Math 178

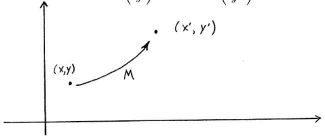
Notes on Affine Transformations (See Section 5.2 in text)

An *affine transformation* w is defined by $w = M + v$ where M is a *linear transformation* and v is the *shift vector*. The linear transformation M can be represented by a matrix of 4 numbers; $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and the vector v can be represented by 2 numbers; $v = \begin{pmatrix} e \\ f \end{pmatrix}$. So all together, 6 numbers a, b, c, d, e, f define an affine transformation (and these are the numbers you use in the computer programs that draw fractals).

M acts acts on points $\begin{pmatrix} x \\ y \end{pmatrix}$ in the plane $\mathbf{R}^2$ through the formula

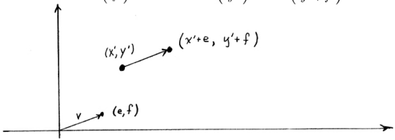
$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \quad \text{where} \quad \begin{matrix} x' & = & ax + by \\ y' & = & cx + dy \end{matrix}$$

That is, M sends the point $\begin{pmatrix} x \\ y \end{pmatrix}$ to the point $\begin{pmatrix} x' \\ y' \end{pmatrix}$;

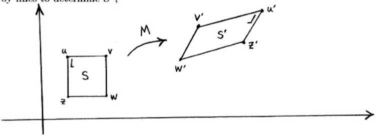


For example, if $M = \begin{pmatrix} -1 & 0.5 \\ 2 & 1.1 \end{pmatrix}$, then $M\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} (-1)2 + (0.5)(-3) \\ (2)2 + (1.1)(-3) \end{pmatrix} = \begin{pmatrix} -3.5 \\ 0.7 \end{pmatrix}$. So this M sends $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ to $\begin{pmatrix} -3.5 \\ 0.7 \end{pmatrix}$.

The vector $v = \begin{pmatrix} e \\ f \end{pmatrix}$ moves the point $\begin{pmatrix} x' \\ y' \end{pmatrix}$ to $v = \begin{pmatrix} x' + e \\ y' + f \end{pmatrix}$;



To visualize the action of an affine transformation M, we can draw what it does to a square S (for example); $M(S) = S'$. Since an affine transformation moves lines into lines, we only need to know where M moves the corners of S, and then join them by lines to determine S';



Here are some examples of linear transformations (see the course webpage for illustrations of these using the VB program *fractal pattern*);

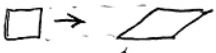
dilations; $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ (shrinks if $|r| < 1$, expands if $|r| > 1$)

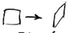
uneven dilations; $\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}$, $r \neq s$

reflection across y-axis; $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

reflection across x-axis; $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

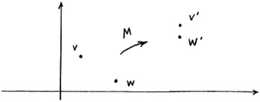
reflection across the diagonal; $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

x-shear; $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ 

y-shear; $\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$ 

rotation by 45^o counter clockwise; $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$; $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ ← rotation by θ

An affine transformation M is a *contraction* if it moves points closer together;



An *Iterated Function System* (IFS) W is a collection of affine transformations $w_1, \dots w_k$;

$$W = w_1 \cup w_2 \cup \cdots \cup w_k$$

Important fact: If each of the affine transformations $w_1, \dots w_k$ are contractions, then the IFS W defines a unique image (fractal) through iteration; $W^n(S) \to \mathcal{F}$ as $n \to \infty$, and this fractal is a *fixed point* of the IFS; $W(\mathcal{F}) = \mathcal{F}$.