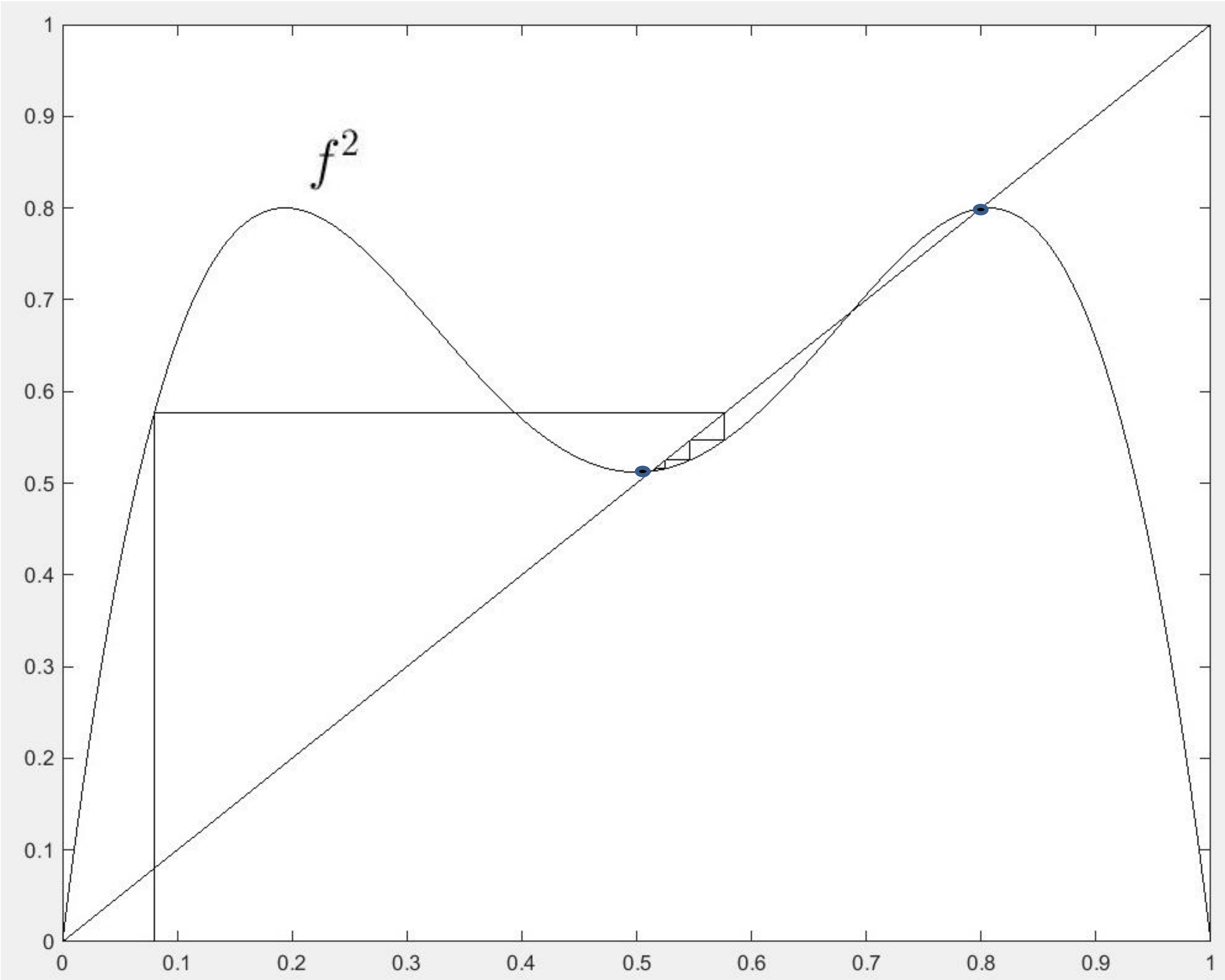
Graphical iteration: attraction to the period 1 orbit



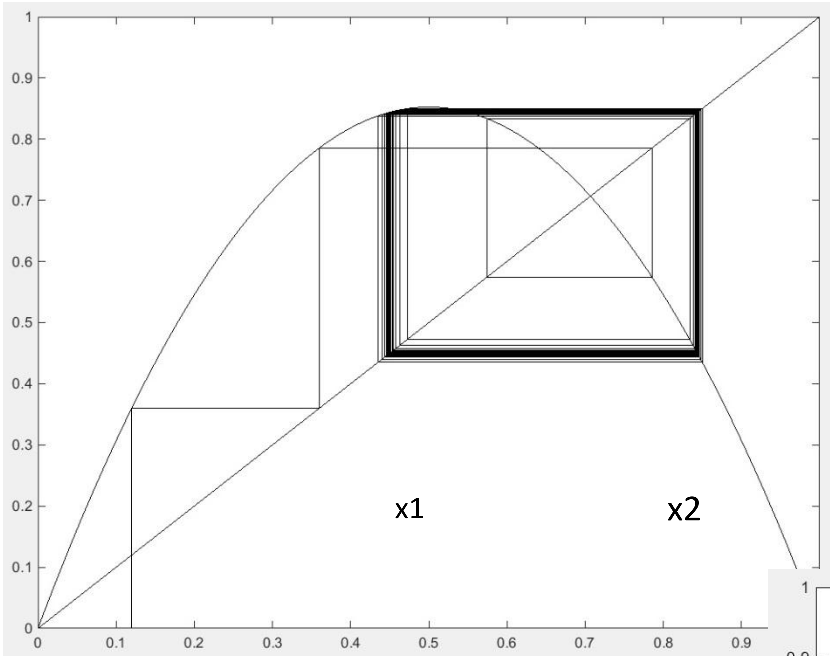
Graphical iteration: attraction to a period 2 orbit $\{x_1, x_2, x_1, x_2, \ldots\}$



Period 2 orbit of f → period 1 orbit of $f^2 \ (= f \circ f)$

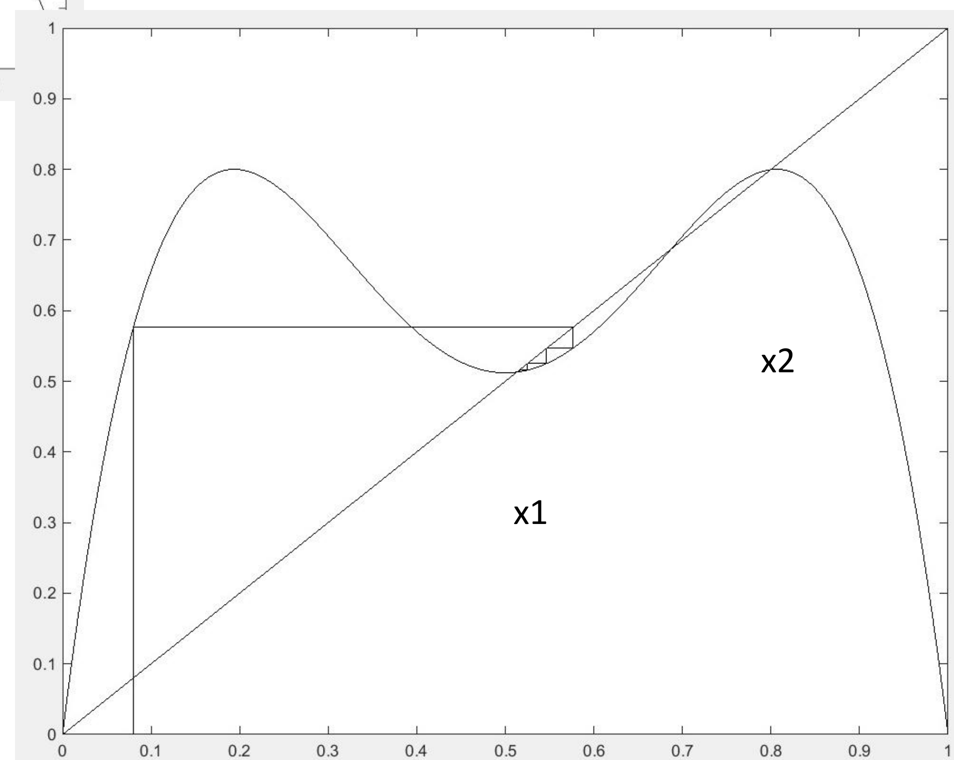


f



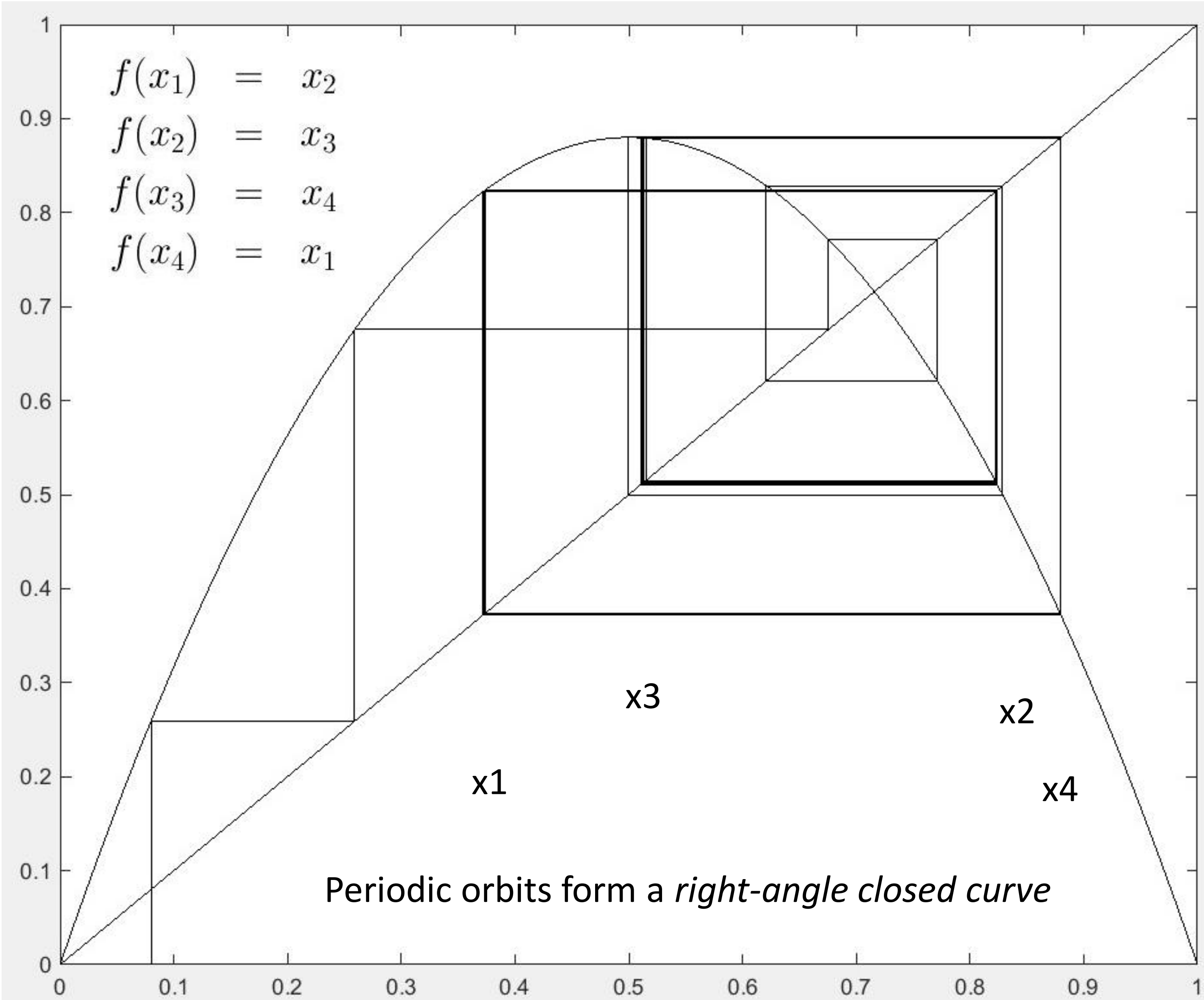
$$f(x_1) = x_2$$
$$f(x_2) = x_1$$

f^2

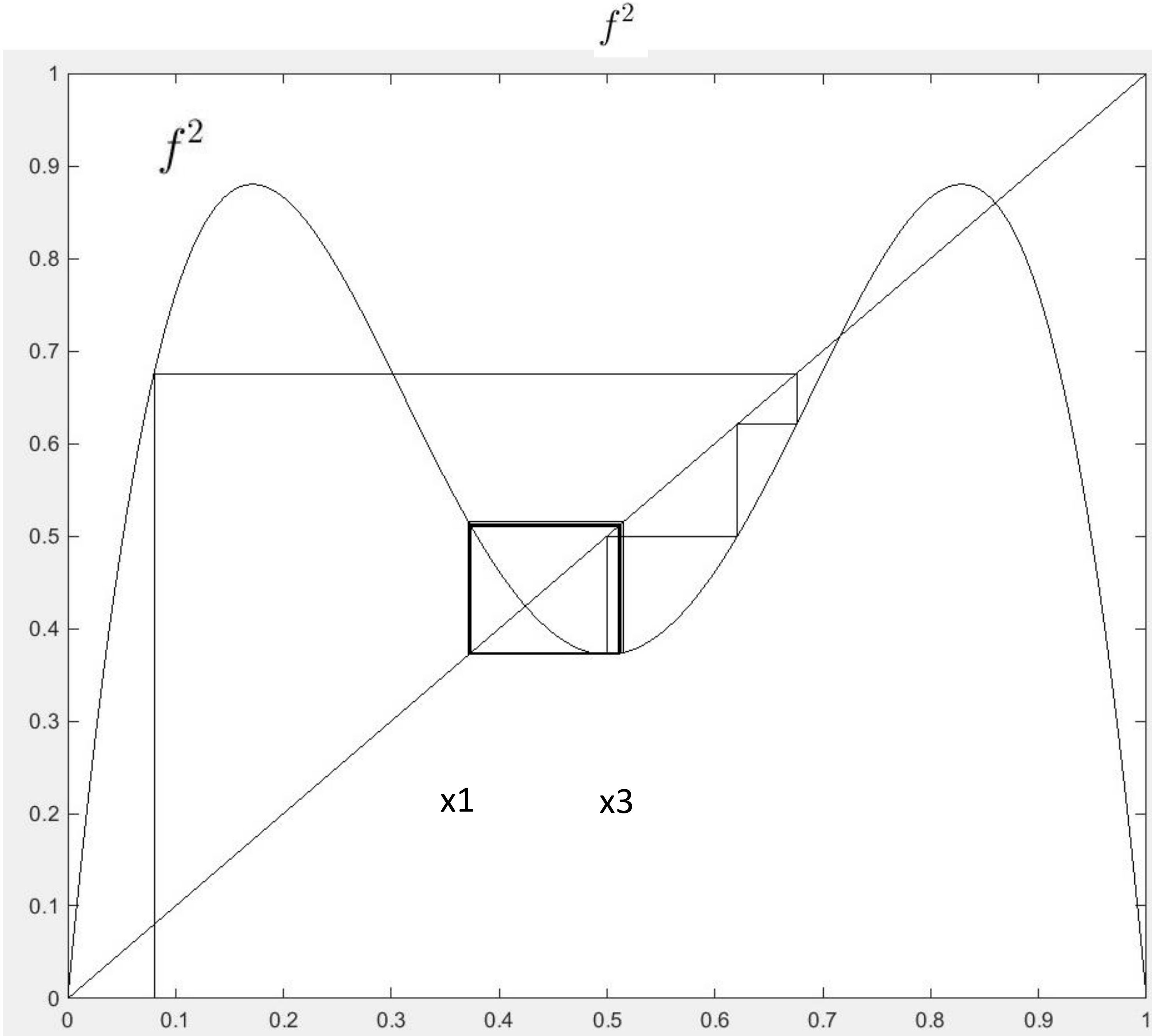


$$f^2(x_1) = x_1$$
$$f^2(x_2) = x_2$$

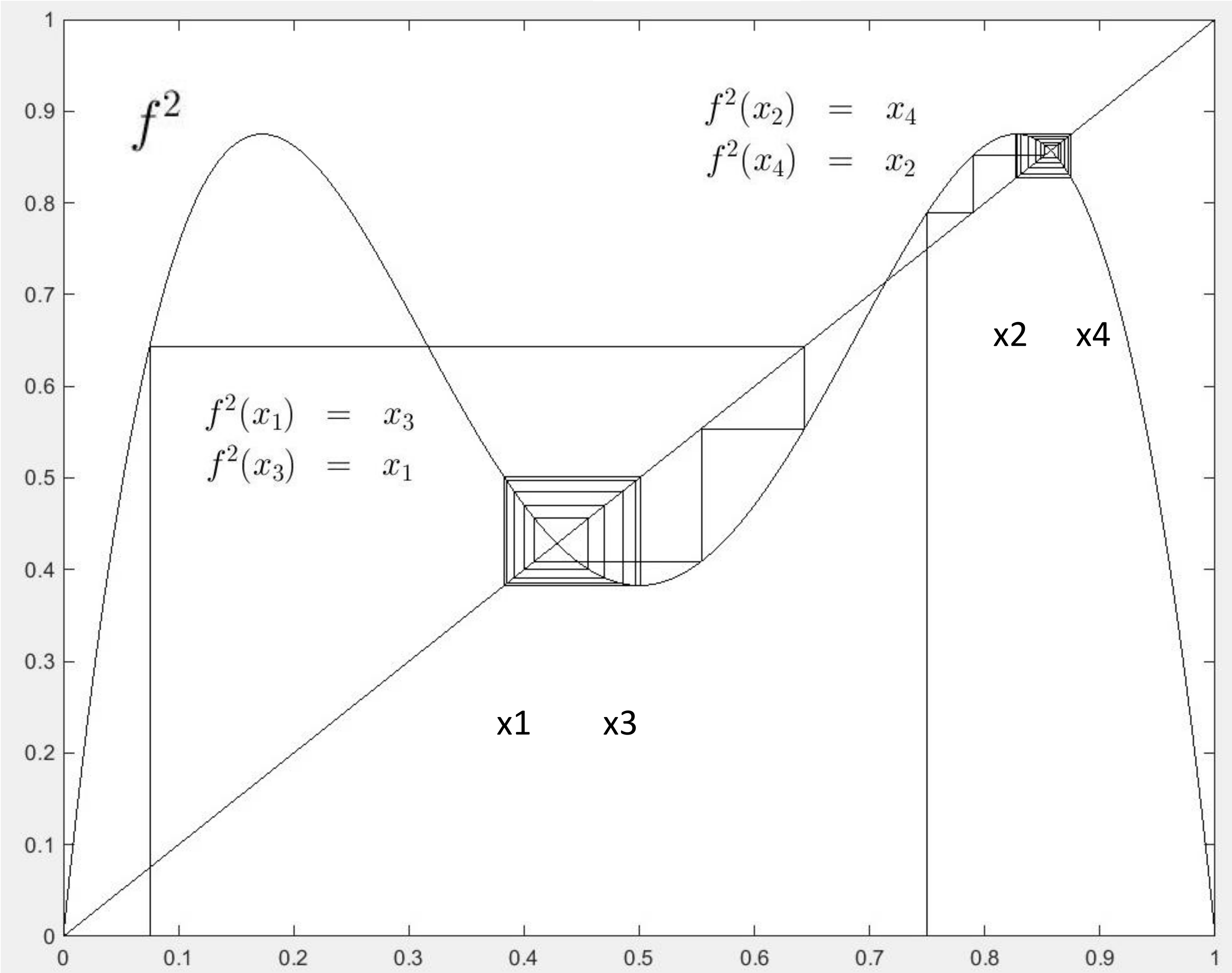
A period 4 orbit of f



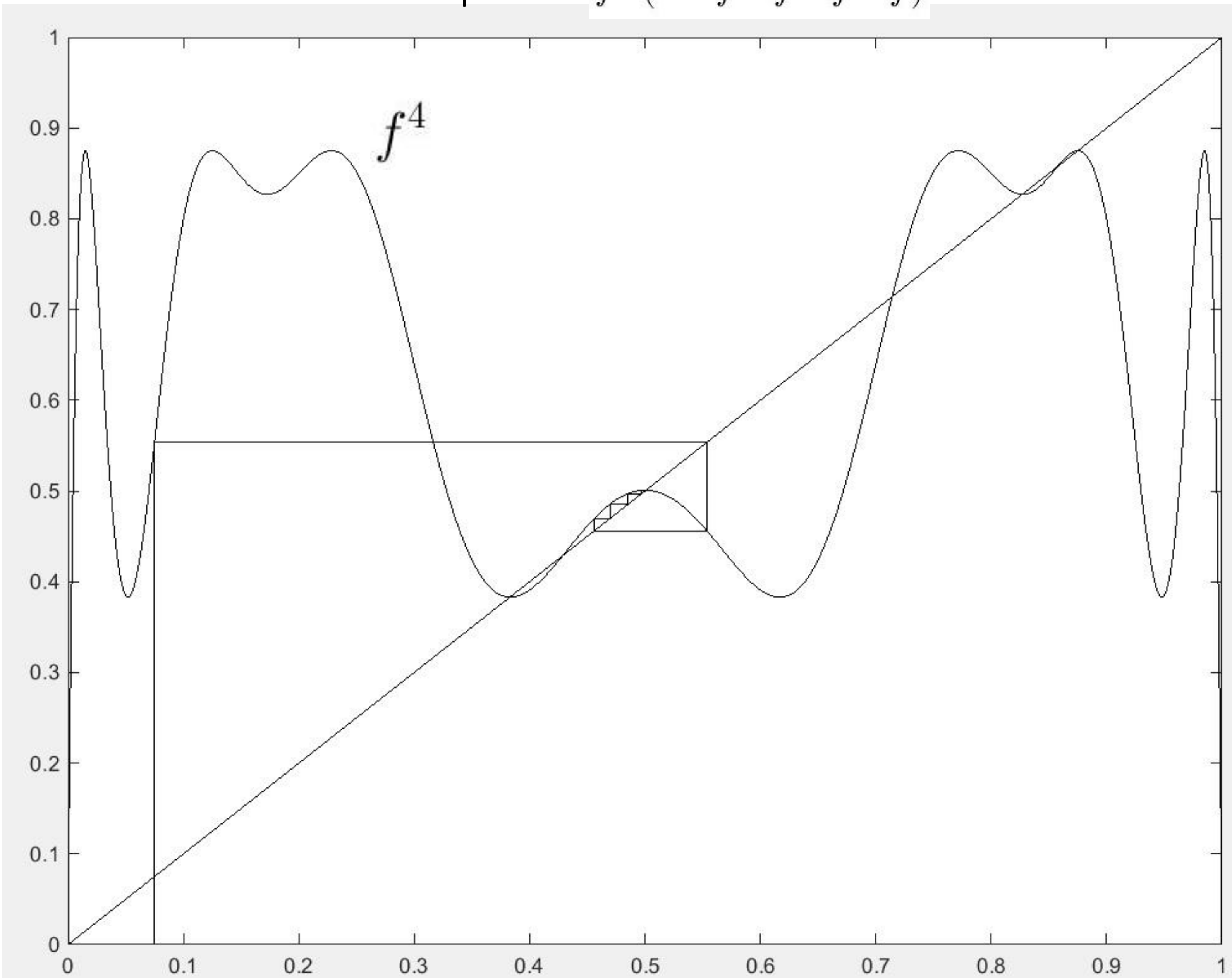
Periodic orbits form a *right-angle closed curve*


