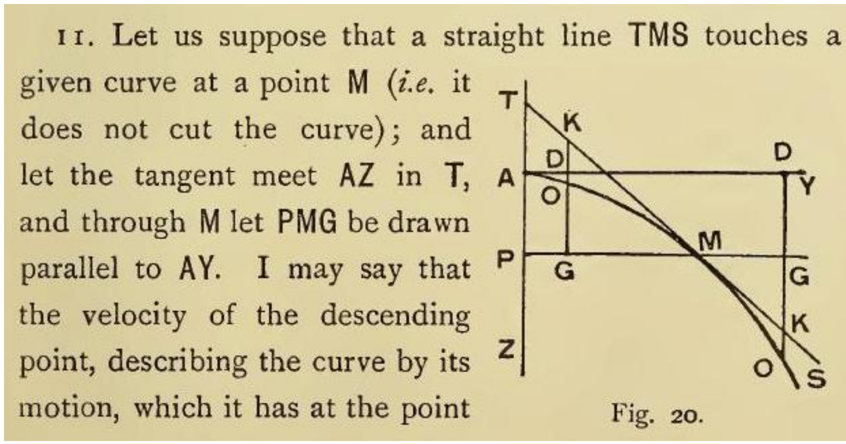
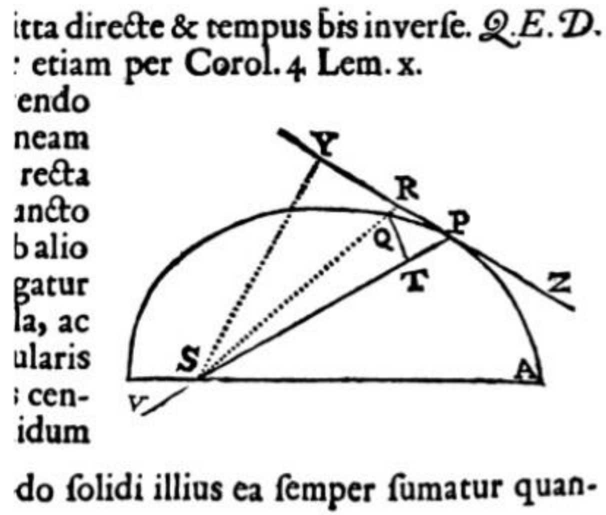
The Calculus of Newton and Maclaurin (circa 1690….)



11. Let us suppose that a straight line TMS touches a given curve at a point M (*i.e.* it does not cut the curve); and let the tangent meet AZ in T, and through M let PMG be drawn parallel to AY. I may say that the velocity of the descending point, describing the curve by its motion, which it has at the point

Fig. 20.



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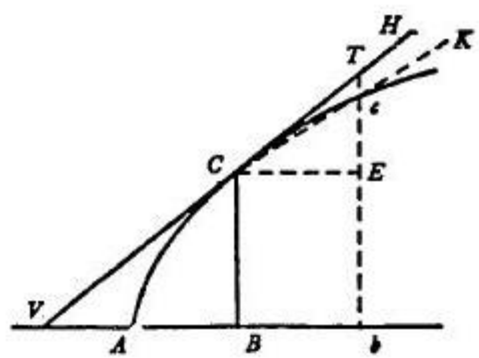


Newton: 1643-1727
Maclaurin: 1698-1746

Life and science in the 17^{th} century….

Renaissance; 14^{th} to 17^{th} centuries (Leonardo da Vinci, Raphael, Michelangelo, …)

Copernicus; astronomer 1473 – 1543
Galileo; astronomer, scientist 1564 – 1642. Careful measurements of motion.
Kepler; astronomer 1571 – 1630 (the first 'data scientist')
Descartes; philosopher, mathematician 1596-1650 ('Cartesian plane')
Leibniz; mathematician and philosopher, 1646-1716 ('abstract' calculus)
The Bernoullis'; mathematicians and physicists 1654 – 1782

Shakespeare; writer 1564 - 1616
J.S. Bach; composer 1685 - 1750

Reformation; 1517 – 1648 (Reformation of the Catholic Church in Europe)
European colonialism; begins in 15^{th} century (England, France, Belgium, Spain, Germany,..)
Great Plague (Bubonic plague); 1665-1666 England

Some physics and mathematics problems at the time:

Why are the orbits of planets ellipses with the sun at one focus?
Why is there Kepler's Three Laws of planetary motion?
The nature of tides.
The nature of gravity (objects falling to Earth)
Mechanics of pendulums, pulleys, projectiles ('practical engineering')
Adding forces, principles of statics and hydrostatics, centre of gravity
Given a force acting on an object, find the path the object follows

How to calculate the slope of a tangent line to a non-straight curve?
Finding the maximum and minimum of functions
Finding the area under a curve
Finding the lengths of curves
Determining the curvature of a curve
Finding roots of polynomials
Trigonometric Laws and identities
The algebraicalization of geometry

<u>Who was Newton?</u>

Issac Newton, 1643-1727 (aged 84)
Lived in England.

