How to draw fractals using an IFS

An Iterated Function System (IFS) is a function W that acts on images; $W : \mathcal{M} \to \mathcal{M}$. Here, $\mathcal{M}$ is the set of all bounded images (that is, images of finite size). The IFS is composed of a number of affine transformations w_i;

$$W = w_1 \cup w_2 \cup \cdots \cup w_k, \quad w_i = A_i + v_i$$

where each A_i is a 2 by 2 matrix and v_i is a vector in two-dimensions;

$$A_i = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad v_i = \begin{bmatrix} e \\ f \end{bmatrix}$$

The IFS acts on an image S by taking the union of all the images of S under the k affine transformations;

$$W(S) = w_1(S) \cup w_2(S) \cup \cdots \cup w_k(S)$$

So, W creates k distorted images of S.

Each affine transformation w_i is a contraction (on $\mathbf{R}^2$) if

$$\|w_i(p_1) - w_i(p_2)\| \le c\|p_1 - p_2\|, \quad \text{for some } 0 < c < 1 \text{ and for all points } p_1, p_2$$

where $\| \bullet \|$ denotes the distance between points. That is, w_i brings points together ($w_i(p_1)$ and $w_i(p_2)$ are closer together than p_1 and p_2).

An important property of such contractions is contained in the Contraction Mapping Principle;

> If $h : \Omega \to \Omega$ is a function defined on a set Ω, and if h is a contraction with respect to some metric (distance function) $\| \bullet \|$ that measures distances between points in Ω, then h has a unique fixed point $\bar{p} \in \Omega$; $h(\bar{p}) = \bar{p}$, and if x is any point in Ω, then $h^k(x) \to \bar{p}$ as $k \to \infty$, where h^k denotes the k^{th} composition of h. (What this means precisely is that $\|h^k(x) - \bar{p}\|$, the distance between $h^k(x)$ and $\bar{p}$, can be made as small as you like by choosing k sufficiently large.)

On the space $\mathcal{M}$ of images, distances are measured by the *Hausdorff distance* $\| \bullet \|_H$, which, roughly speaking, measures the (total) area of the regions where two images do not coincide (so if $\| \bullet \|_H$ is small, the two images look similar). It can be proven than if each w_i is a contraction on $\mathbf{R}^2$, then the IFS W is a contraction on $\mathcal{M}$ with respect to $\| \bullet \|_H$. Thus, according to the Contraction Mapping Principle, if S is any image, and if $\mathcal{F}$ is the fixed point of the IFS W, (so that $W(\mathcal{F}) = \mathcal{F}$), then $\|W^k(S) - \mathcal{F}\|_H \to 0$ as $k \to \infty$, or in other words, $W^k(S) \to \mathcal{F}$ as $k \to \infty$. That is, if we iterate the IFS beginning with any image S, it looks more and more like the fixed point image $\mathcal{F}$ as we iterate.

So, to draw a given fractal $\mathcal{F}$ you take its IFS W (so $W(\mathcal{F}) = \mathcal{F}$) and iterate beginning with any image S; $W^k(S)$. After iterating enough times your resulting image will look like the fractal. However, if the contraction rate of one or more lenses is too large, the number of iterations required to produce the fractal may be too large for a computer to draw the fractal in a reasonable amount of time.

We see too that the IFS determines the self-similarity of the fractal; $W(\mathcal{F}) = w_1(\mathcal{F}) \cup \cdots \cup w_k(\mathcal{F}) \to \mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_k$.