f^2
f^2
x1
x3
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

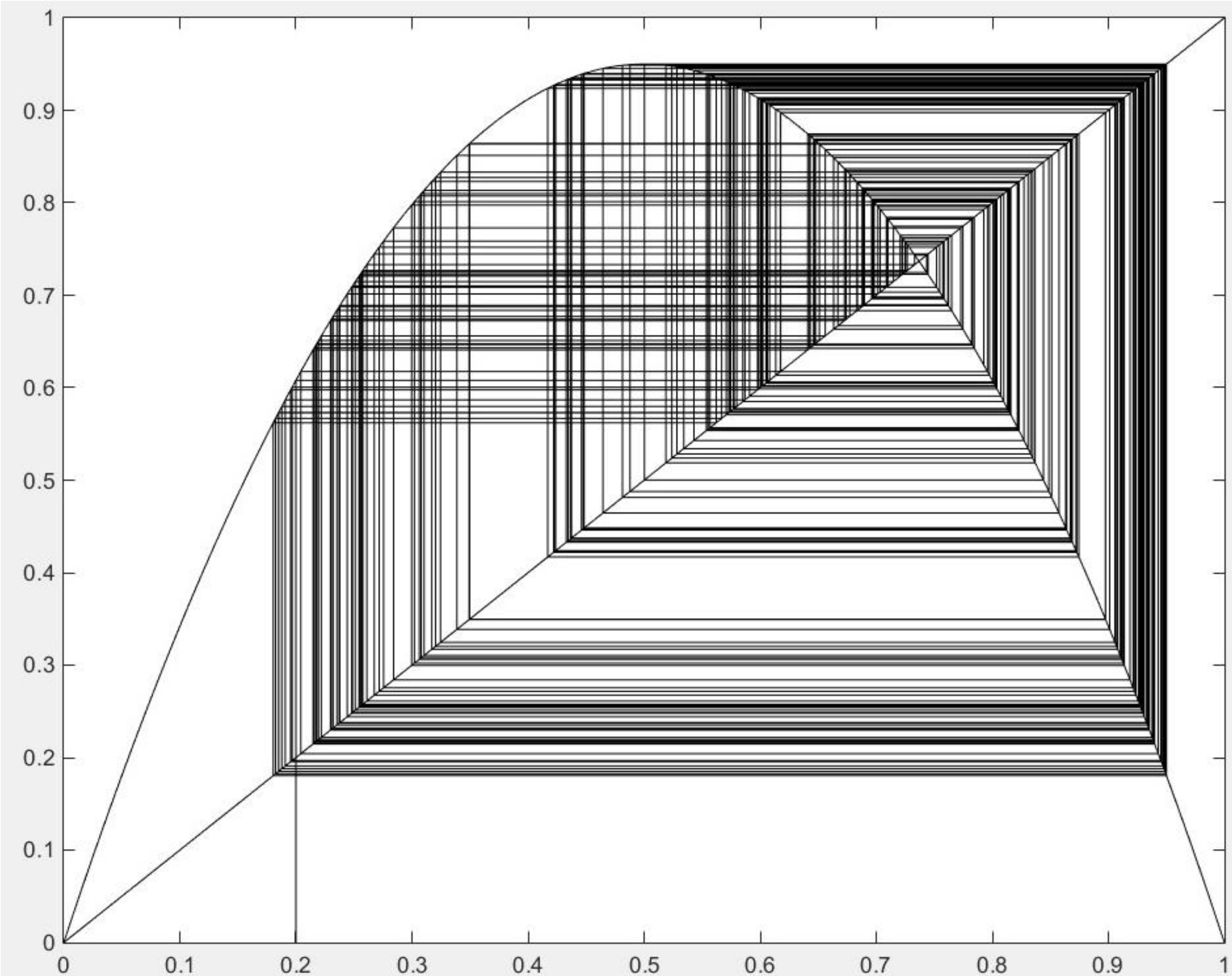
.. is a period 2 orbit of $f^2 \; (= \; f \circ f)$



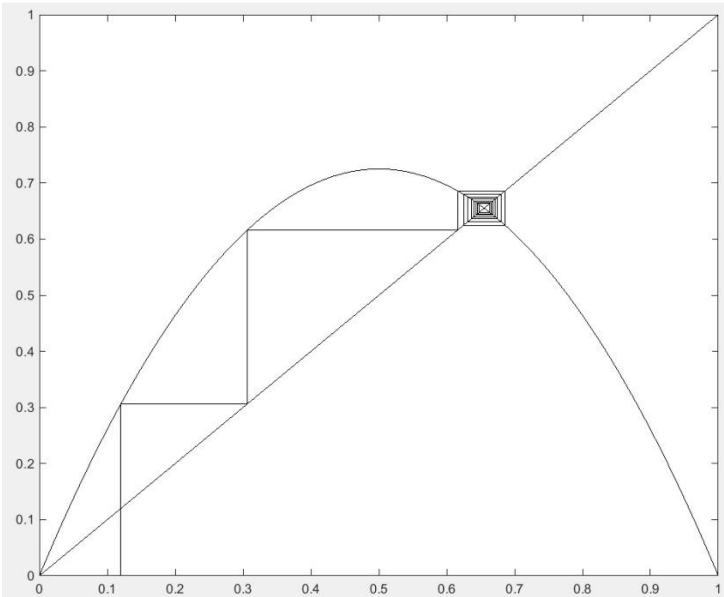
… and a fixed point of $f^4\ (=\ f \circ f \circ f \circ f)$

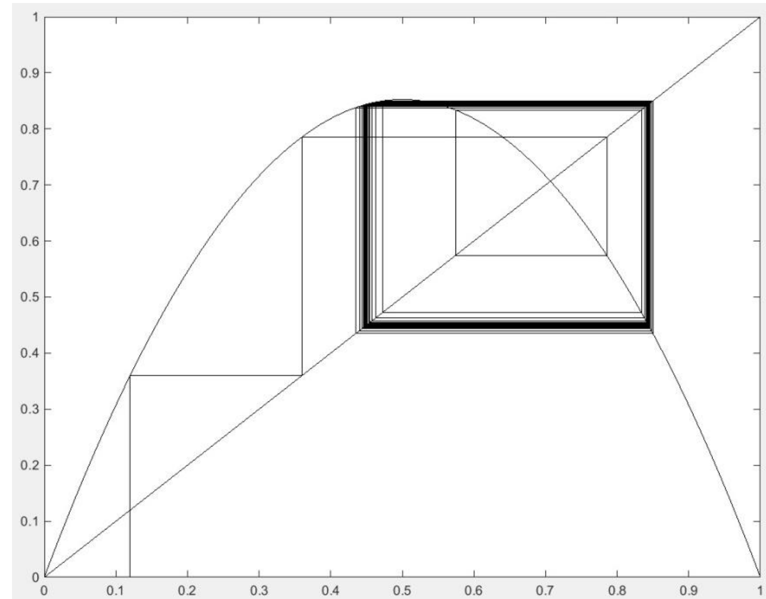


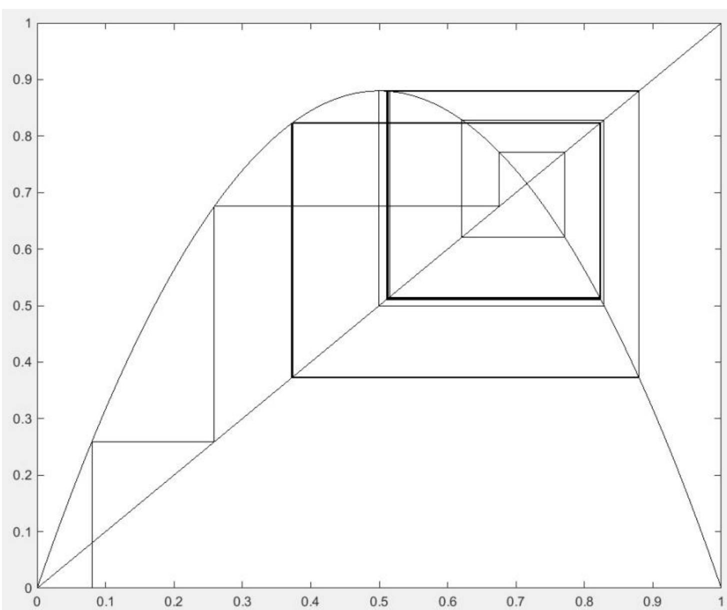
Graphical iteration: aperiodic orbit

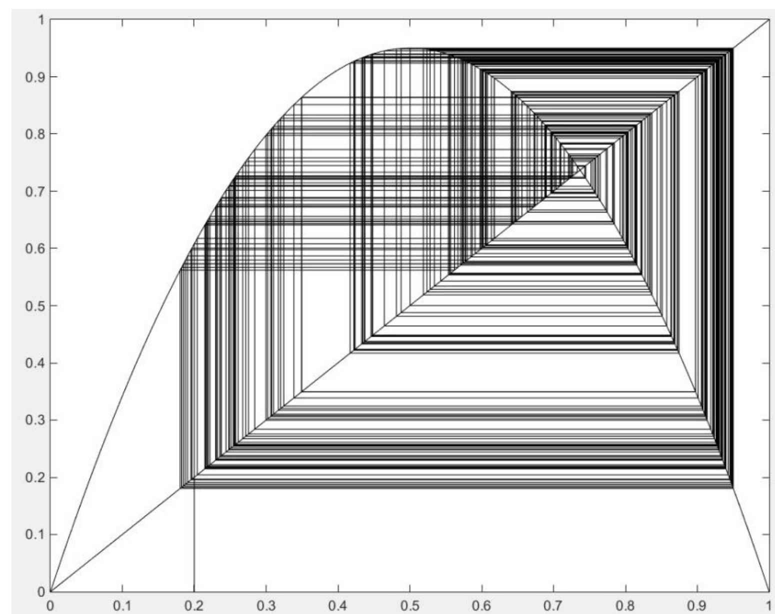


As a varies, the orbital structure of $f_a(x) = ax(1-x)$ changes…..









As a varies, the orbital structure of $f_a(x) = ax(1-x)$ changes.

We say $\bar{a}$ is a **bifurcation point** of f_a if the orbital structure f_a changes at $\bar{a}$

To determine bifurcation points, we can try to find periodic points analytically…

A **bifurcation curve** is a plot of the periodic points p as a function of *a*; p(*a*).
Plotting the bifurcation curves on the *a*-x plane we obtain a **bifurcation diagram**.

We begin by finding the periodic points of the logistic equation. For fixed points (period 1), we solve $ax(1-x) = x \;\rightarrow\; ax^2 + (1-a)x = 0$. The solutions of this are $x = 0$ and $x = (a-1)/a$.

For period 2 points we solve $f_a^2(x) = x$;

$$a[ax(1-x)](1-[ax(1-x)]) = x$$
$$a^3x^4 - 2a^3x^3 + a^2(a+1)x^2 + (1-a^2)x = 0$$

We know $x = 0$ is a solution as well as $x = (a-1)/a$ (the fixed points). Factoring these terms out of the equation we obtain

$$a^2x^2 - a(1+a)x + 1 + a = 0$$

Applying the quadratic formula to this we find that the roots are given by

$$\frac{(1+a) \pm \sqrt{(a+1)(a-3)}}{2a}$$

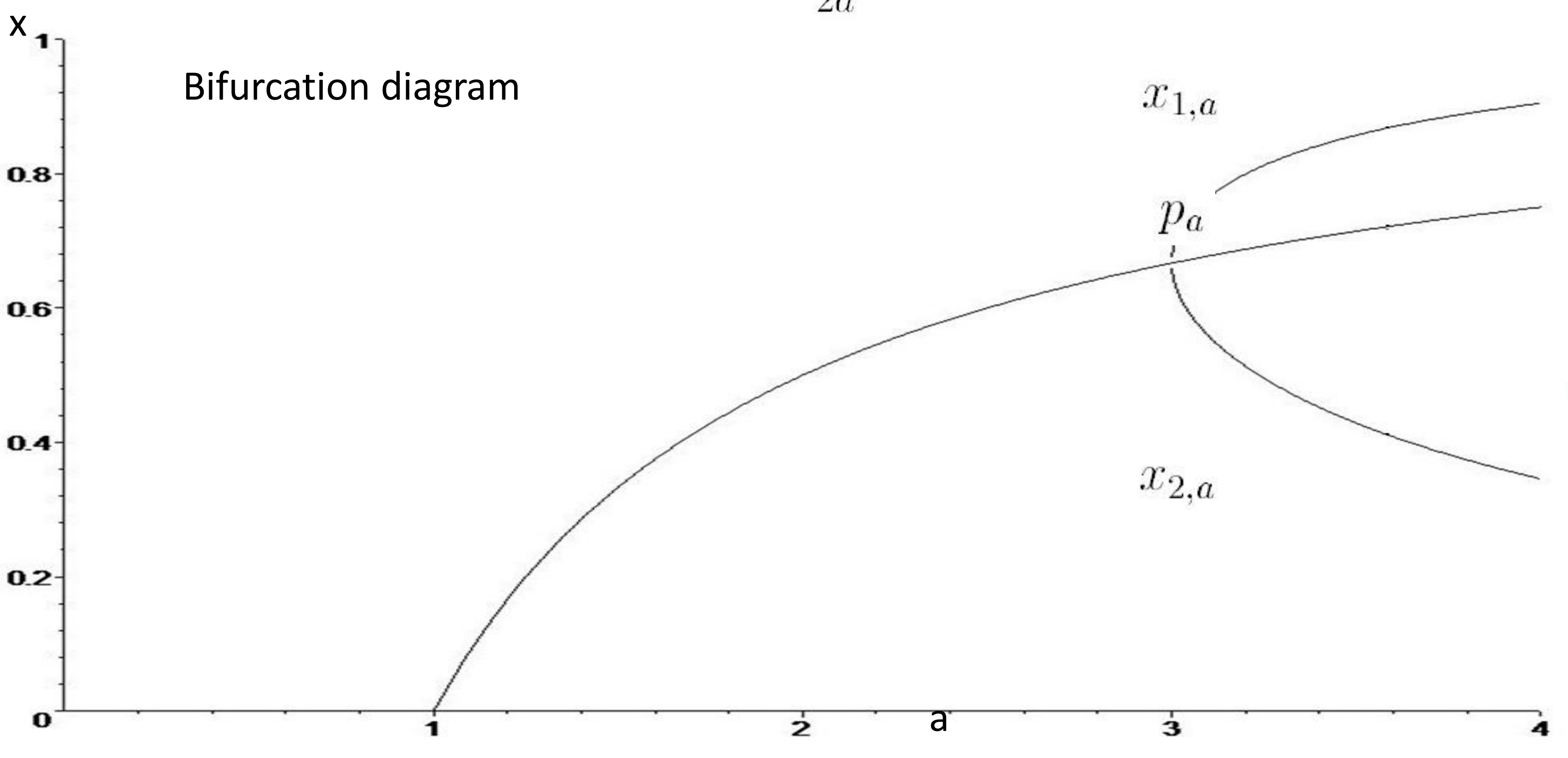
Note that these period 2 points occur only when a>=3

Here is a plot of the bifurcation curves for the period 1 and period 2 orbits (ignoring their stability types);

$$p_a = \frac{a-1}{a}$$

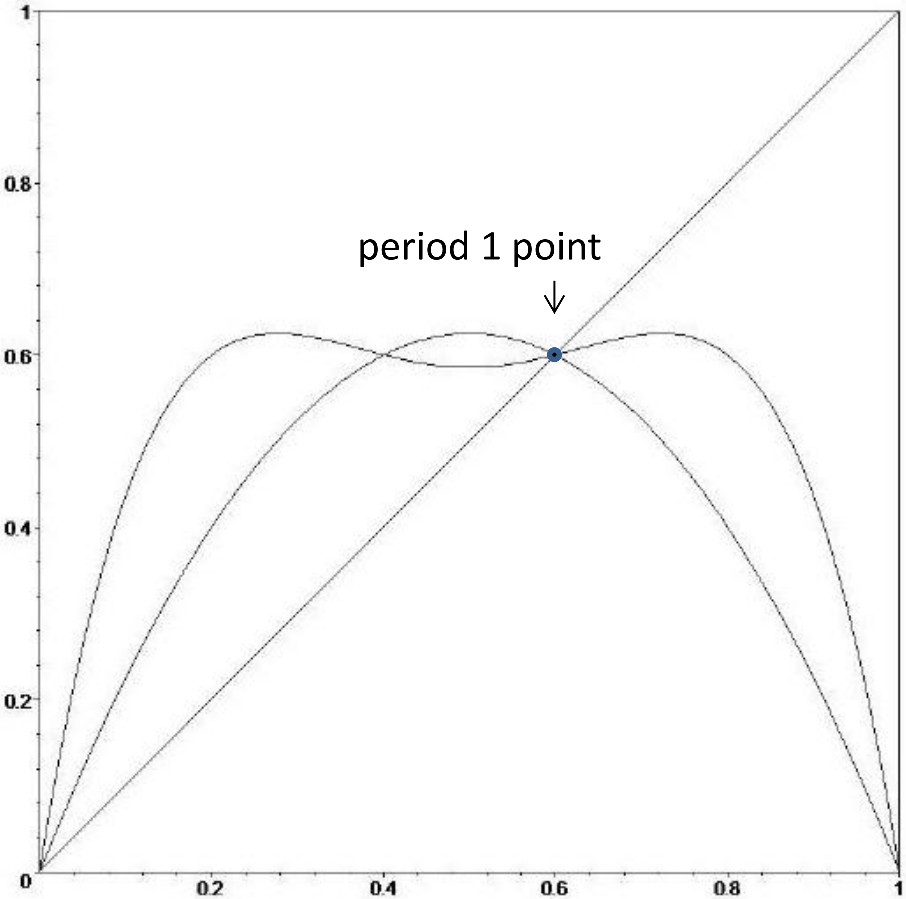
$$x_{1,a} = \frac{a+1+\sqrt{(a+1)(a-3)}}{2a}$$

$$x_{2,a} = \frac{a+1-\sqrt{(a+1)(a-3)}}{2a}$$

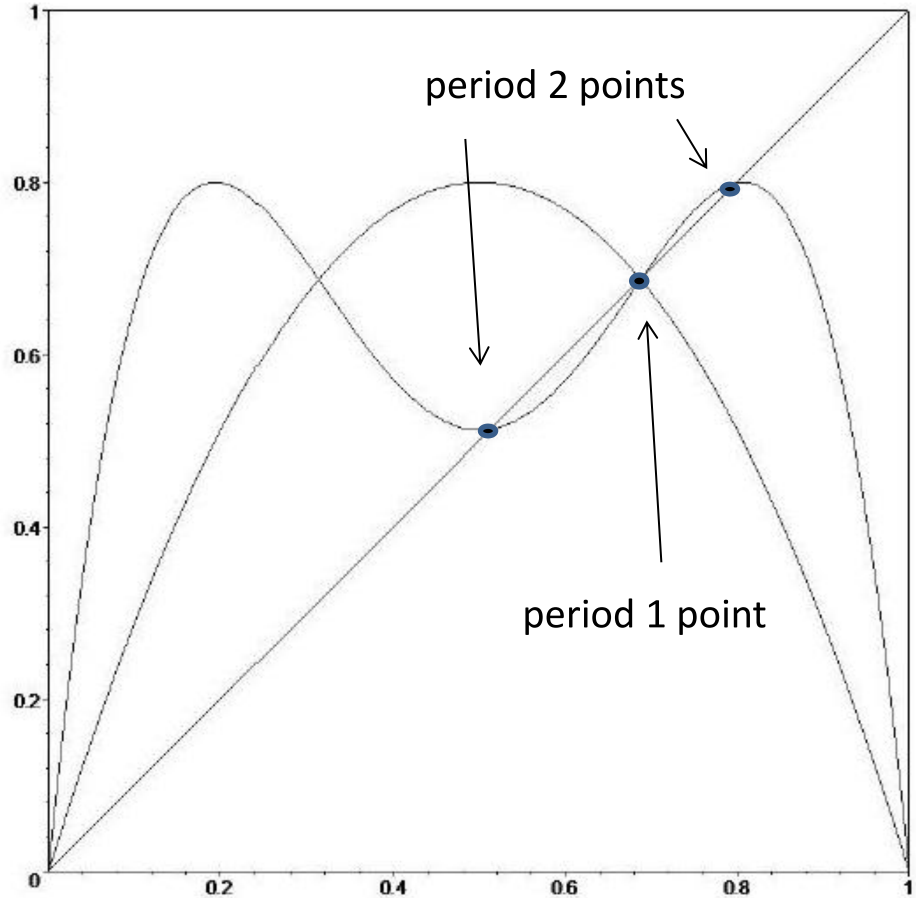


Graphical analysis of the period doubling bifurcation.