Studied natural science at Cambridge (Aristotle, Descartes, Galileo, …)
BA 1665, then plague hits England. Goes off to the countryside for 2 years and develops calculus and gravitational law (falling apple 1666….)

<u>Major works</u>: 1687 Principia (*Principles of Natural Philosophy*); mechanics, gravitation, tides, planetary orbits, fluid dynamics, … Needed the calculus.
1704 Opticks; refraction, reflection (also fluxions).
Invented the *Newtonian telescope*
1707 Arithmetica Universalis *(Universal Arithmetic)*
1736 Method of Fluxions (written 1671)

Studied the physics and chemistry of matter (e.g. Newton's Law of Cooling)

Also interested in theology and alchemy (e.g. tried to assign dates to biblical events)

Became warden and (later) Master of the Royal Mint in 1696 (*the Great Recoinage*)

<u>Who was Maclaurin?</u>

Colin Maclaurin, 1698-1746 (aged 48)
Lived in Scotland

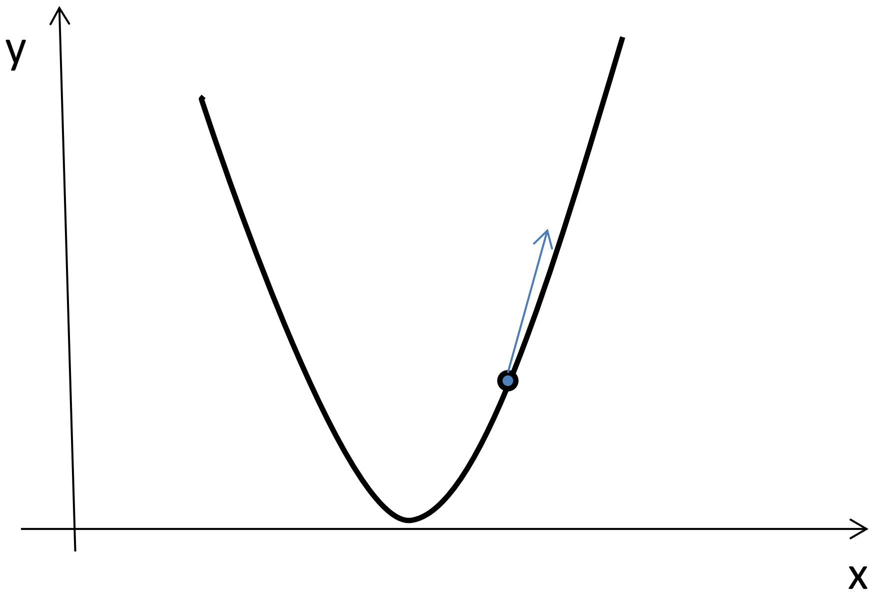
Begins studies at the University of Glasgow at age 11, completes his MA at age 14.
At age 19 is appointed Professor of Mathematics University of Aberdeen.
Travelled to England 1719-1721 and meets Newton.
Takes (unannounced) leave of absence from U of Aberdeen to live in London for 3 yrs.
Returns to Scotland and is appointed Professor of Mathematics University of Edinburgh
(Newton writes a letter of recommendation, even offers to pay for Maclaurin's salary)

<u>Major works</u>: 1742 Treatise on Fluxions (power series, max/min of functions, *Maclaurin spheroids,* numerical integration, elliptic integrals…)
1748 Treatise on Algebra ('Cramer's Rule', etc)

Joint founder of the Royal Society of Edinburgh, 1731 (then called the Medical Society of Edinburgh)

Newton and Maclaurin (and others) looked at mathematical problems as if they involved *dynamic, changing quantities*.

For example, when studying a curve they considered a point's x and y coordinates changing as it moved along it. From this they could discuss the curvature of the curve and tangent lines to the curve (among other things).



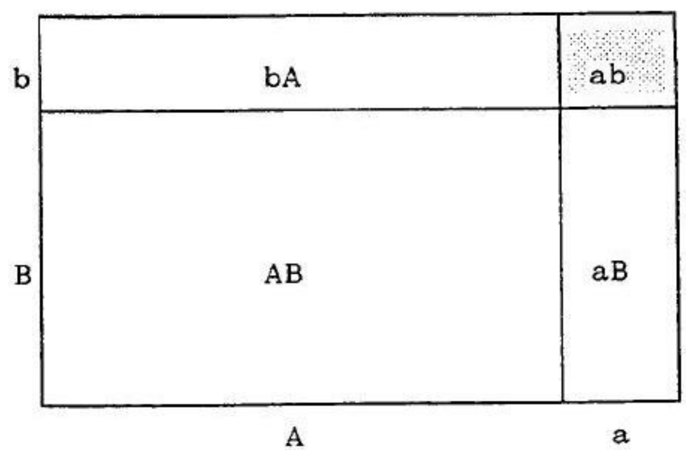
Furthermore, these problems were analyzed *in the small.* That is, by looking at how these quantities changed for very small increments (in time).
Not only was this 'microscopic' analysis enough to solve deep mathematical and physical problems, it also simplified them so that they *could* solve them.

Some terminology used by Newton and Maclaurin :

Fluent; something that changes ('moves'), e.g. points, lines, planes
Fluxion; the velocity at which the fluent is moving
Moment; the amount a fluent changes in a small amount of time due to its fluxion; moment = fluxion × time

Example. Suppose A and B are two fluents with moments a and b.
What is the moment of their product AB, *when a and b are very small*?

A increases to A+a and B increases to B+b. How much did AB increase?



The product AB increases by bA+ab+aB. But for very small moments a,b we neglect the product ab and only consider the change that is *proportional* to a and b; bA+aB.

So, in the limit of small a and b,
the moment of AB is bA + aB.
This is not exactly the same as the amount that AB changed, but is nearly so when a and b are small.

Fundamental problems of the calculus:

Given a relation between two fluents, find the relation between their fluxions, and conversely.

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For example, if x and y are related by $y = f(x)$, and x has fluxion (velocity) l, what is the fluxion of y?

Suppose fluxion (velocity) of y is m.
Then for time interval t, moment of x is lt and moment of y is mt.

Fundamental hypothesis; when the time interval is arbitrarily small, the moment of y is proportional to the moment of x.

This implies that the fluxion of y is proportional to the fluxion of x (at that time). That is, the rate of change of y with respect to time (its velocity m) is proportional to the rate of change of x with respect to time (its velocity l).

Let, α = moment of y/moment of x when the time interval is arbitrarily small.
Then α = mt/lt = m/l = *velocity of y/velocity of x* = $\frac{change\ in\ y}{change\ in\ t} \times \frac{change\ in\ t}{change\ in\ x} = \frac{change\ in\ y}{change\ in\ x}$

In modern terminology, α is the derivative of f; $\alpha = \frac{dy}{dx} = f'(x)$, *the rate of change of y with respect to x.*

This is Calculus I

What is the geometric significance of the derivative? →

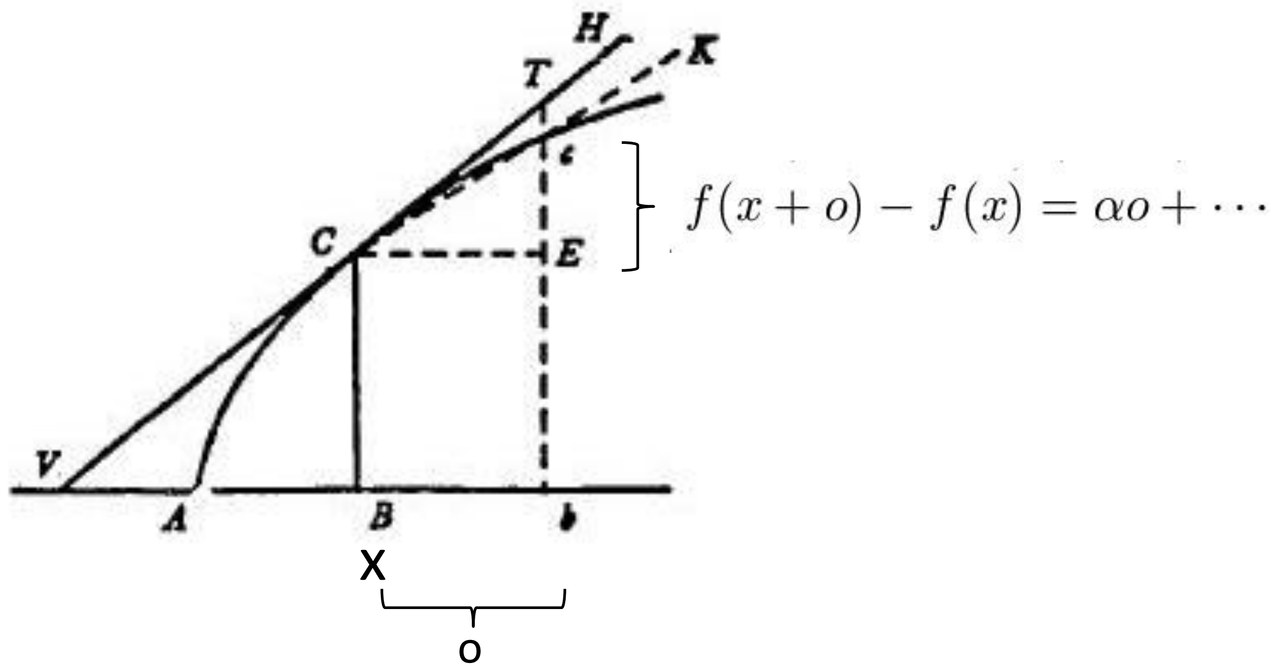
Fluxion of $y = f(x)$;