Period 2 orbit appears at *a = 3*

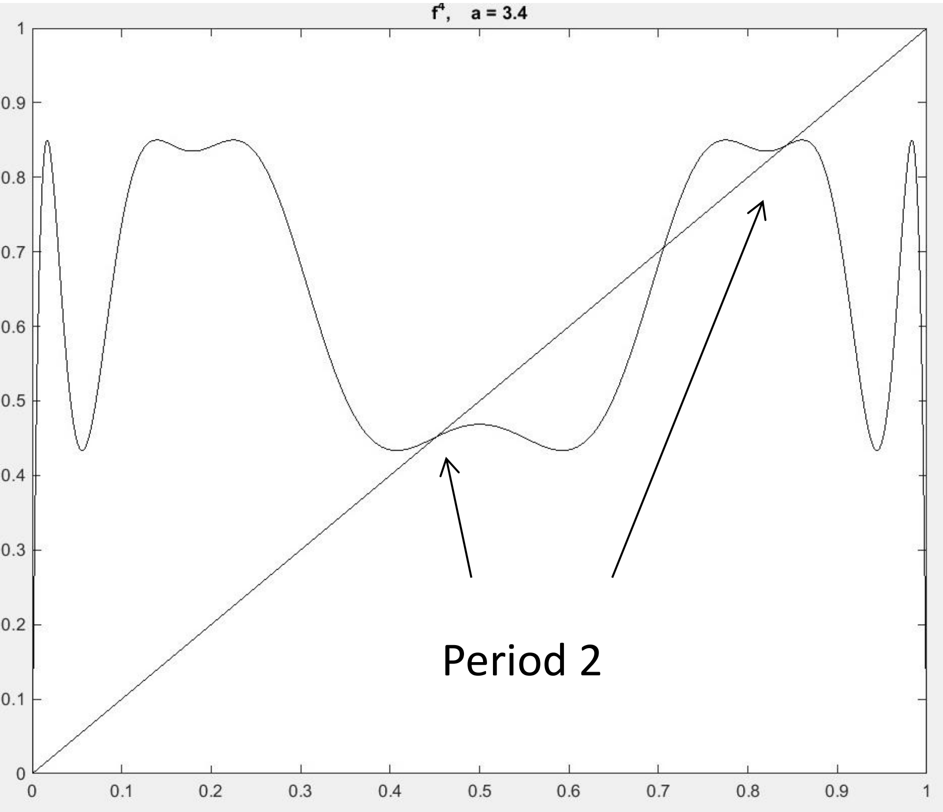


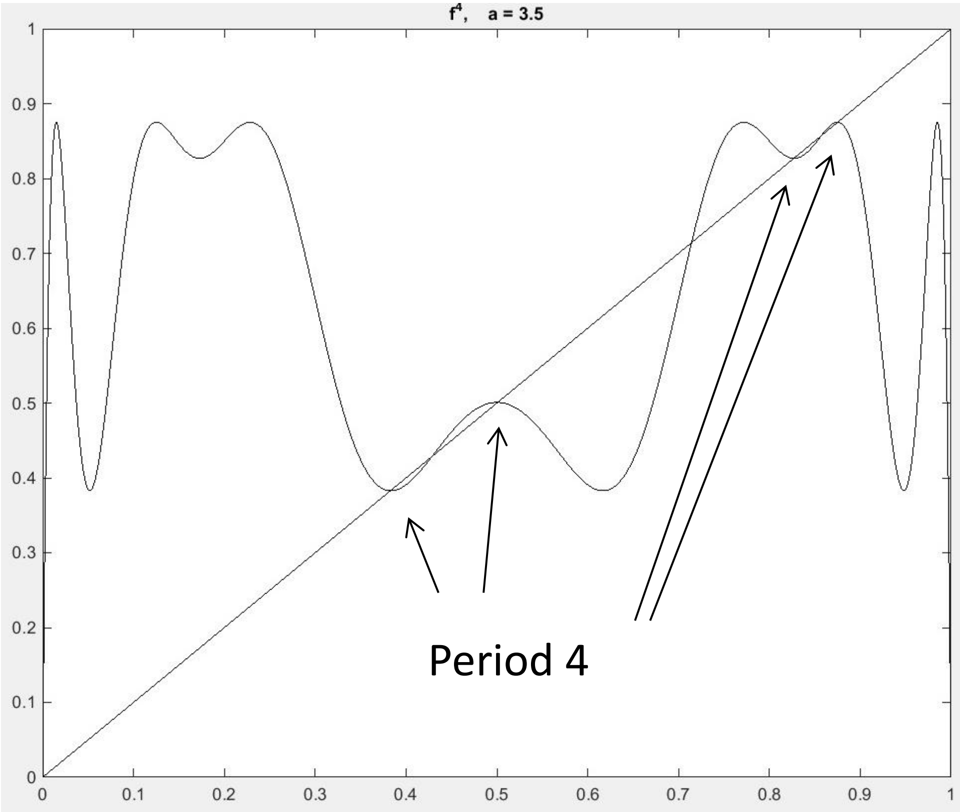
$f_a(x)$ and $f_a^2(x)$ for $a = 2.5$



and for $a = 3.3$

And similarly, the period 2 orbit bifurcates into a period 4 orbit….





a = 3.4

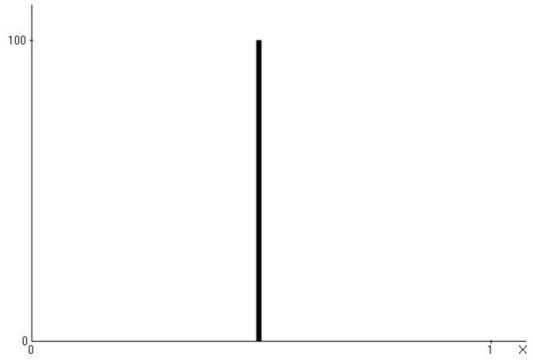
f^4

a = 3.5

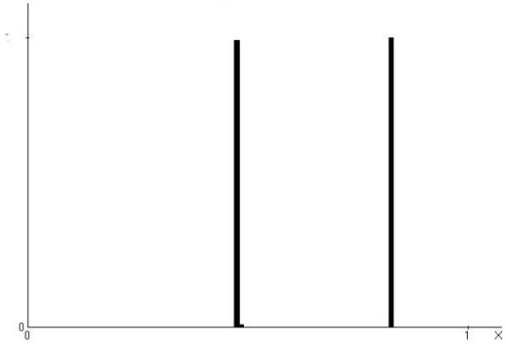
Another way to obtain a kind of bifurcation diagram is to look at the orbits numerically for various values of *a* and try to identify periodic orbits….. **Final State Diagram**

Here's what we do: Choose an *a*. Then numerically plot the orbit starting at some point *x0*. Throw away the first 1000 points in the orbit and then plot the next 1000. If there is a (stable) periodic point then the last 1000 points will settle in on it.

Here's what the histograms look like;



Period 1



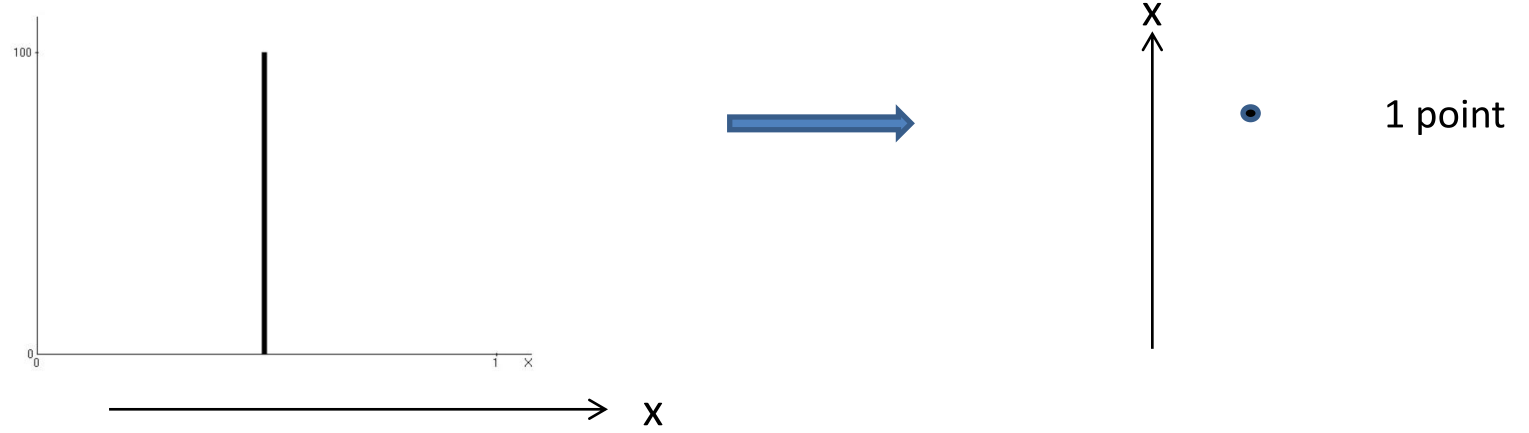
Period 2

Period 4

Now look down on these, from above.

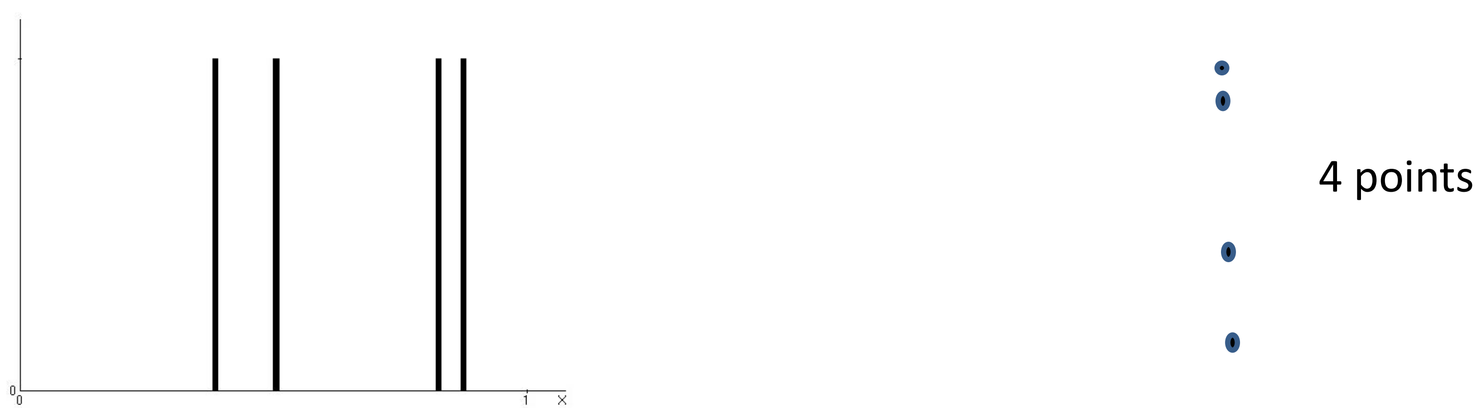
Here's what you see:

Period 1

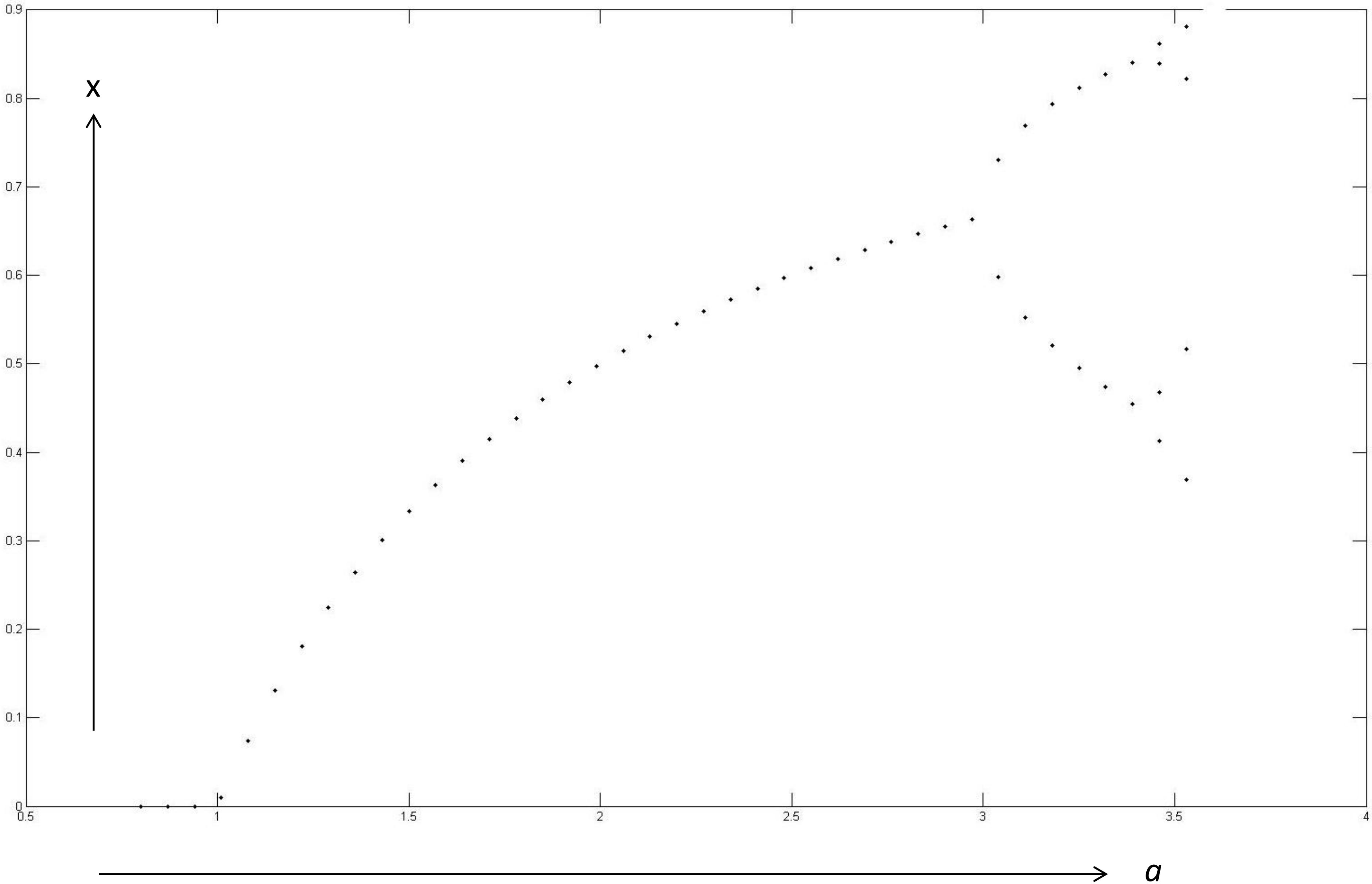


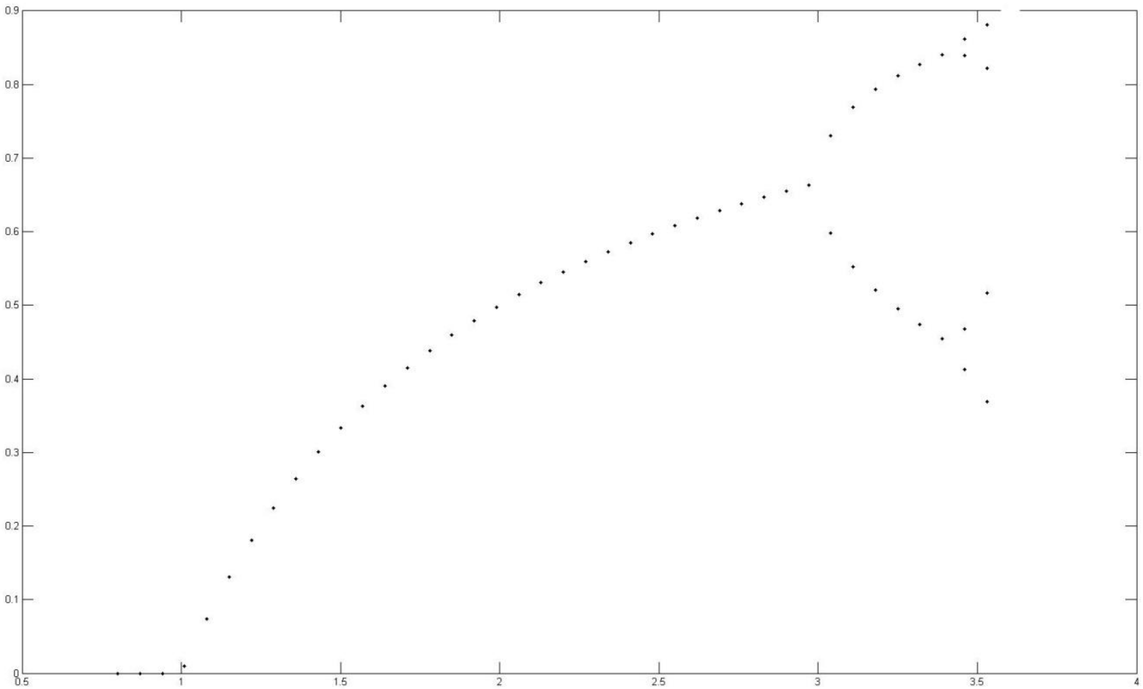
Period 2

Period 4

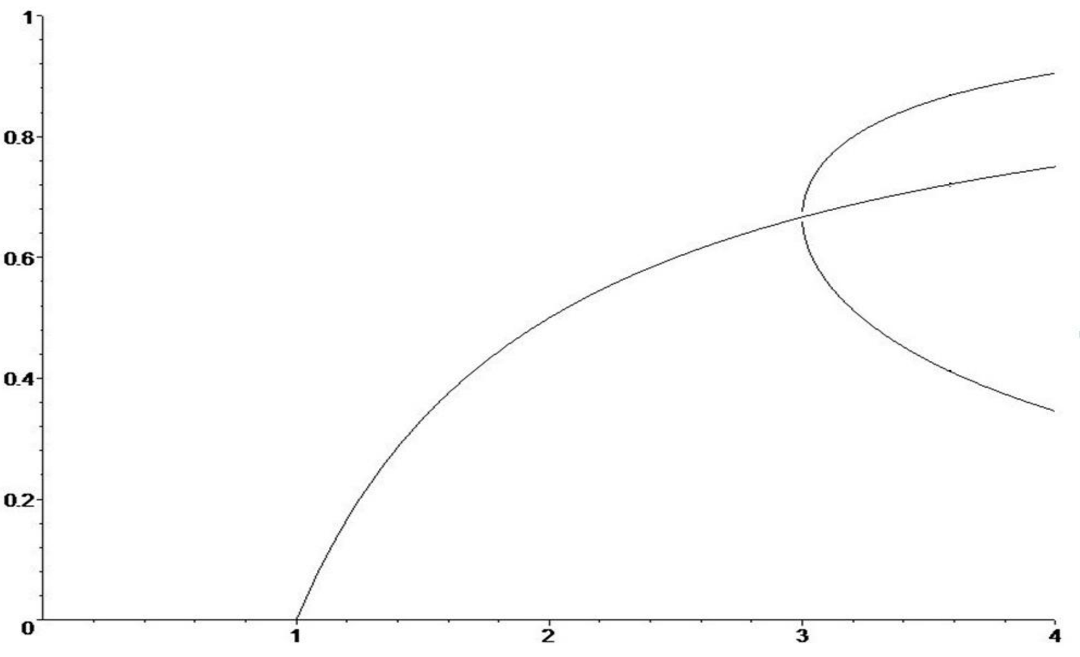


Line these up, along the *a*-axis



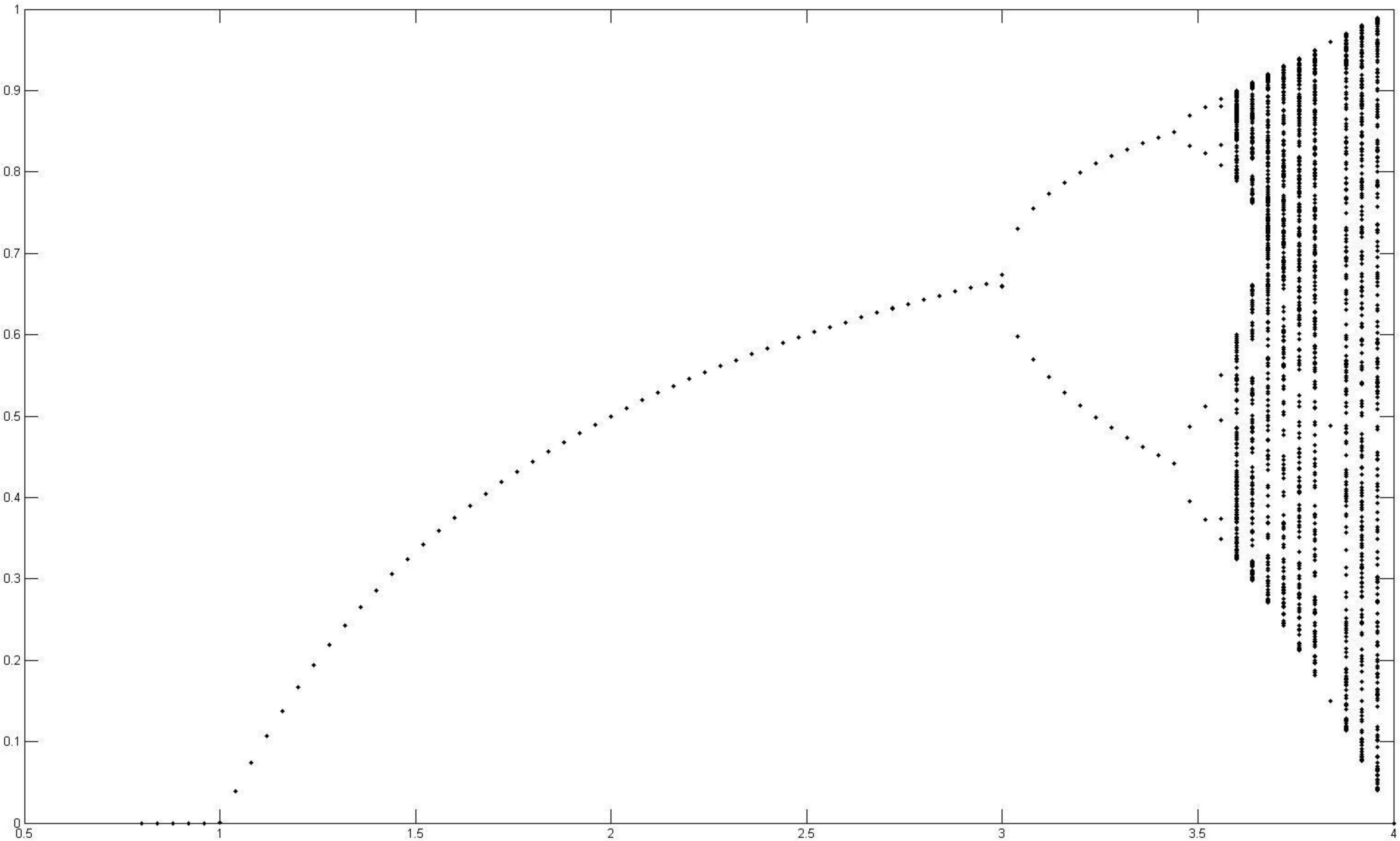


Final state diagram

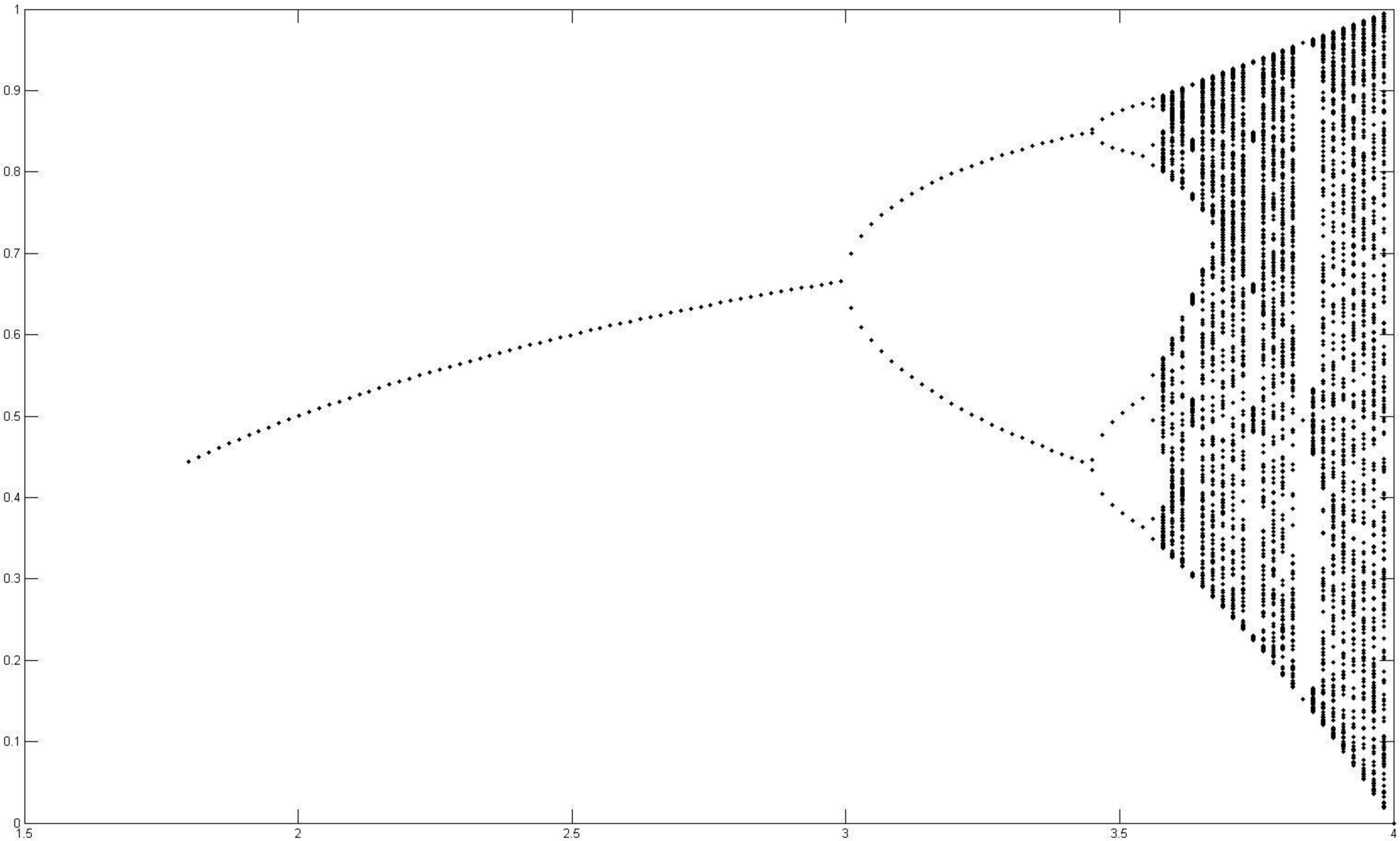


Bifurcation diagram

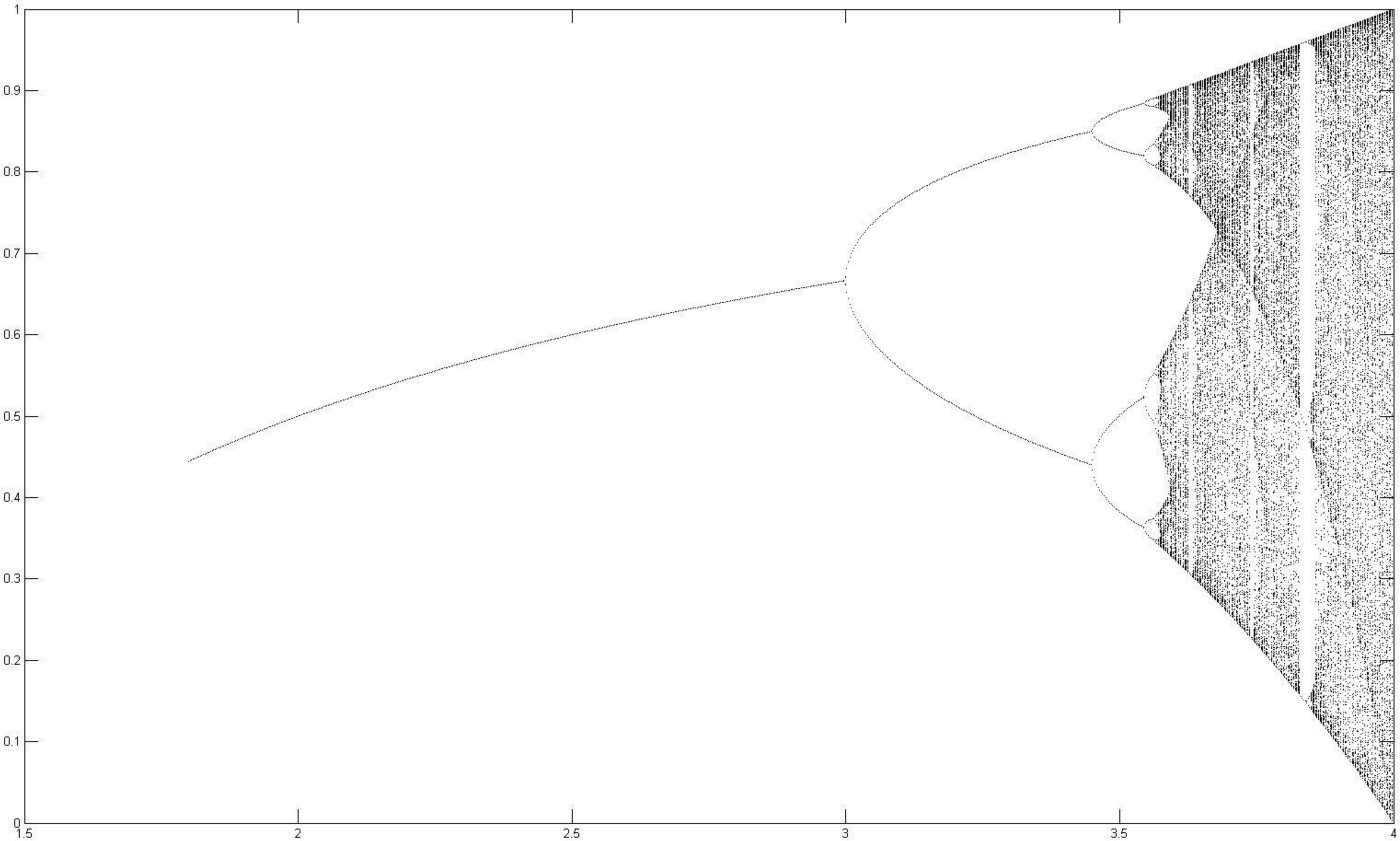
Let’s do it for more *a* values;



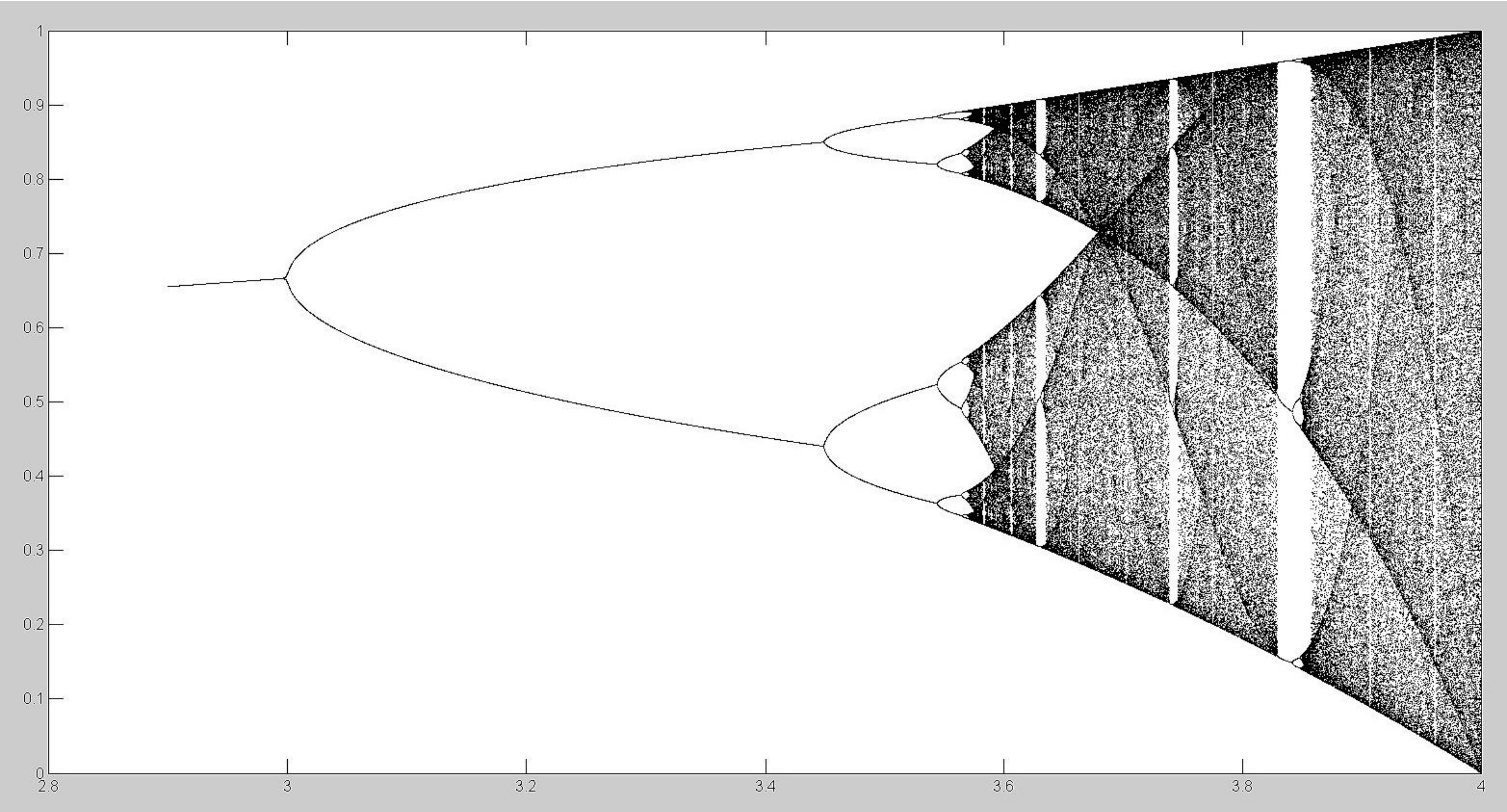
And more;



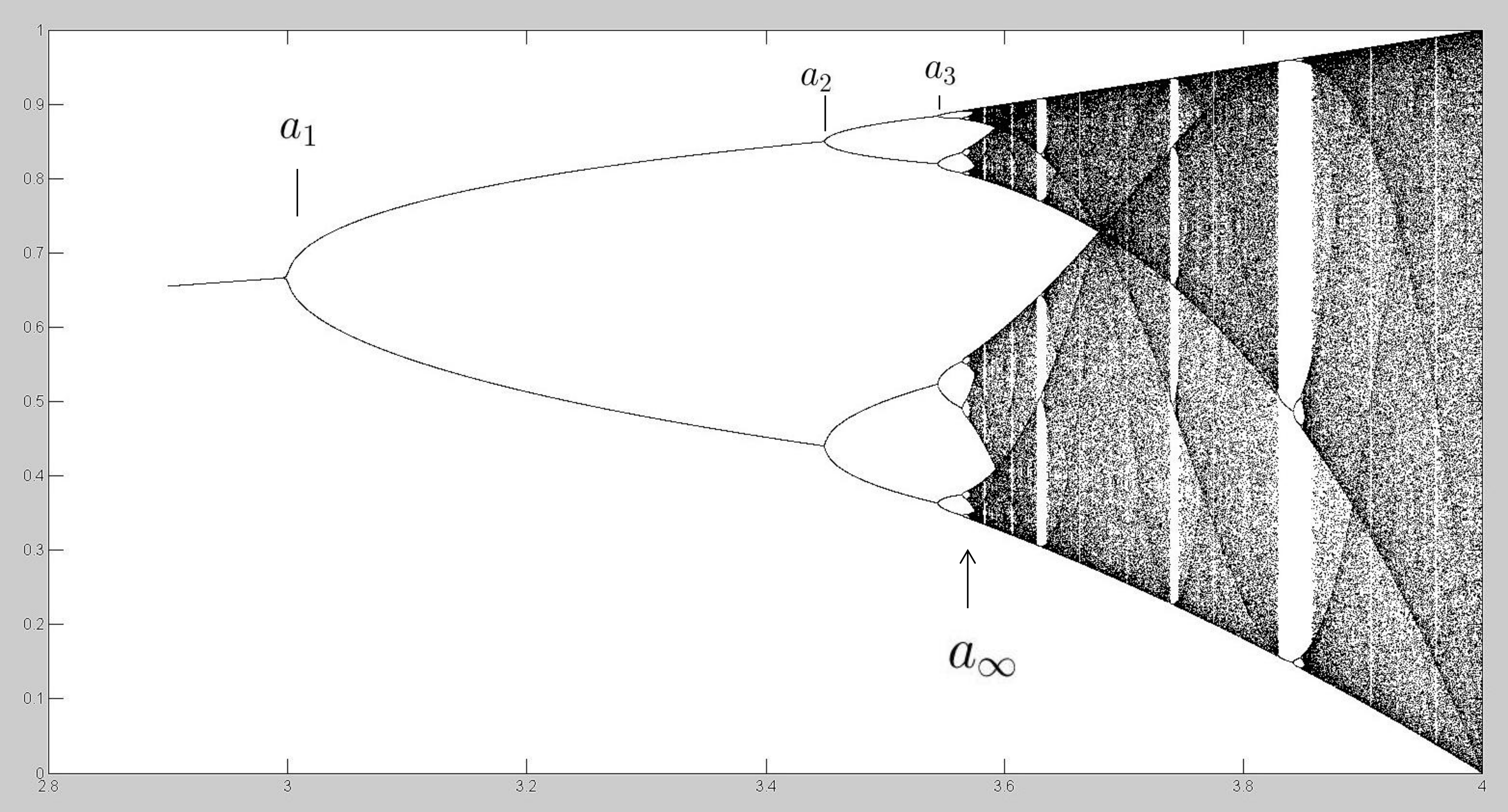
And more;

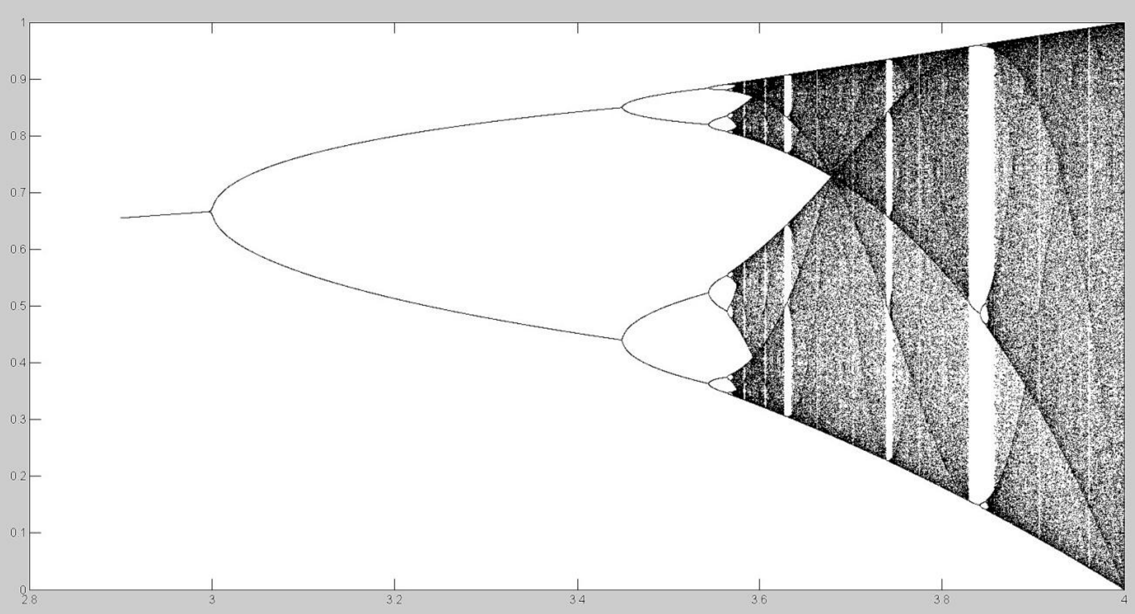


Final state diagram for logisitic equation, *2.8 < a < 4*



The period doubling bifurcations accumulate to $a_\infty = 3.5699...$

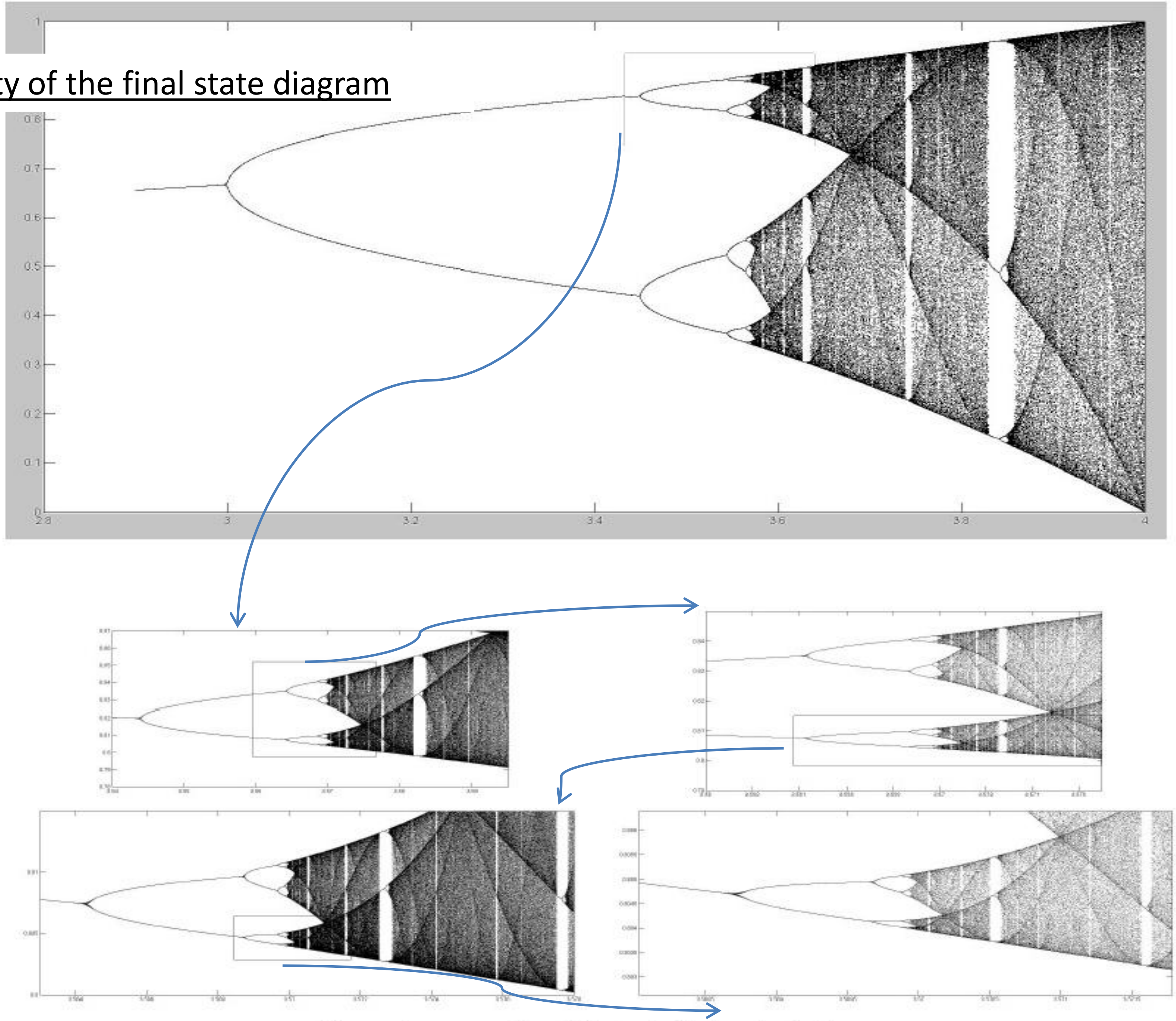




Some features of the final state diagram for the logistic equation:

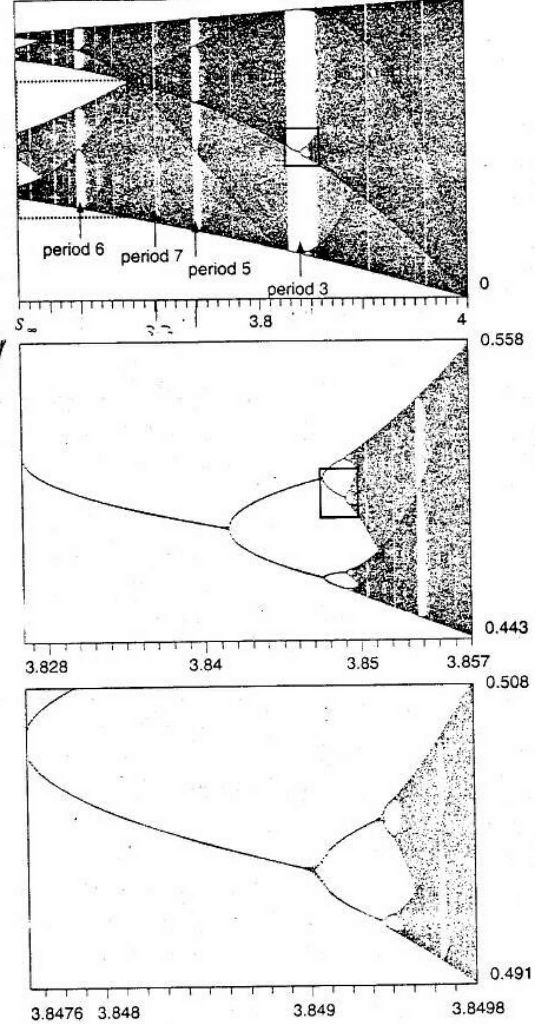
1. Self similarity
2. ‘Shadow’ lines
3. Ordering of periodic orbits