o is the moment of x, so αo is the moment of y

$x \mapsto x + o,$

$$f(x+o) = f(x) + \alpha o + \cdots$$

$$f(x+o) - f(x) = \alpha o + \cdots$$



As o vanishes, triangles CET and CEc are similar

$\rightarrow$ slope of tangent line is $\alpha o / o = \alpha$

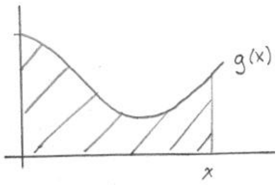
Thus we see that the derivative of *f(x) at x, which is α,* is the slope of the tangent line to the graph of *f(x) at x.*

Given a relation between two fluents, find the relation between their fluxions, and conversely.

Now, given a relation between the fluxions of *x* and *y*, find a relation between *x* and *y*.

For example, suppose *x* has fluxion *l* and *y* has fluxion *m* such that $\frac{m}{l} = g(x)$. Find $y = f(x)$.
In modern terminology this amounts to solving the differential equation
dy/dx = g(x) for y; This is Calculus II (and Math 310, Differential Equations)

solution is: $y = f(x) = \underbrace{\int_0^x g(s)ds}_{\text{this is the (signed) area under the curve } g(x)}$



Why? Because the derivative of *f(x)* is *g(x)* →

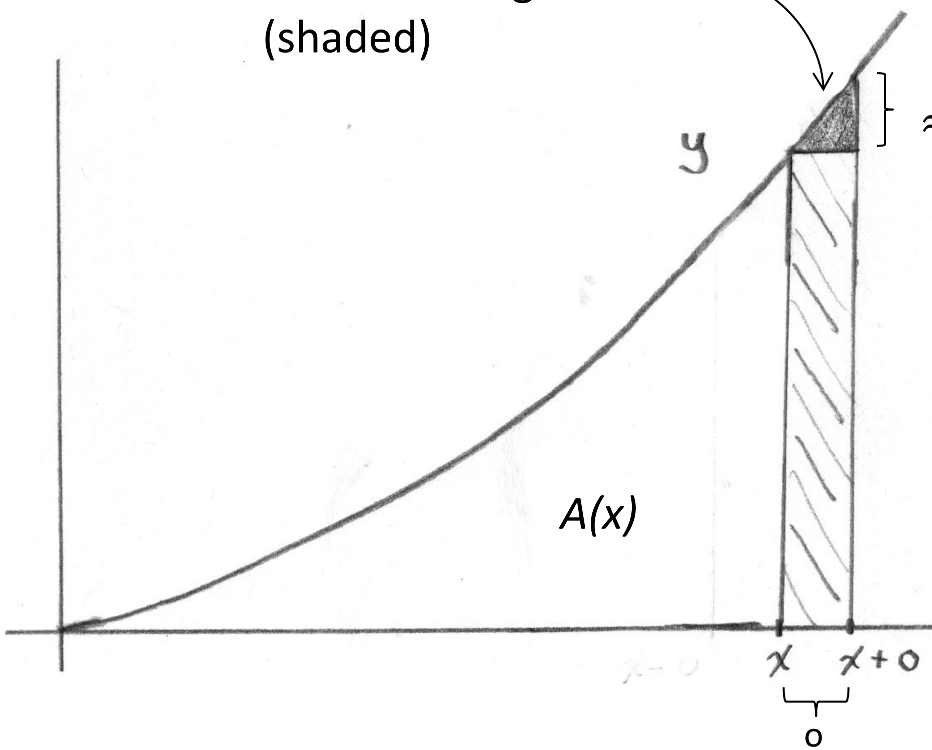
(remember; the derivative of y is the ratio $\frac{fluxion\ of\ y}{fluxion\ of\ x}$)

Derivative of the area A(x) under a curve y=g(x);

Moment of A(x) is the change of area when x changes from x to x+o.

The derivative of A(x) is the ratio of the moments.

$$A(x) = \int_0^x g(s)ds$$



Area of rectangle $= y.o$

Exact change in area is part below the graph (rectangle + error)

Exact change in area under graph (moment) is *y.o* + error

Error $\approx m\ o^2$

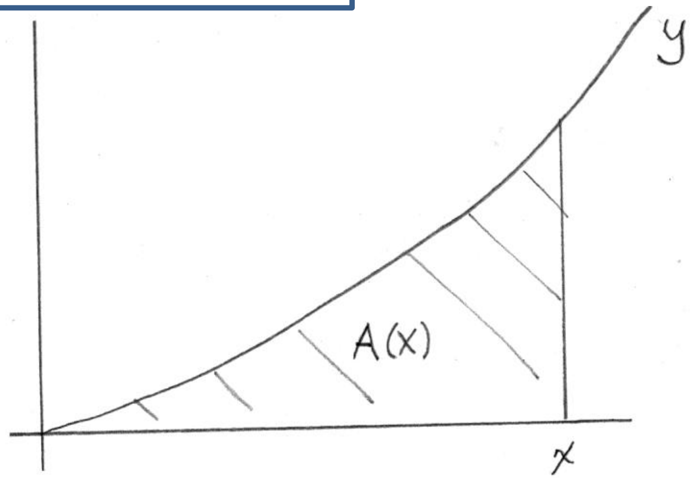
So as o becomes smaller and smaller, the error disappears more quickly than the area itself → moment of A(x) becomes almost equal to *y.o,* and so the derivative of *A(x)* is $\frac{y.o}{o}$ = *y* = *g(x).*

An example.... →

The derivative of the area A(x) under the curve of f(x) is f(x)=y

Example: Suppose $A(x) = x^n$

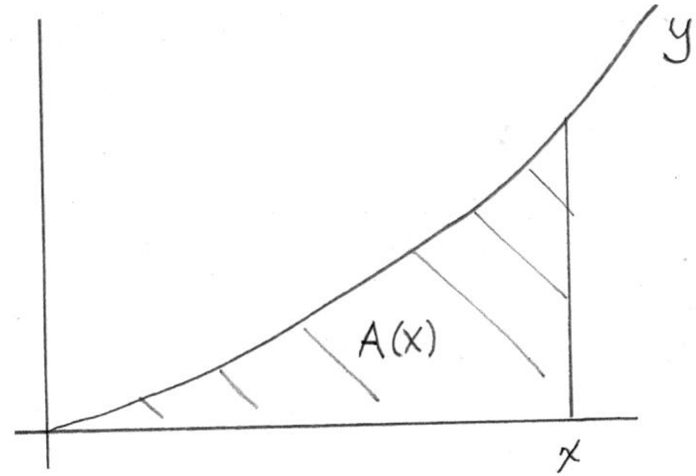
What is y?



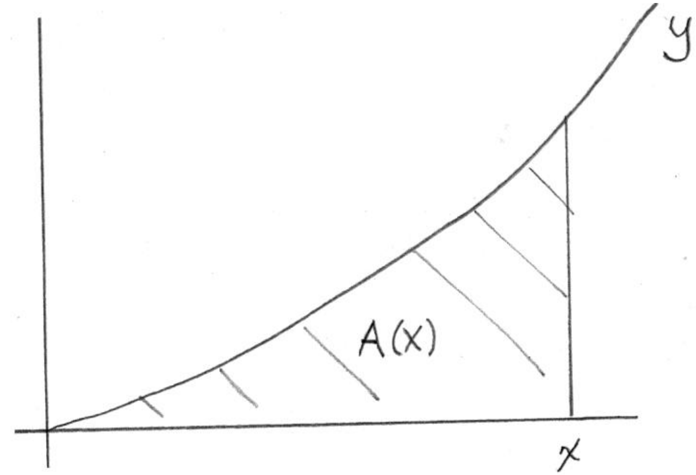
Example: Suppose $A(x) = x^n$

What is y?

From previous, we know that y = A'(x)
We compute A'(x) by the method of fluxions.



Example: Suppose $A(x) = x^n$



$$
\begin{aligned}
(x+o)^n &= x^n + nx^{n-1}o + \frac{(n(n-1)}{2}x^{n-2}o^2 + \cdots + nxo^{n-1} + o^n \\
&= x^n + o\left\{nx^{n-1} + \frac{(n(n-1)}{2}x^{n-2}o + \cdots + nxo^{n-2} + o^{n-1}\right\} \\
&= x^n + o\left(nx^{n-1}\right) + o^2(\ \cdots\ )
\end{aligned}
$$

$\Longrightarrow$ The ratio of fluxions is nx^{n-1}. That is $A'(x) = nx^{n-1} = y$