Self-similarity of the final state diagram

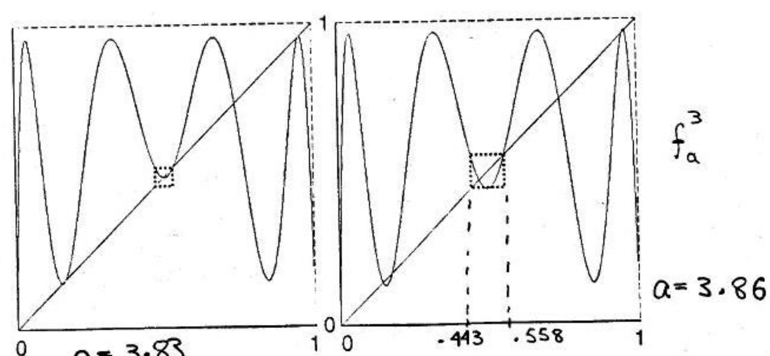


Successive zoom ins; left to right, top to bottom

Explaining the self-similarity of the periodic windows



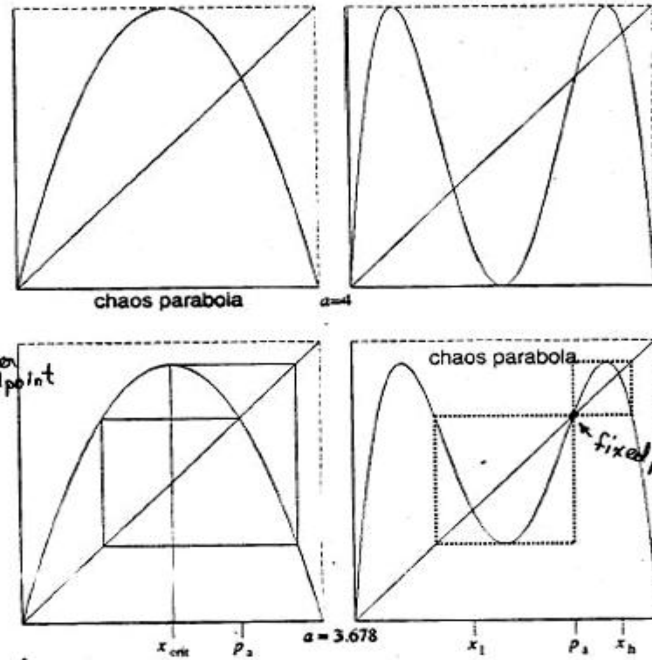
Note that the part of the graph of f_a^3 inside the invariant square as 'a' changes from 3.83 to 3.86, looks similar to how the graph of the logistic eqn. fa changes as 'a' goes from 3 to 4.



a = 4

a = 3.68

Explaining the self-similarity of the bands



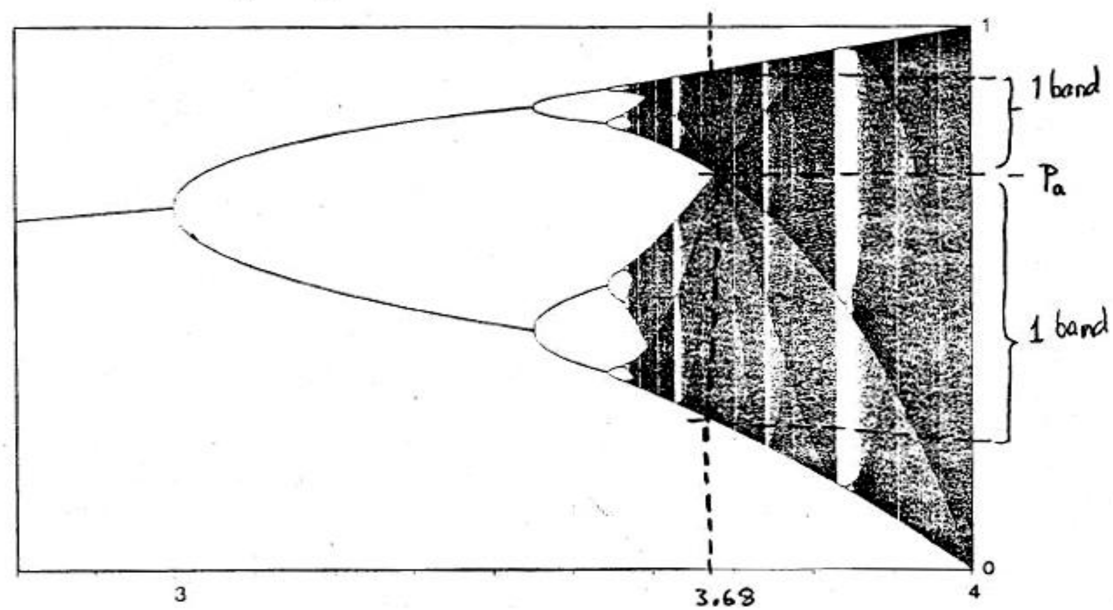
f_a and f_a^2

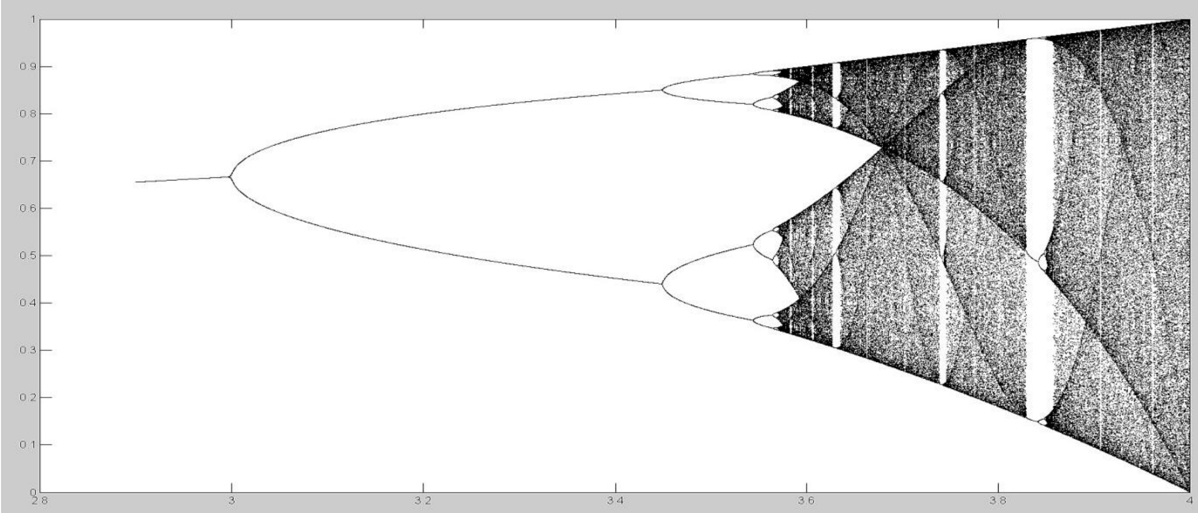
Comparing f_a and f_a^2 at $a = 4$ and at $a = m_1$. f_a^2 forms two small versions of graphs similar to the parabola for $a = 4$ (enclosed by the dashed square).

one corner a fixed point

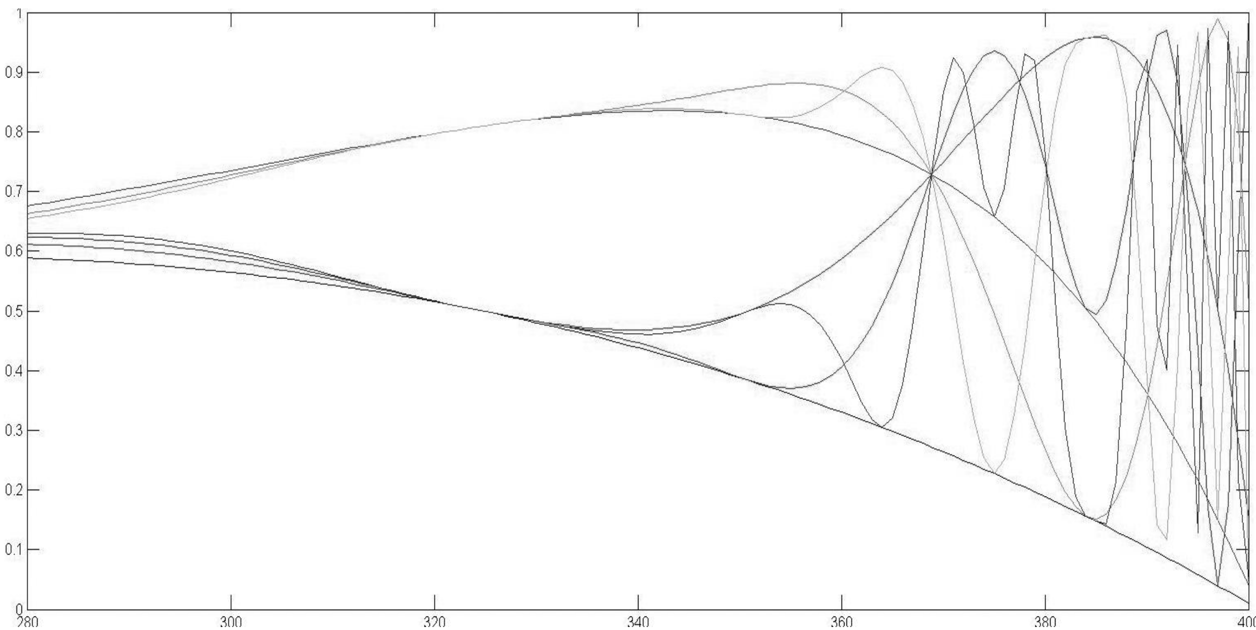
If you can draw a square around the graph of $f_a^k(x)$ and $y = x$, then that will be an 'invariant square'; orbits starting in there will remain in there. So you will have a 'copy' of the logistic function in miniature (and all the dynamics we observe for the logistic equation).

If x_0 is in one of these two small squares, then the entire orbit of x_0 stays within the square. Any orbit will end up in one of the two squares.

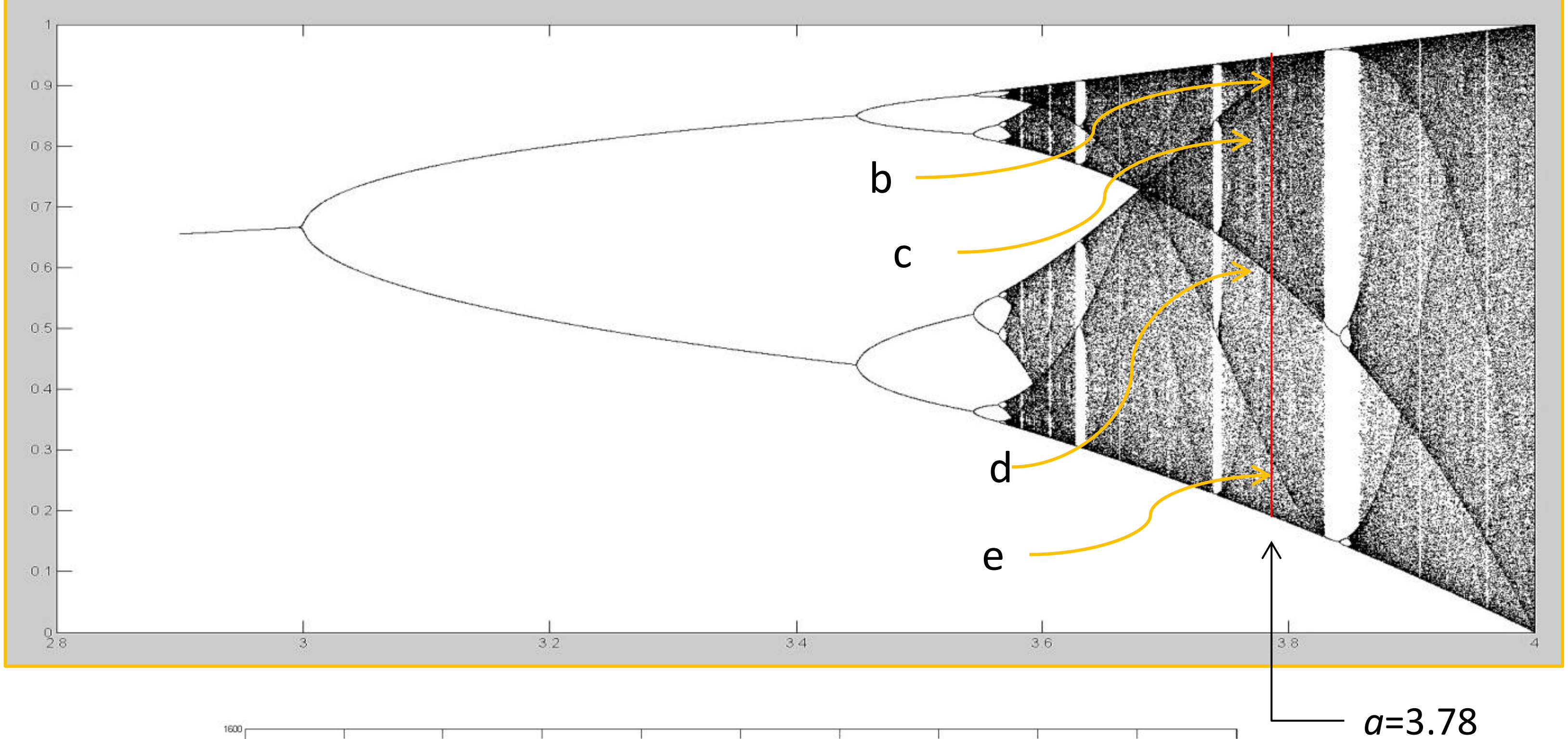


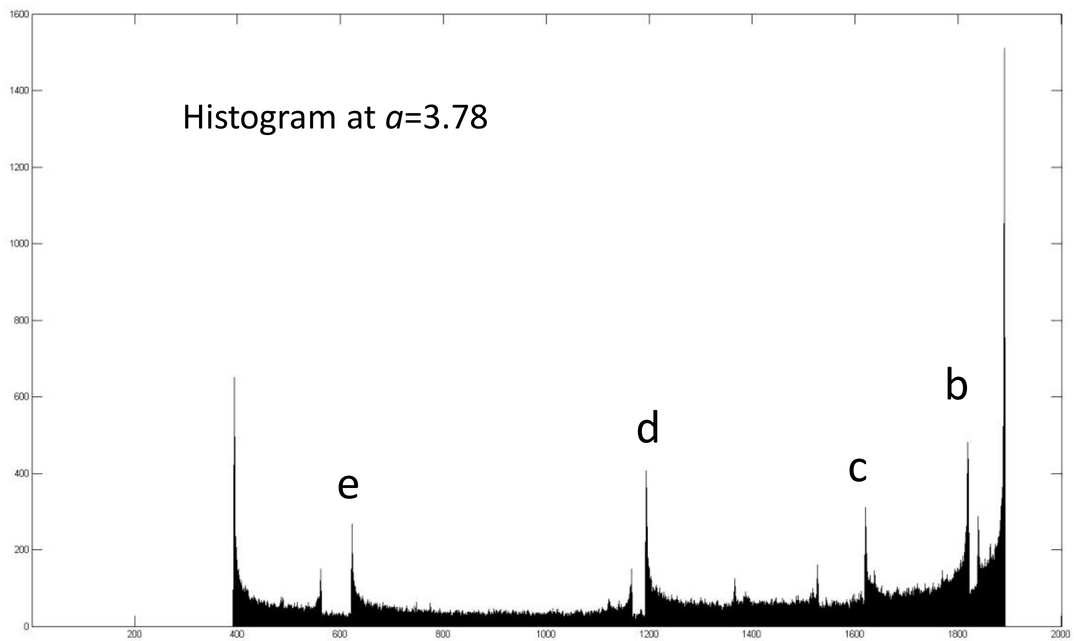


Shadow lines……



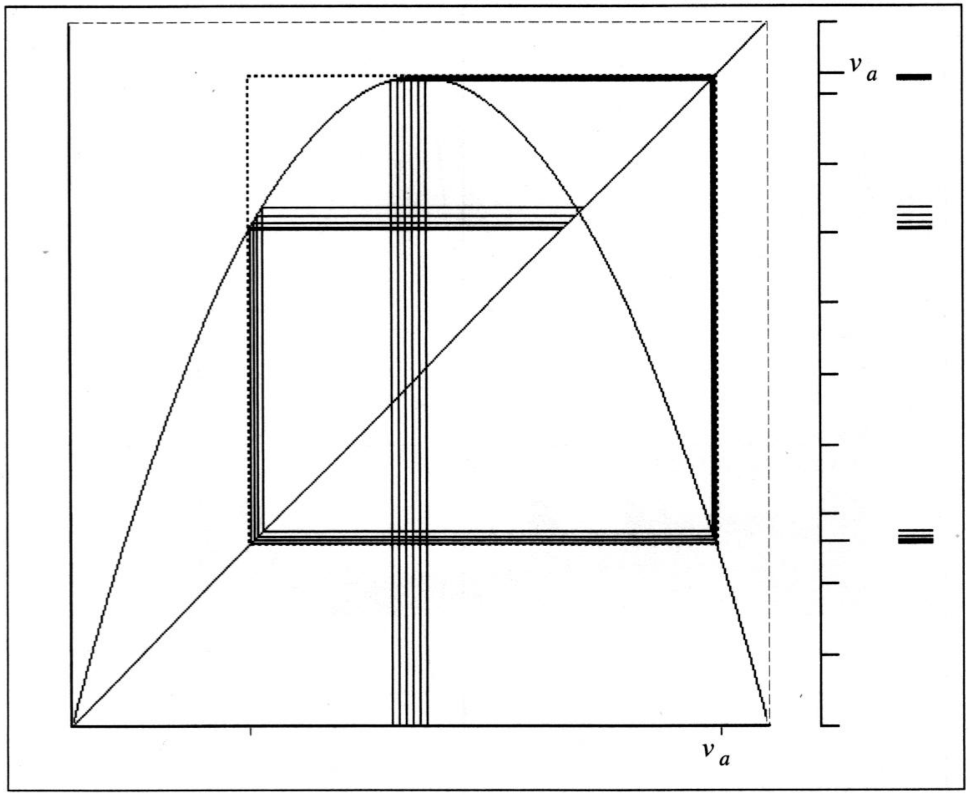
Can be computed





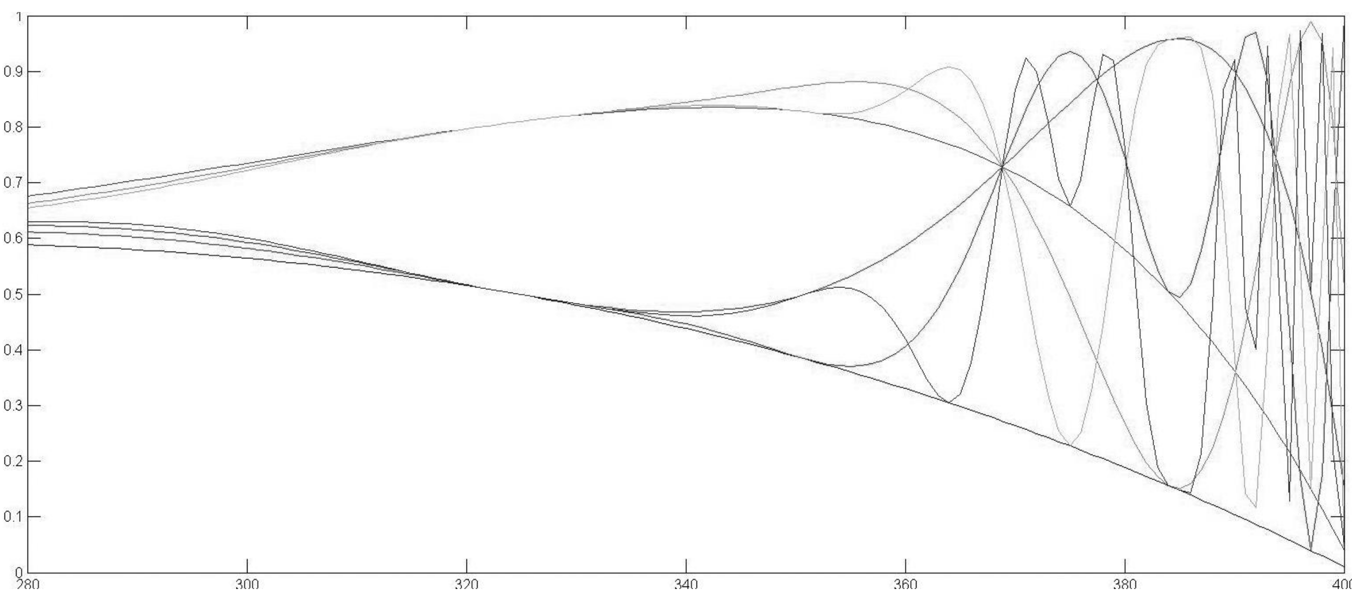
Shadow lines……

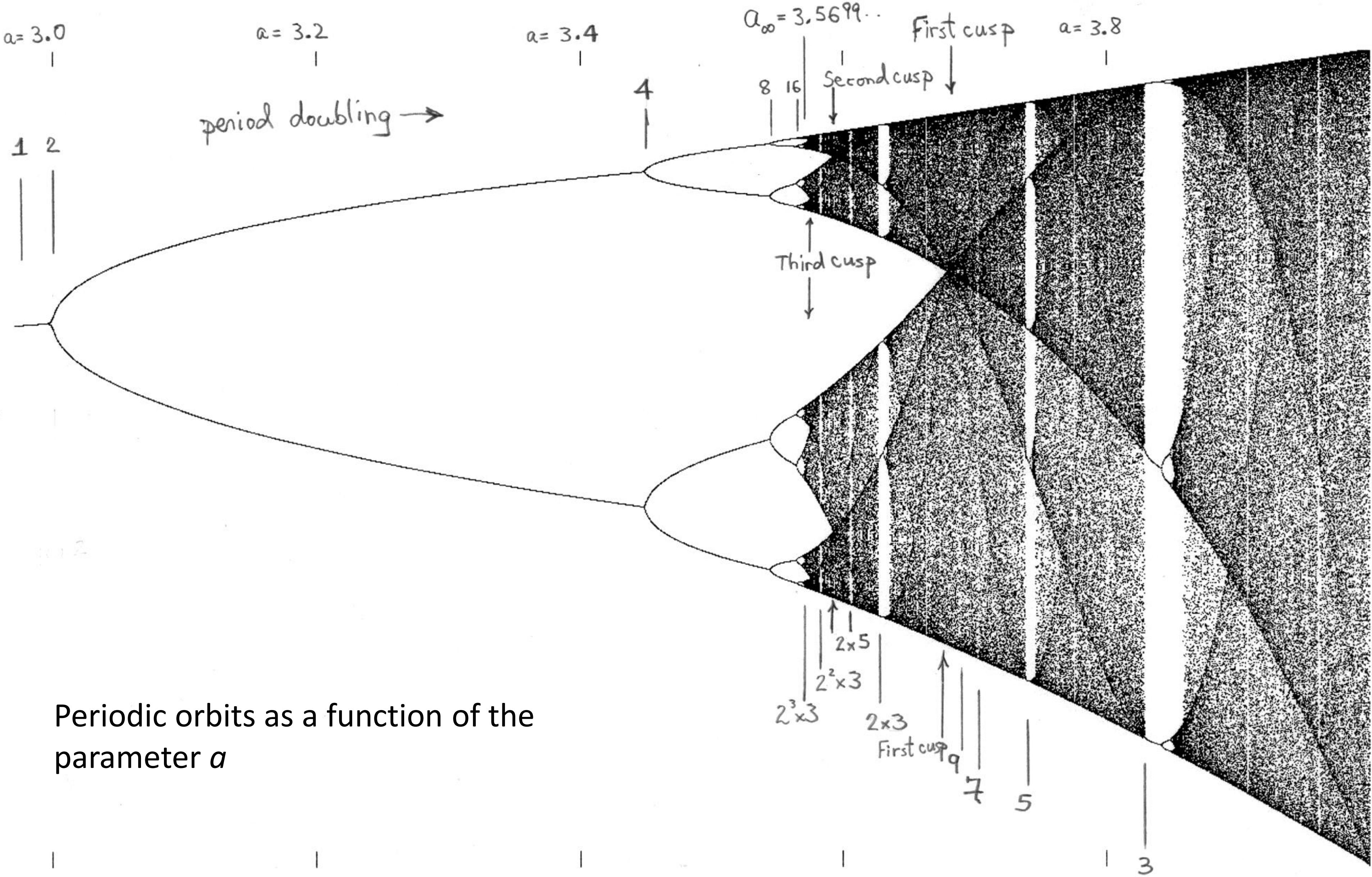
What are they?
They are peaks in the histogram caused by ‘squeezing’ of the points in the orbit due to the peak in $f(x)$….



ν_a is the image of points near the peak at $x = 0.5$ Subsequent iterates, $f_a^2(\nu_a), f_a^3(\nu_a), \ldots$ produce weaker peaks in the histogram.

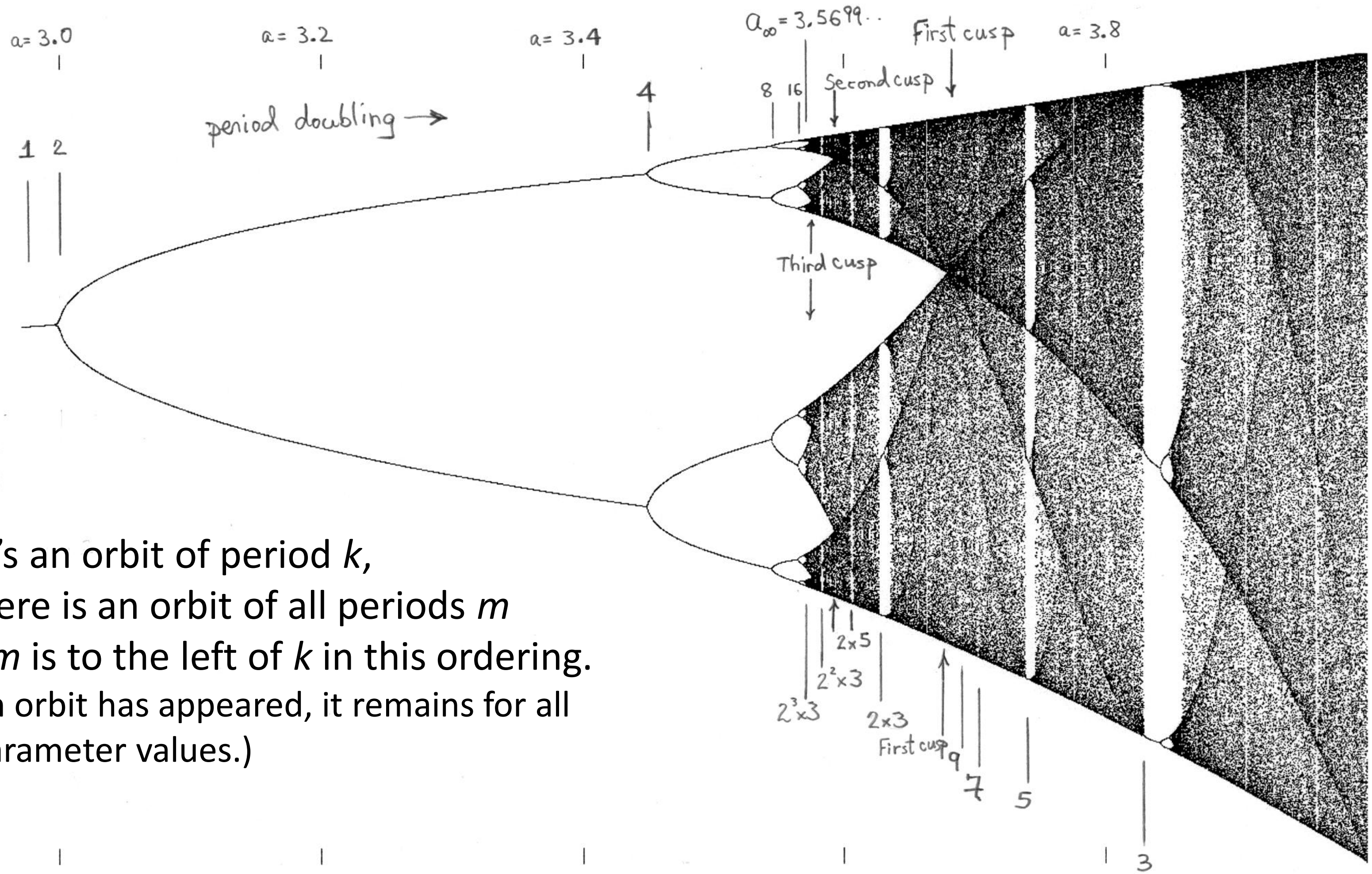
Plots of graphs of $f_a(\nu_a), f_a^2(\nu_a), \ldots, f_a^5(\nu_a)$ as a function of a.





Periodic orbits as a function of the parameter *a*

From left of a_∞ have period doublings; 1, 2, 4, 8, ….
From right of first cusp have all the odd integers; 3, 5, 7, ….
From right of second cusp, have all 2x(odd) integers; 6, 10, 14, ….
From right of third cusp, have all 4x(odd) integers; 12, 20, 28, …..



If there's an orbit of period *k*,
then there is an orbit of all periods *m*
where *m* is to the left of *k* in this ordering.
(Once an orbit has appeared, it remains for all larger parameter values.)

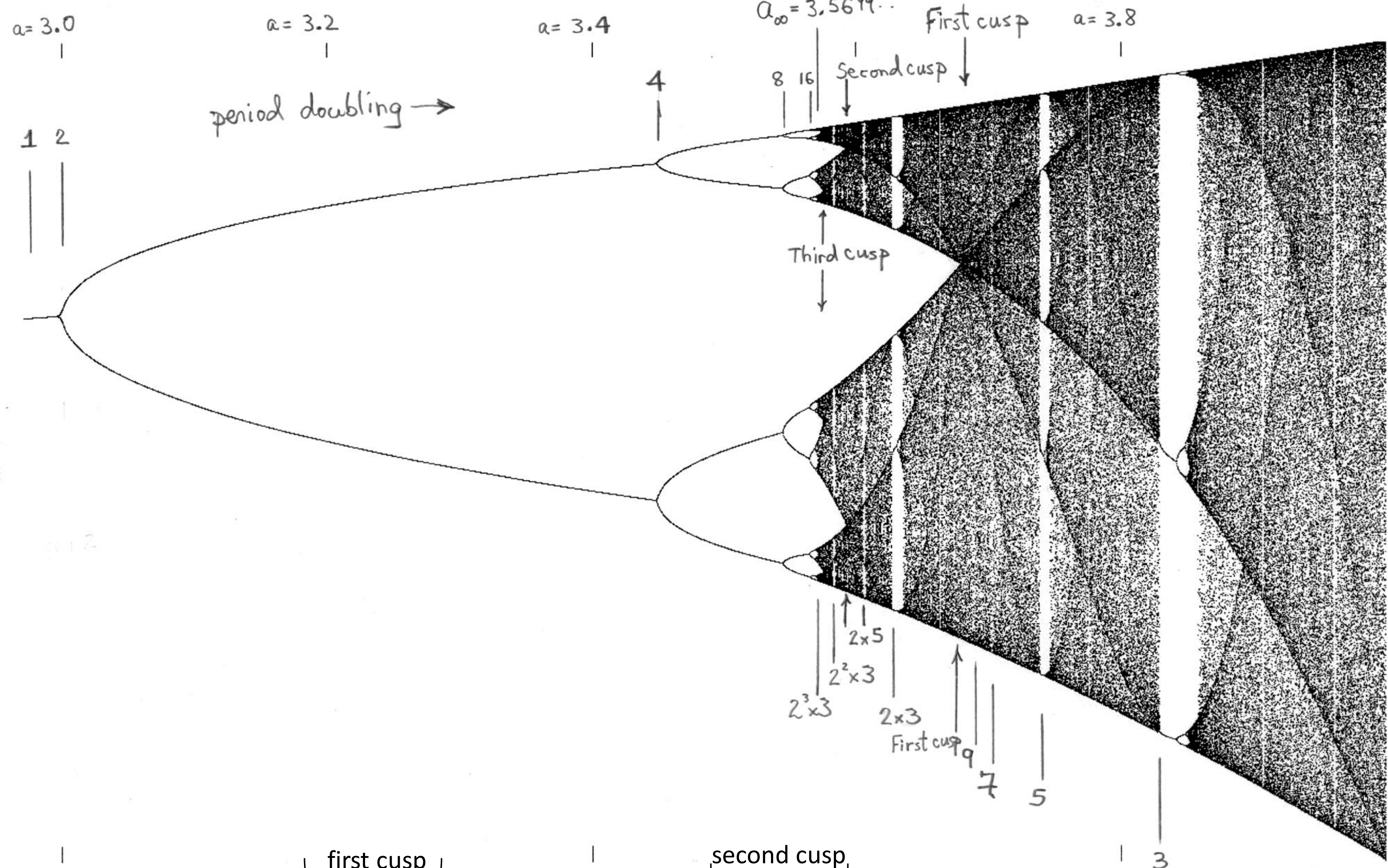
The Charkovsky ordering of the positive integers;

$$3 >> 5 >> 7 >> 9 >> ... >> (2)3 >> (2)5 >> (2)7 >> ... >> (2^2)3 >> (2^2)5 >> (2^2)7 >> ...$$
$$>> (2^3)3 >> (2^3)5 >> ...... >> (2^n)3 >> (2^n)5 >> ...... >> 2^5 >> 2^4 >> 2^3 >> 2^2 >> 2 >> 1$$

Charkovsky's Theorem:

If f is a continuous function that transforms an interval I onto itself (i.e., $f(I)$ is contained in I), and if f has a periodic point of period k, then f has a periodic point of period m for every m such that $k >> m$ in the Charkovsky ordering.

"Period 3 implies Chaos"……



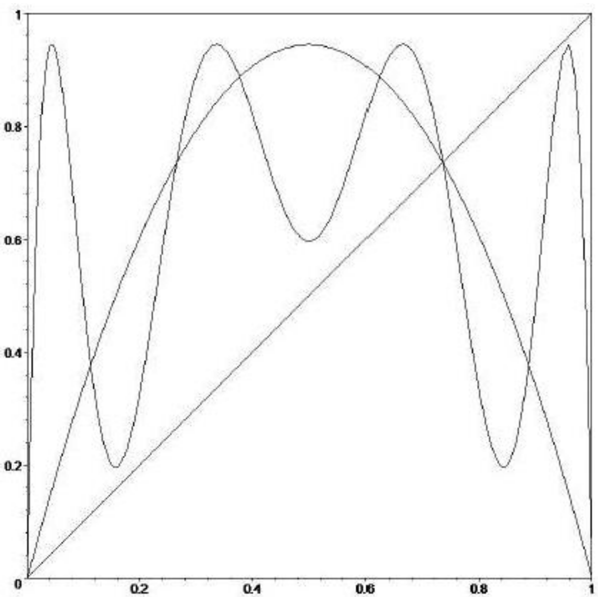
first cusp — second cusp

$3 >> 5 >> 7 >> 9 >> \ldots >> (2)3 >> (2)5 >> (2)7 >> \ldots >> (2^2)3 >> (2^2)5 >> (2^2)7 >> \ldots$ third cusp

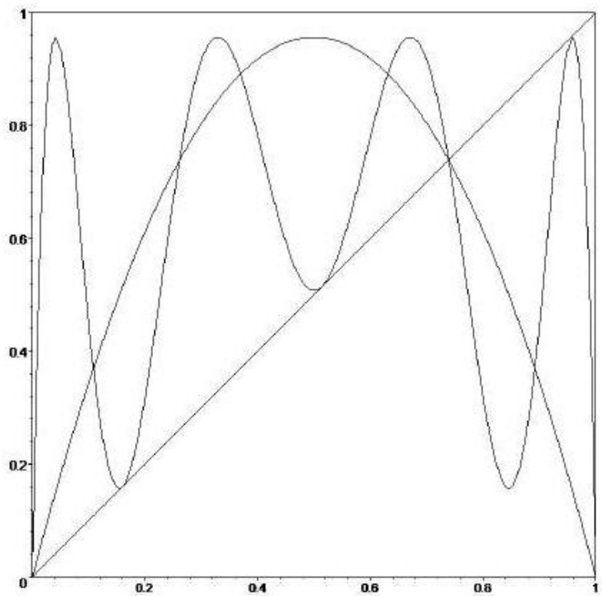
$>> (2^3)3 >> (2^3)5 >> \ldots\ldots >> (2^n)3 >> (2^n)5 >> \ldots\ldots >> 2^5 >> 2^4 >> 2^3 >> 2^2 >> 2 >> 1$

a_∞

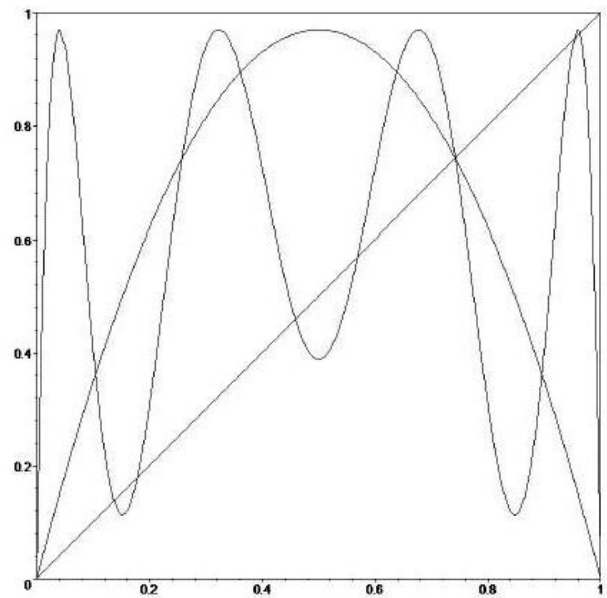
The period 3 orbit first appears at a=3.828…..