The derivative of the area under the curve of $f(x)$ is $f(x)$;

$$A(x) = \int_0^x f(x)\,dx \ \longrightarrow A'(x) = f(x)$$

Fundamental Theorem of Calculus

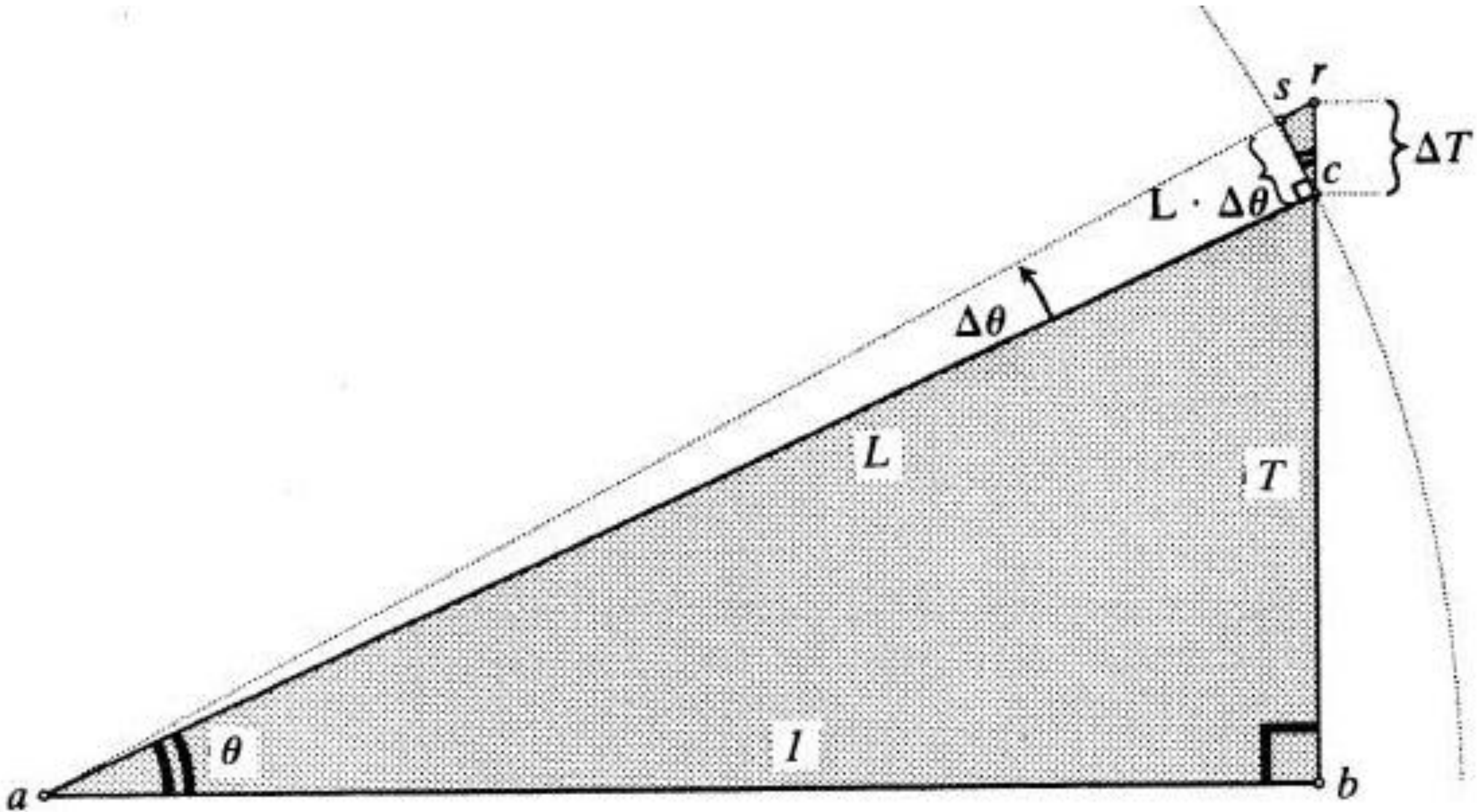
Two methods for computing derivative of *f(x)* by the method of fluxions;

1. Algebraically; e.g., expand *f(x+o)* in a power series in powers of *o* (Maclaurin) and look for the coefficient ('modulus') of the term linear in *o*

2. Geometrically; e.g., cos, sin, tan, ….

$$\tan\theta = T, \qquad \sec\theta = L$$
$$\tan(\theta + \Delta\theta) = T + \Delta T$$