$f_a(x)$ and $f_a^3(x)$ for $a = 3.78$

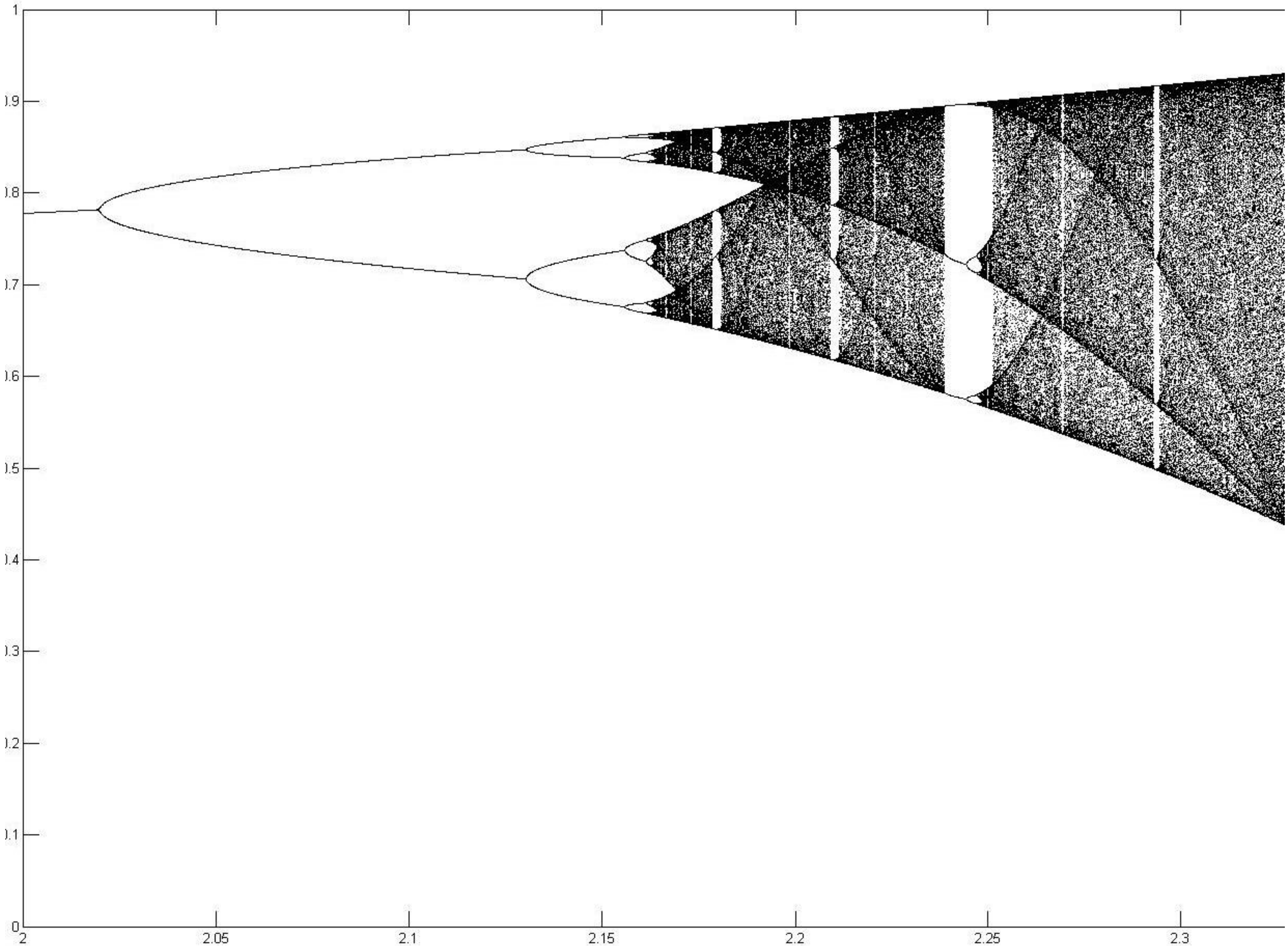


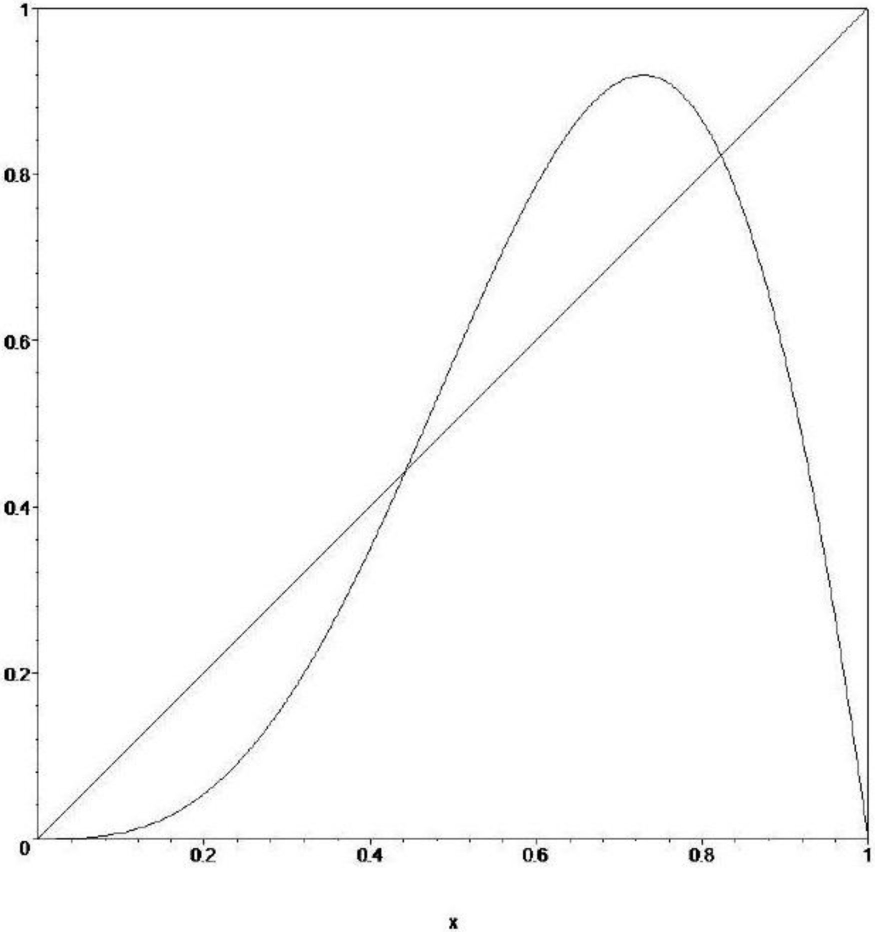
and for $a = 3.828$

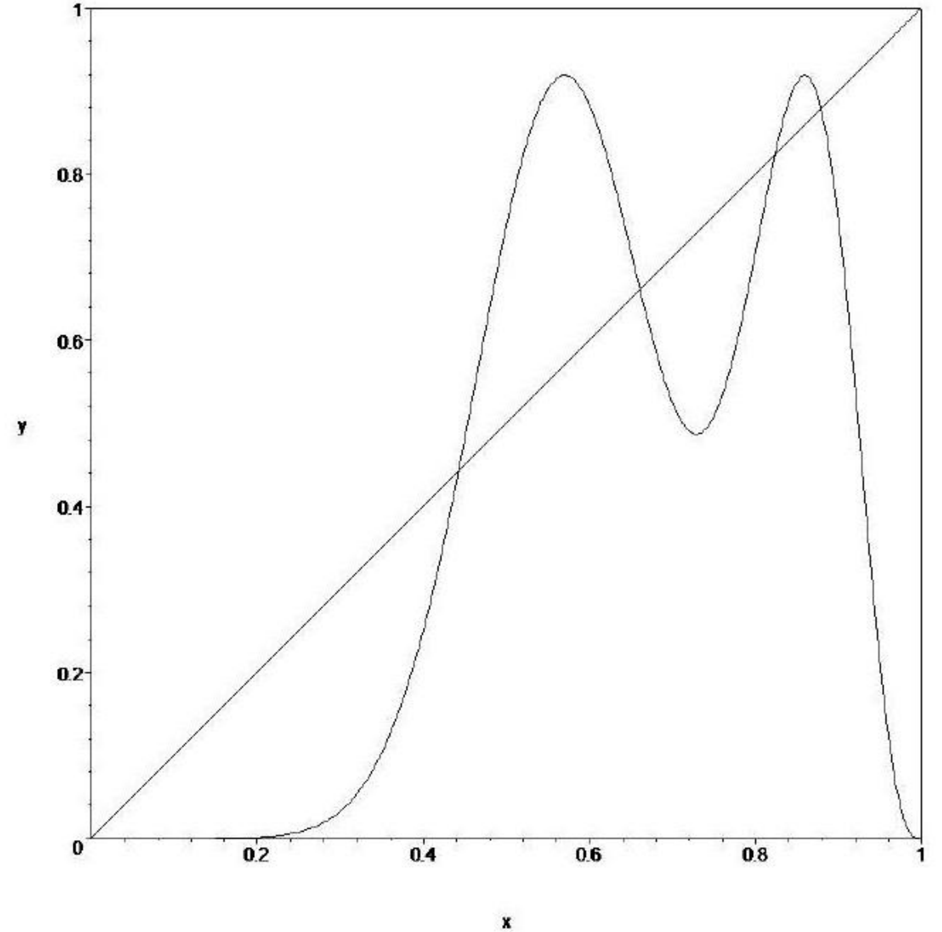


and for $a = 3.88$

Final state diagram for $g_a(x) = ax^2 \sin(\pi x)$

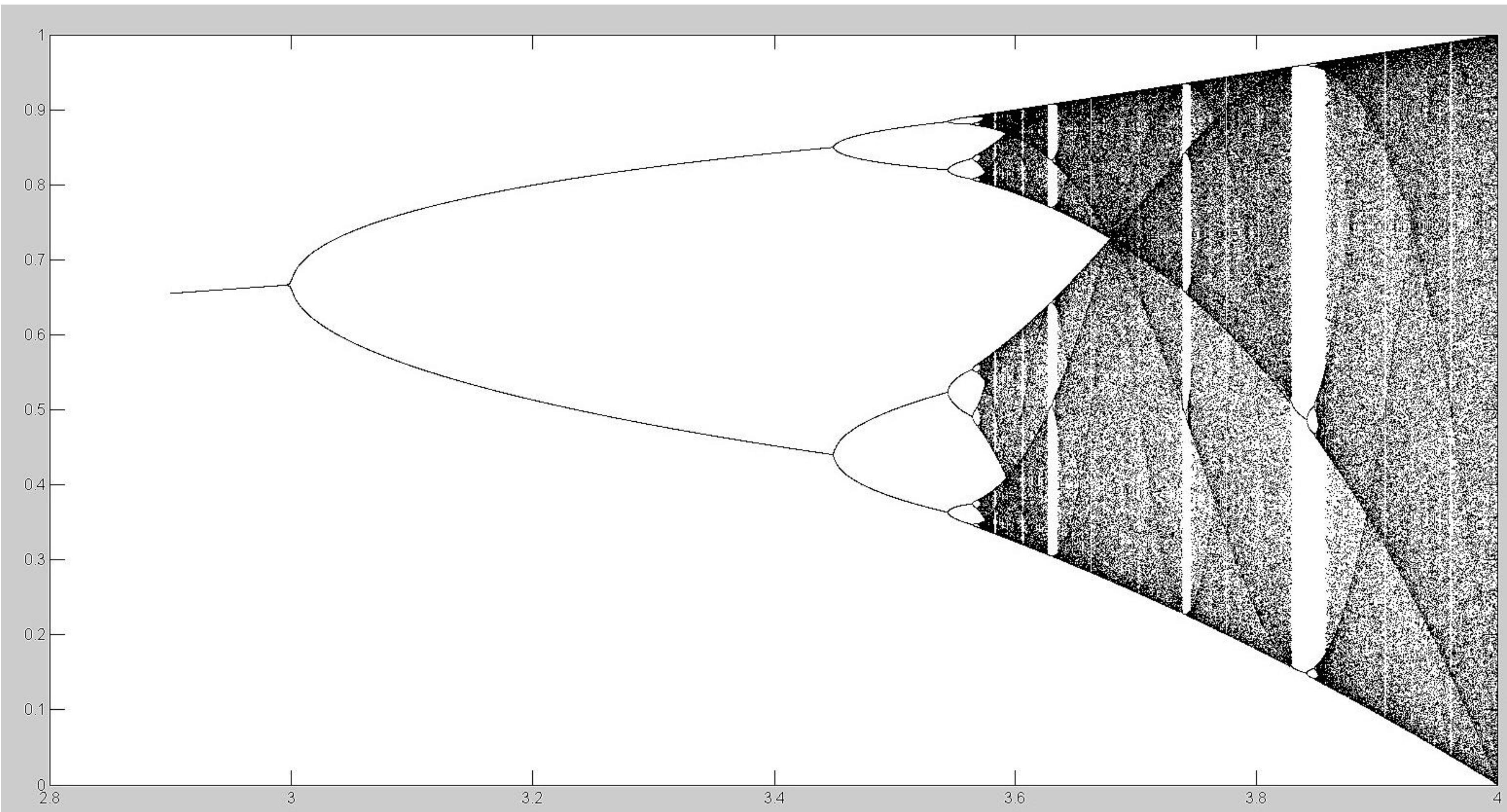


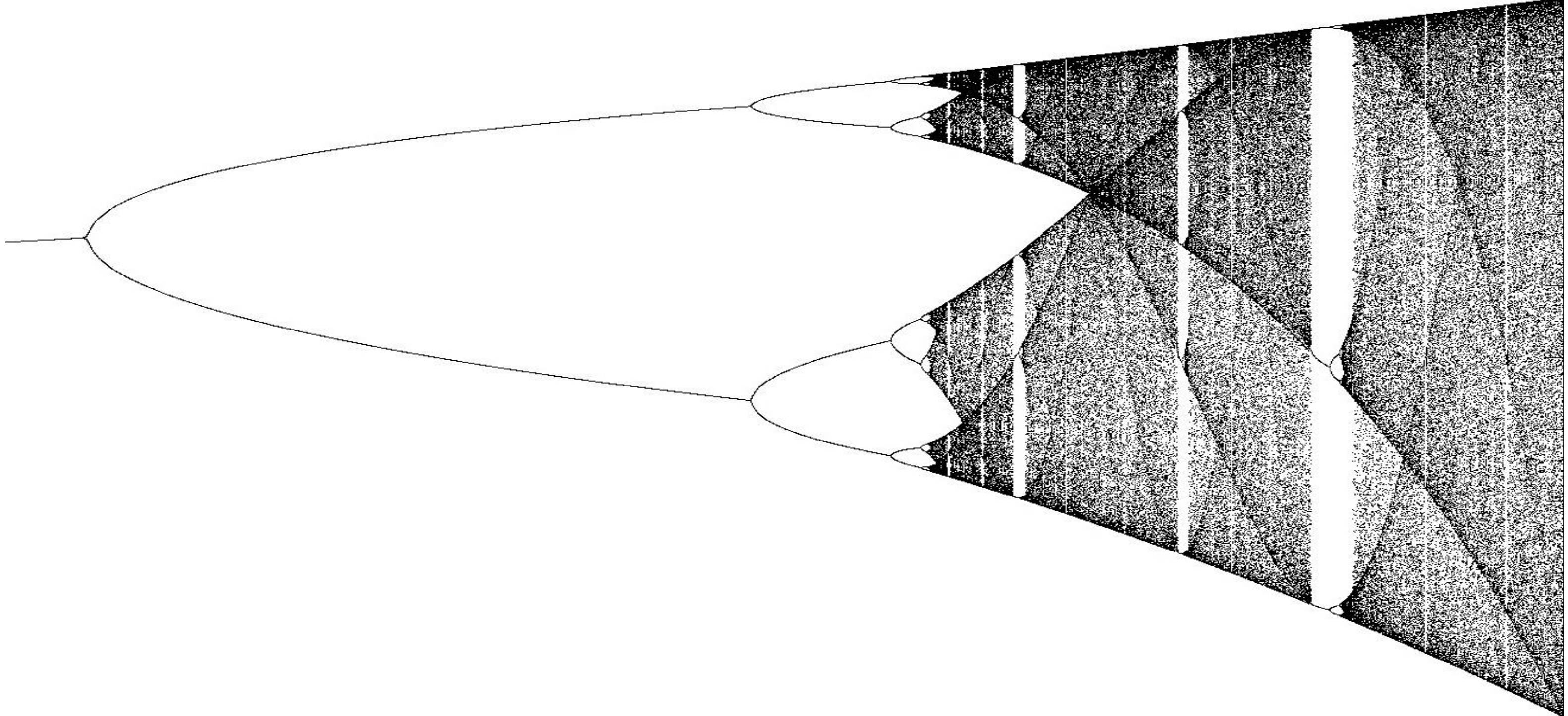




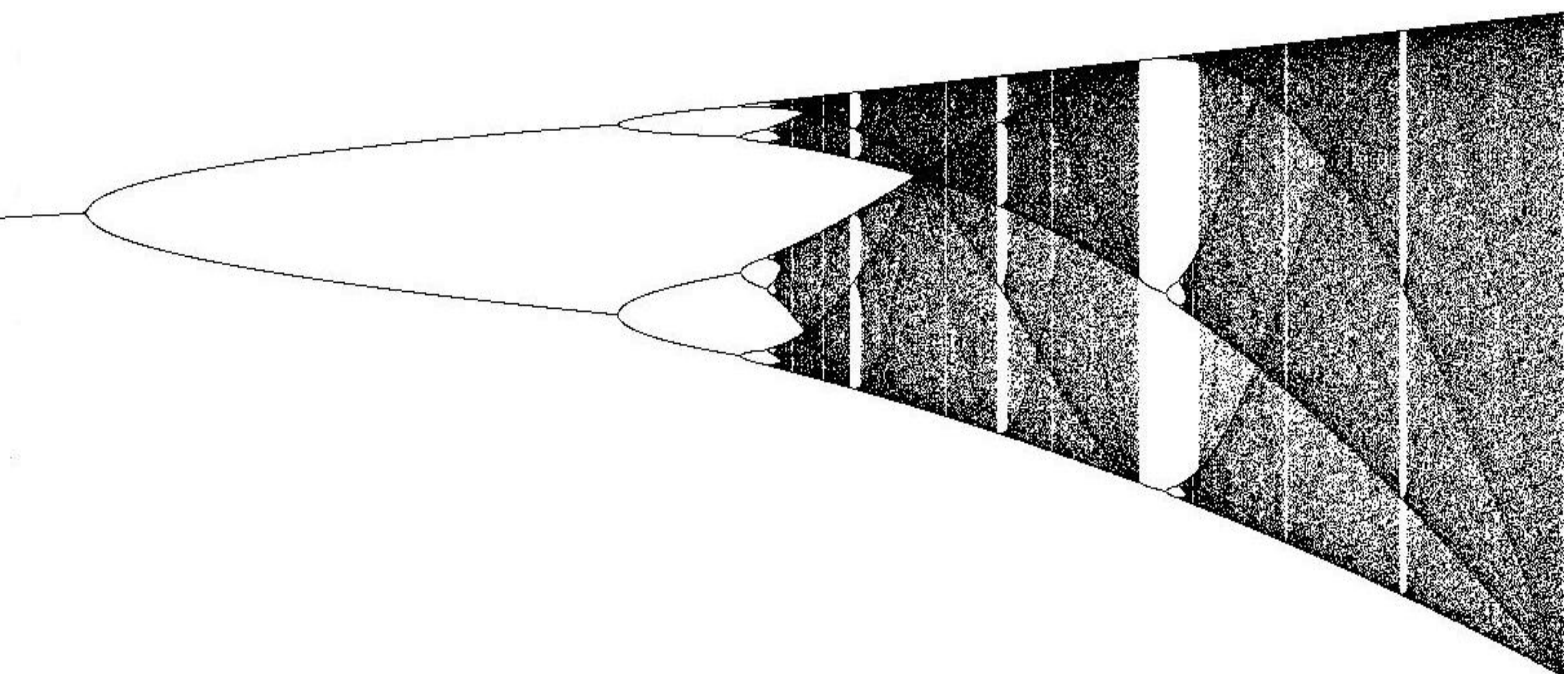
Graph of $g_a(x) = ax^2 \sin(\pi x)$ for $a = 2.3$ and $g_a^2(x)$

In fact, this final state diagram is **universal** for all such ('uni-modal') functions….

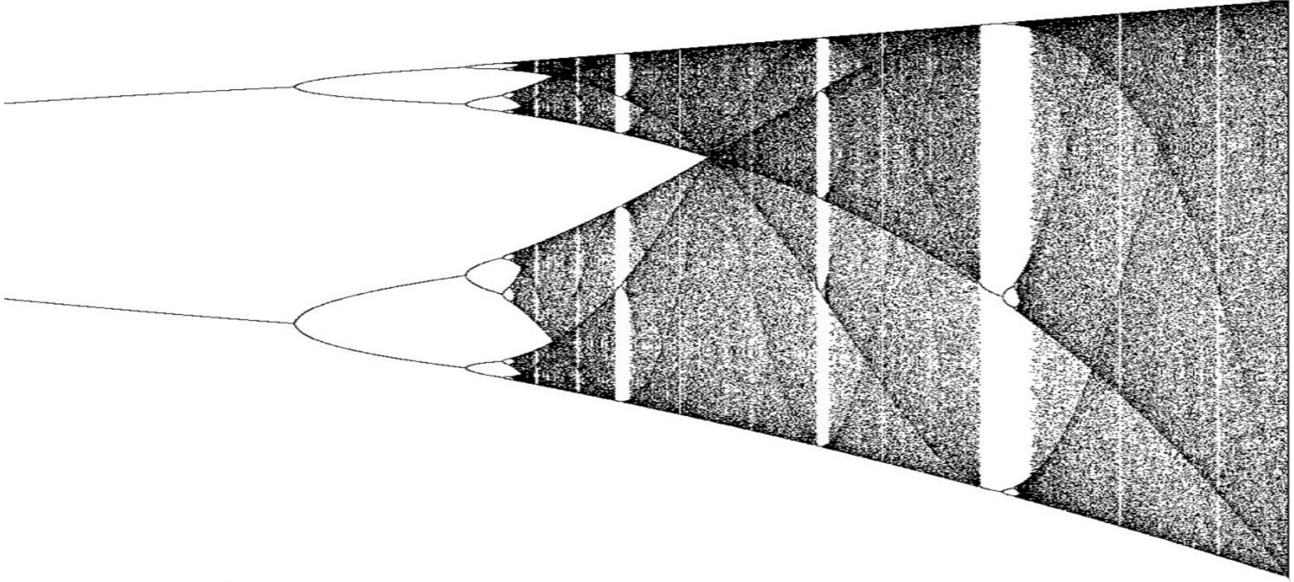


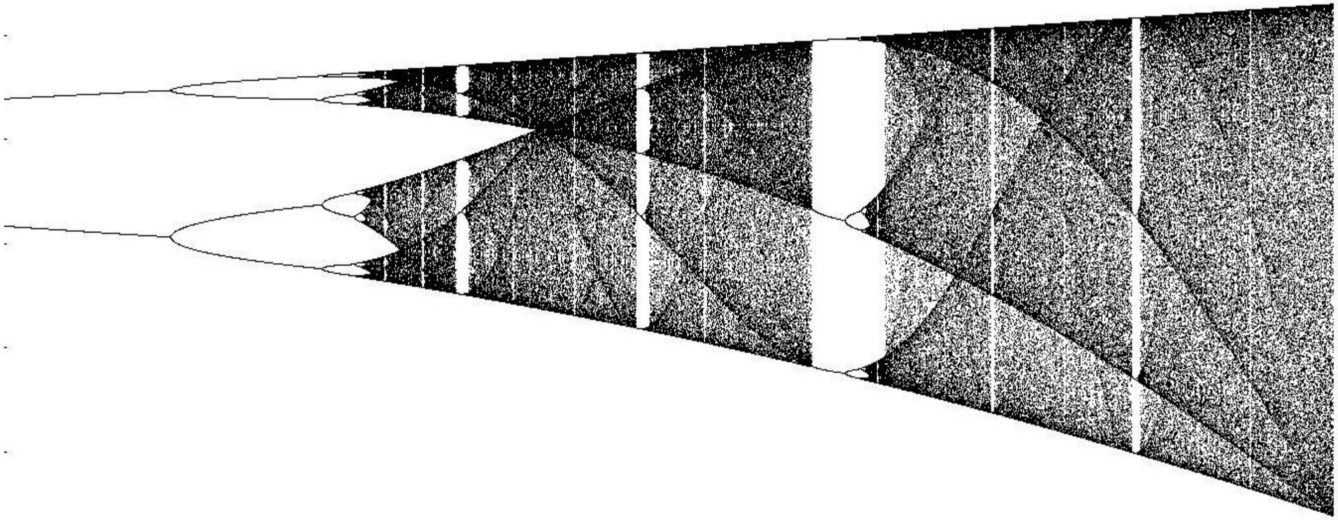


$ax(1-x)$



$ax^2\,\sin(\pi x)$





But there’s more; the *<u>rate</u>* at which the period doubling bifurcations take place is the same!

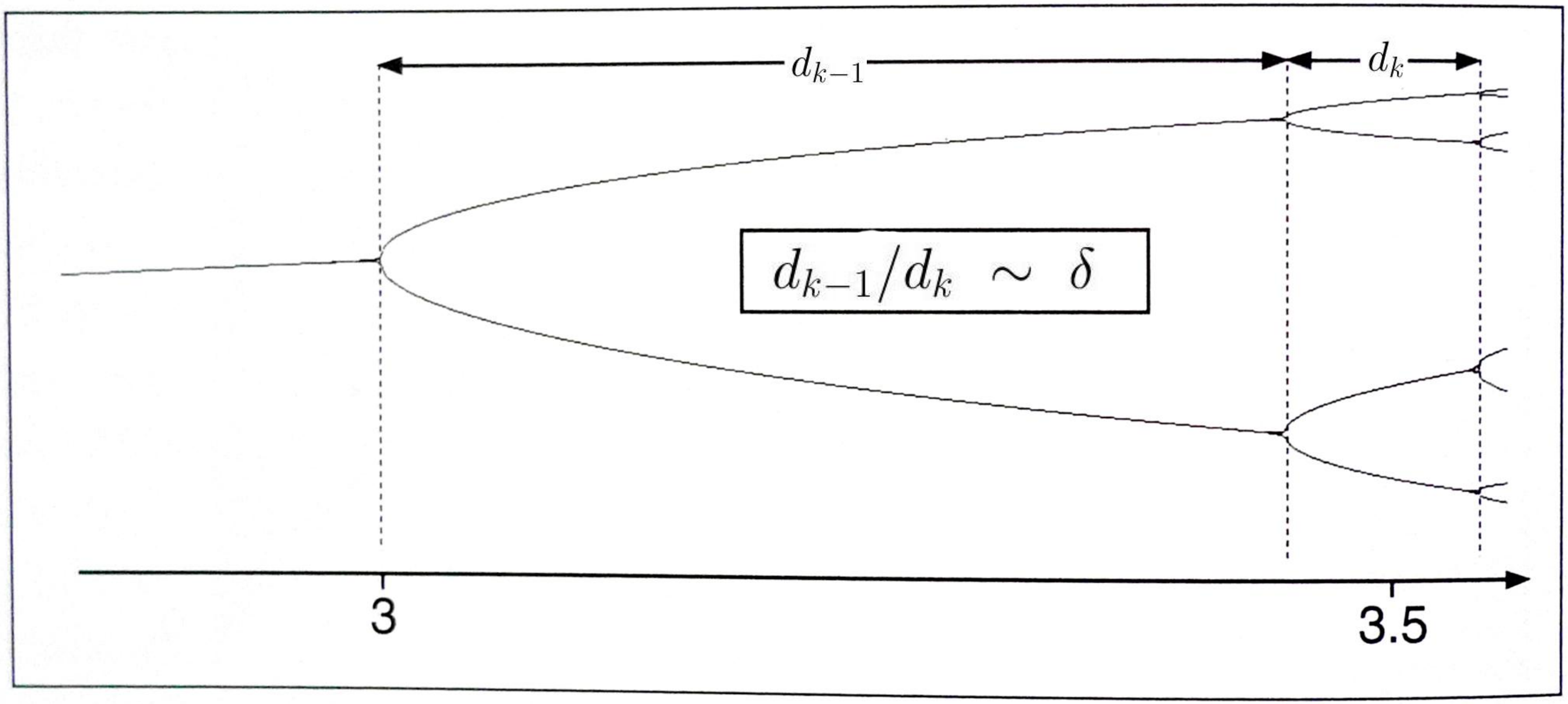
Bifurcation point	Period
$a_1 = 3.0$	1
$a_2 = 3.449489$	2
$a_3 = 3.544090$	4
$a_4 = 3.564407$	8
$a_5 = 3.568759$	16
$a_6 = 3.569692$	32
$a_7 = 3.569891$	64

Difference	Ratio
$d_1 = a_2 - a_1 = 0.44949$	
$d_2 = a_3 - a_2 = 0.94611$	$d_1/d_2 = 4.7514$
$d_3 = a_4 - a_3 = 0.020316$	$d_2/d_3 = 4.6562$
$d_4 = a_5 - a_4 = 0.0043521$	$d_3/d_4 = 4.6682$
$d_5 = a_6 - a_5 = 0.00093219$	$d_4/d_5 = 4.6687$
$d_6 = a_7 - a_6 = 0.00019964$	$d_5/d_6 = 4.6693$

$$\delta_k = d_k/d_{k+1}, \qquad \delta_k \;\to\; \delta = 4.66....$$

Feigenbaum’s constant

The Feigenbaum constant δ specifies the rate at which period doubling bifurcations take place



Universal behaviour

Experimental Measurements of Period-Doublings		
Experiment	Number of period doublings	δ
Hydrodynamic:		
water	4	4.3 ± 0.8
helium	4	3.5 ± 0.15
mercury	4	4.4 ± 0.1
Electronic:		
diode	5	4.3 ± 0.1
transistor	4	4.7 ± 0.3
Josephson	4	4.4 ± 0.3
Laser:		
laser feedback	3	4.3 ± 0.3
Acoustic:		
helium	3	4.8 ± 0.6

Feigenbaum’s constant is universal

A dynamical system depends on a parameter a. Initially, you observe a steady state (i.e., a period 1 orbit). As a increases you observe a period 2 oscillation appearing at $a = a_1 = 7$. Then at $a = a_2 = 10$ you observe that the period 2 orbits splits into a period 4 orbit. As a continues to increase a series of period-doublings occurs. Assuming Universality, at what a value would you expect to observe the onset of chaos?

Qualitative features of the period doubling scenerio for 'uni-modal maps' can be understood by graphical analysis.

But the quantitative features, the universality of the rate of bifurcations (Feigenbaum's constant δ) needs much more work…

This was understood by Feigenbaum in 1975 using methods from Renormalization (physics).

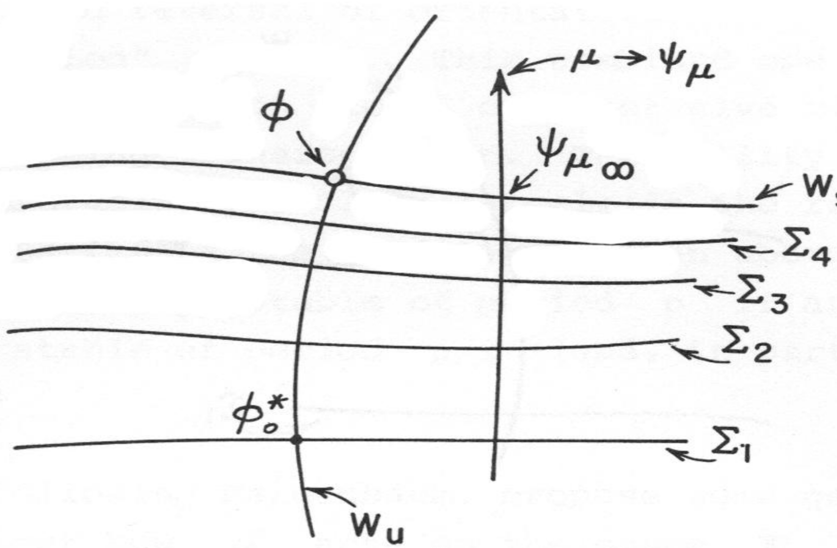
Qualitative features of the period doubling scenario for ‘uni-modal maps’ can be understood by graphical analysis.

But the quantitative features, the universality of the rate of bifurcations (Feigenbaum’s constant δ) needs much more work…

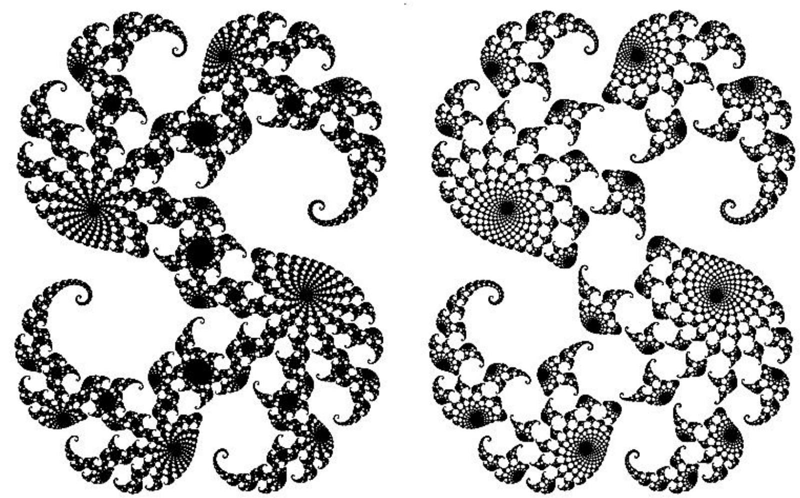
This was understood by Feigenbaum in 1975 using methods from Renormalization (physics).

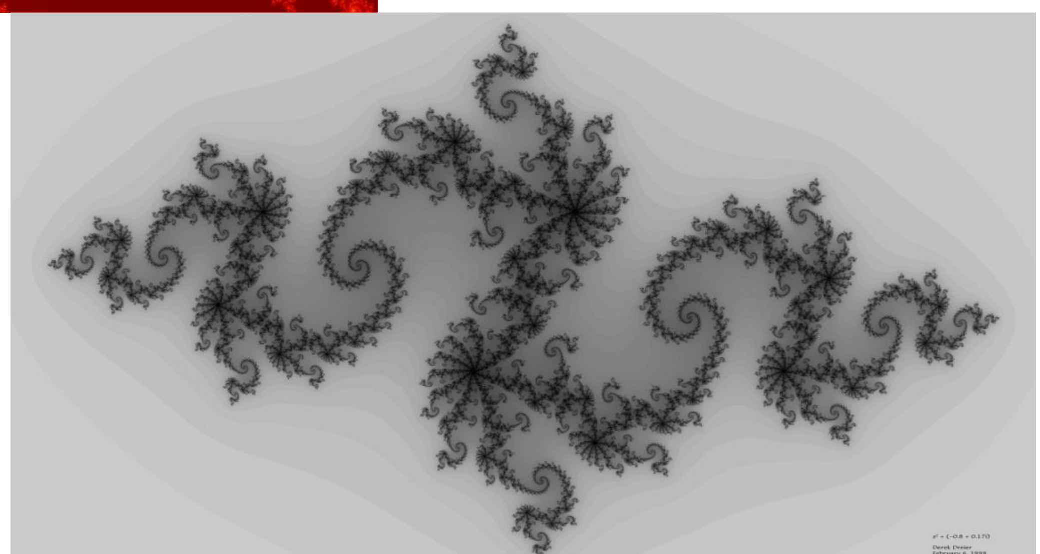
Renormalization in a nutshell….

In the space of unimodal functions on $[0, 1]$, the ’period-doubling’ map $\mathcal{F}$ has a fixed point ϕ. The linearlized map at ϕ has a single eigenvalue greater than one in absolute value, $\delta = 4.66...$, and the rest of the spectrum lies inside the unit circle. It has a one dimensional stable manifold W_s and a co-dimension one unstable manifold W_u. Σ_1 is the set of unimodal functions that have a stable (prime) period 2 orbit. $\Sigma_j = \mathcal{F}^{-j+1}(\Sigma_1), j = 2, \ldots$ are unimodal functions with period 2^j orbits. ψ_μ is a one-parameter family of maps. Period doubling bifurcations take place at μ_j, where $\psi_{\mu_j} \in \Sigma_j$.



More iteration: Julia sets



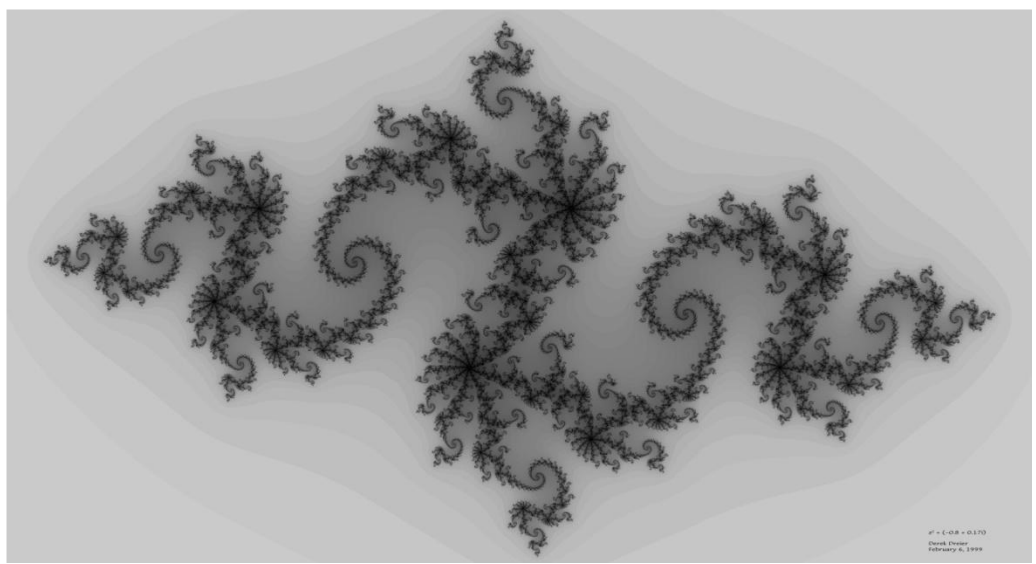


More iteration: Julia sets

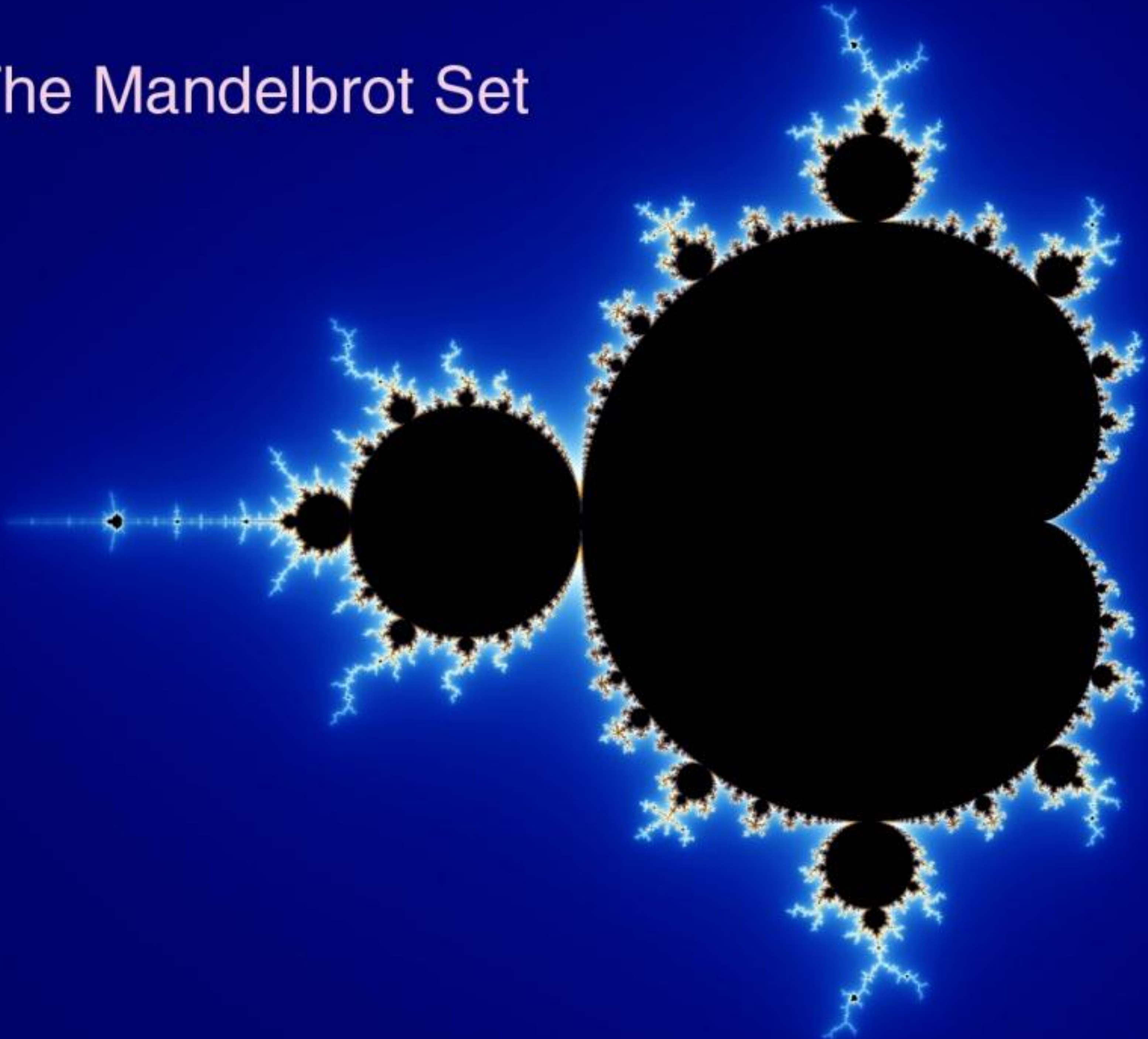
Complex iteration: $q_c(z) \;=\; z^2 + c, \quad z, c \in \mathbf{C}$

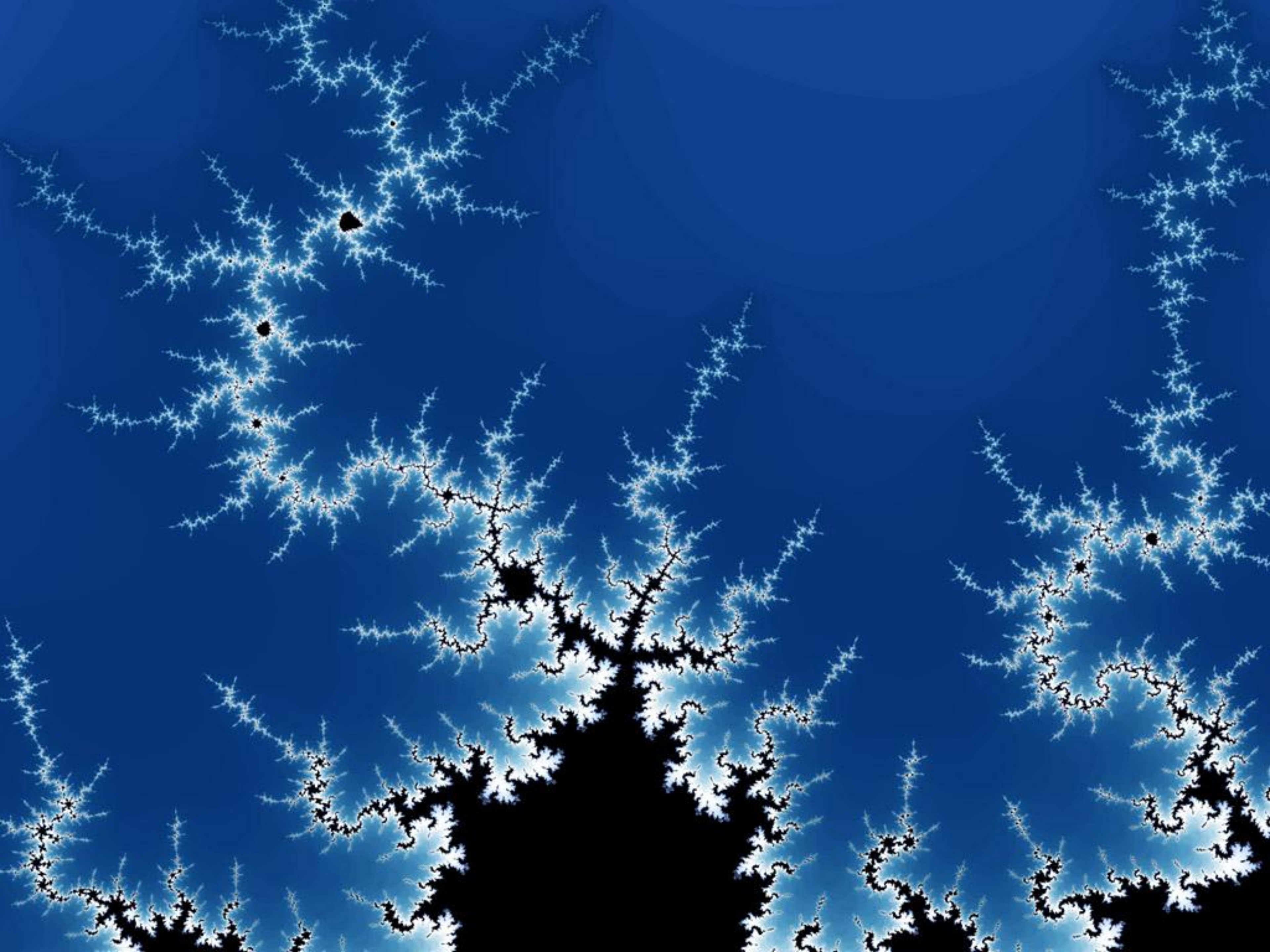
'Prisoner set' P_c; set of complex numbers whose orbits are bounded;
$P_c \;= \{z \in \mathbf{C} \mid \|q_c^n(z)\| < M \quad \forall n = 1, 2, 3, \ldots\}.$
Julia set J_c is the boundary of P_c.

Julia sets are either completely disconnected (`dust') or are connected (one piece).
The values of *c* for which the Julia set is connected form the Mandelbrot set…..

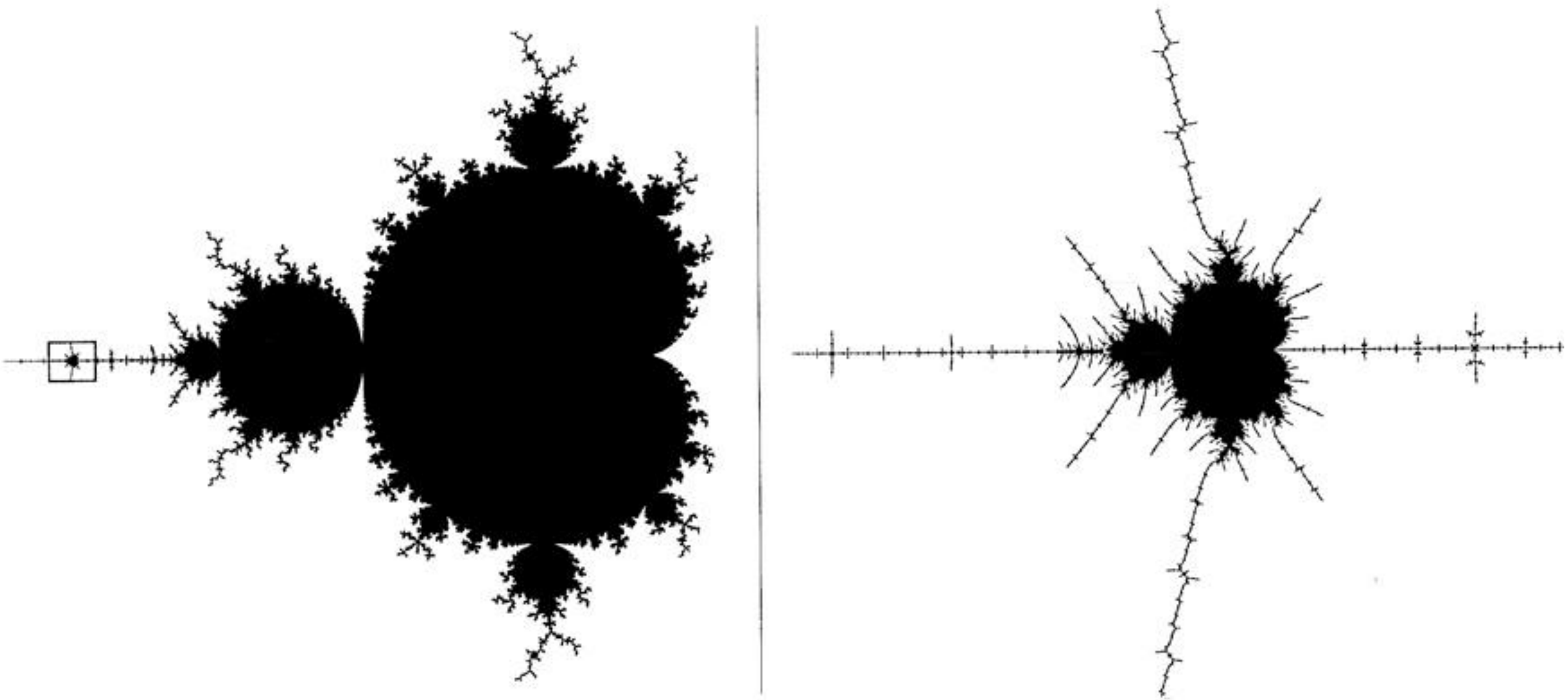


The Mandelbrot Set

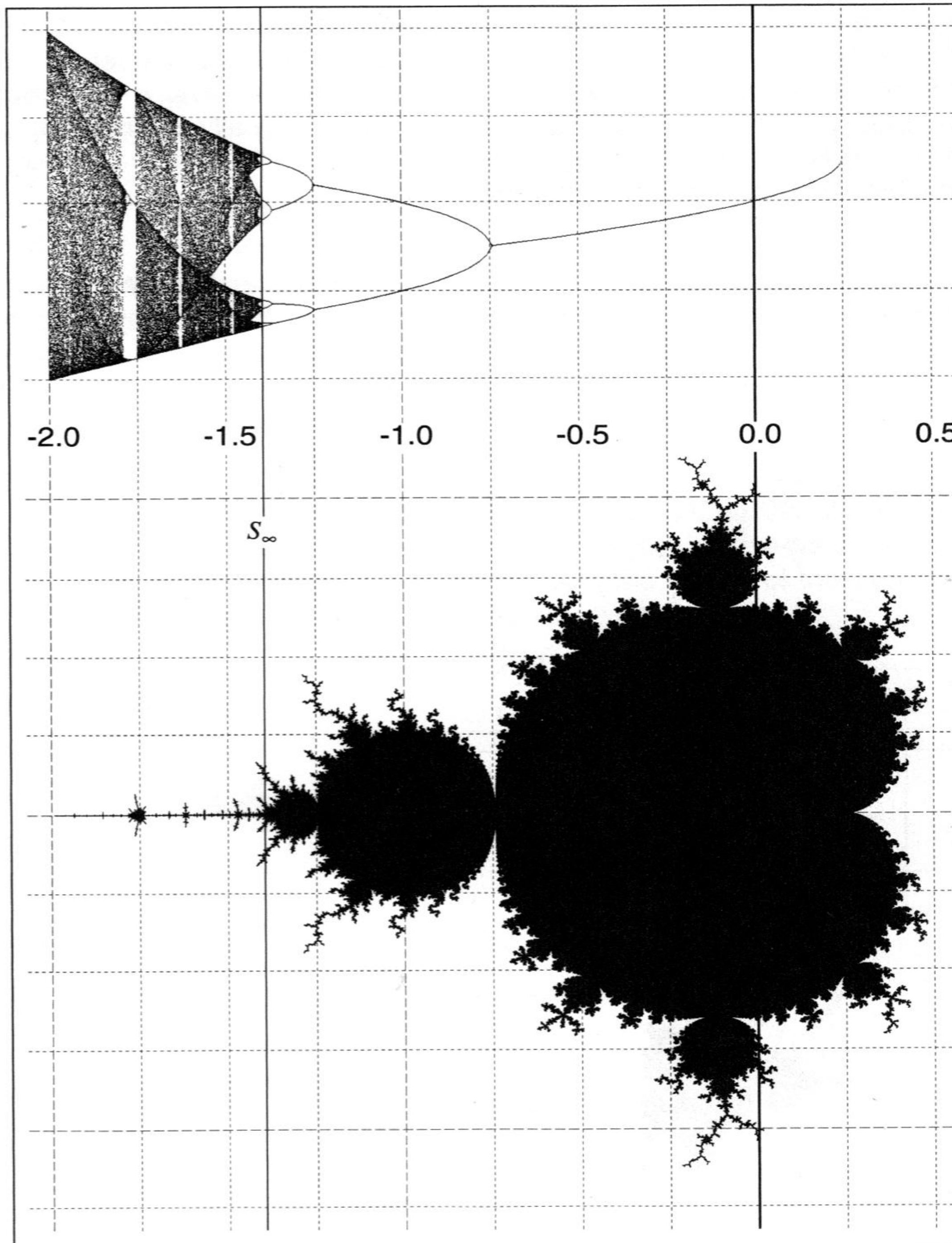




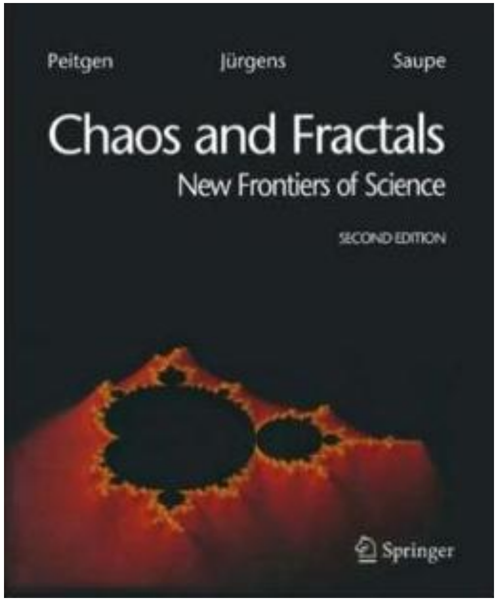
Self-similarity of the Mandelbrot set



Relation of the Mandelbrot set with the final state diagram for the logistic function



References:



Chaos and Fractals, by Peitgen, Jurgens, Saupe

Iterated Maps on the Interval as Dynamical Systems, by Pierre Collet and Jean-Pierre Eckmann. (Technical)

More resources on my webpage;   www.sfu.ca~rpyke/fractals