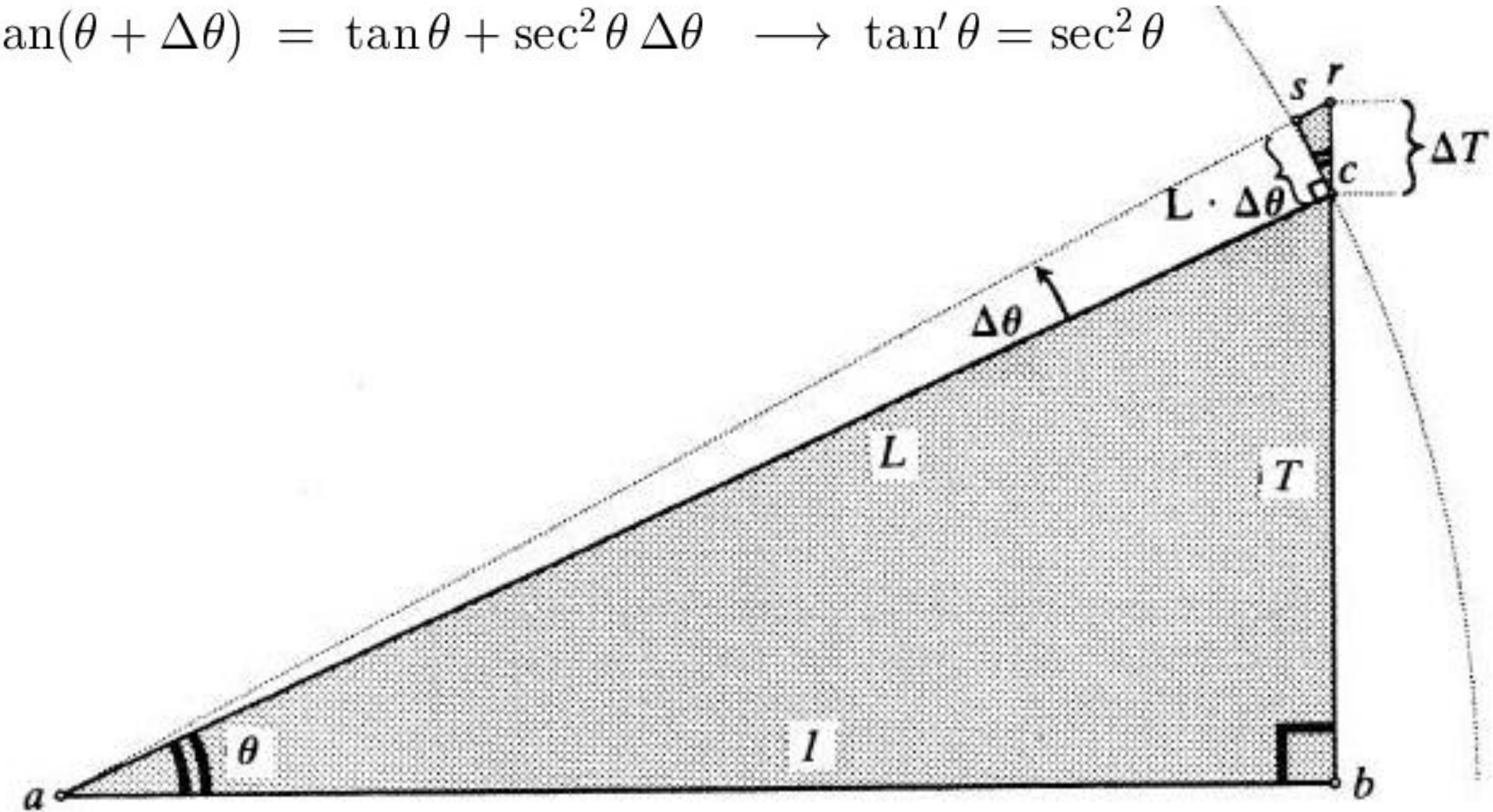
(From T. Needham, 1993)



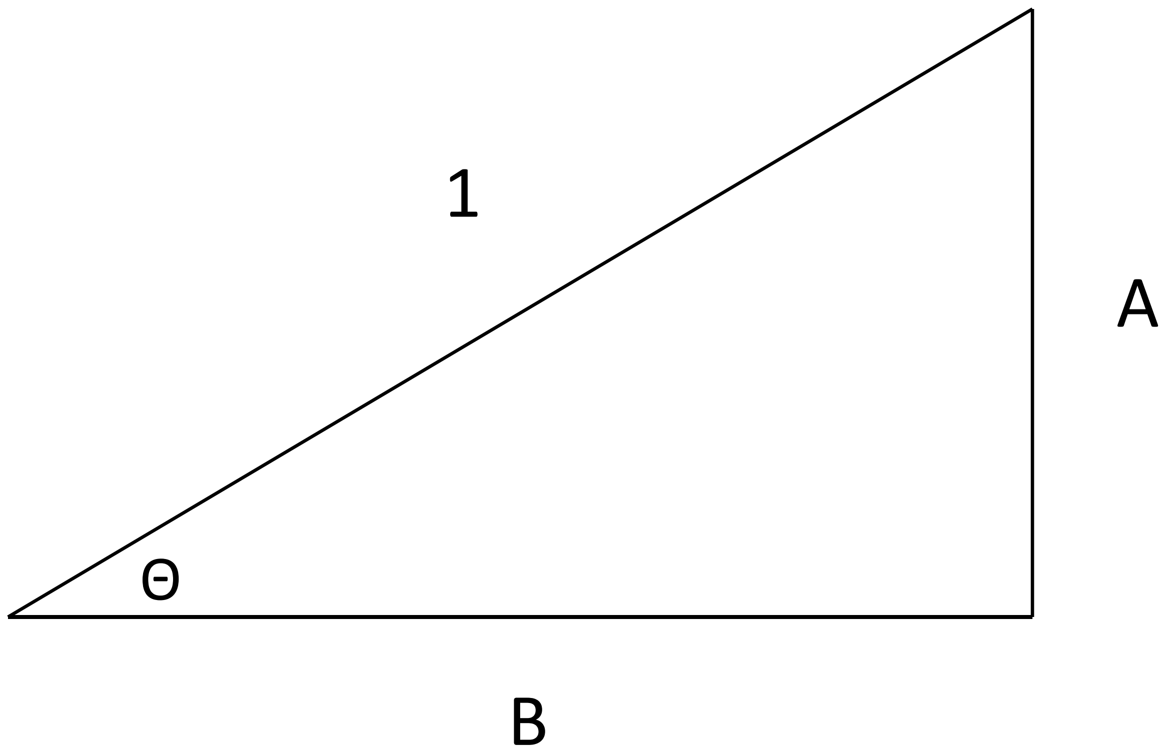
$$\tan\theta = T, \qquad \sec\theta = L$$
$$\tan(\theta+\Delta\theta) = T + \Delta T$$

$$\text{Similar triangles; crs} \sim \text{acb} \implies \frac{L}{1} = \frac{cr}{cs} \approx \frac{\Delta T}{L\Delta\theta}$$
$$\implies \Delta T \approx L^2\Delta\theta = \sec^2\theta\,\Delta\theta$$

$$\tan(\theta+\Delta\theta) = \tan\theta + \sec^2\theta\,\Delta\theta \longrightarrow \tan'\theta = \sec^2\theta$$



HW: Deduce that $\cos'\theta = -\sin\theta$ and $\sin'\theta = \cos\theta$;



Newton deduced the Product, Quotient, and Chain Rules of differentiation

Product Rule: $h(x) = f(x)g(x)$. We want to determine α in

$$h(x+o) = h(x) + \alpha o + \cdots$$

We have that

$$\begin{aligned} f(x+o) &= f(x) + \alpha_1 o + \cdots \quad & \alpha_1 = f'(x) \\ g(x+o) &= g(x) + \alpha_2 o + \cdots \quad & \alpha_2 = g'(x) \end{aligned}$$

So,

$$\begin{aligned} h(x+o) &= f(x+o)g(x+o) \\ &= [f(x) + \alpha_1 o + \cdots][g(x) + \alpha_2 o + \cdots] \\ &= f(x)g(x) + o\alpha_1 g(x) + o\alpha_2 f(x) + \alpha_1\alpha_2 o^2 + \cdots \\ &= h(x) + o\left(\alpha_1 g(x) + \alpha_2 f(x)\right) + \cdots \end{aligned}$$

$$\Rightarrow \quad \alpha = h'(x) = f'(x)g(x) + f(x)g'(x)$$

Newton’s and Maclaurins fluxion approach to calculus was superseded by the algebraic approach of Leibniz and others in the 18th century (‘differentials’, ‘infinitesimals’). This lead to great successes in the field of mechanics and celestial mechanics.

In the 19th century Augustine-Louise Cauchy formulated the ‘modern’ approach to calculus by developing the formal limit concept (‘analysis’). This is the calculus you learn now in textbooks.

This presentation: www.sfu.ca/~rpyke --> presentations; Fluents and Fluxions

Some references:

T. Needham, "*Newton and the Transmutation of Force*". The American Mathematical Monthly, Vol. 100, No. 2 (Feb., 1993).

J.V. Grabiner, "*Was Newton's Calculus a Dead End? The Continental Influence of Maclaurin's Treatise of Fluxions*". The American Mathematical Monthly, Vol. 104, No. 5 (May, 1997).

M. Kline, "*Mathematical Thought from Ancient to Modern Times*". Oxford University Press, 1964.

E.R. Sageng, "*Colin Maclaurin and the Foundations of the Method of Fluxions*", PhD Thesis, Princeton University, 1989.