Math 178 Examination Summary

Please bring a ruler and calculator to the exam. The exam is Saturday Aug 11 8:30am - 11:30 in WMC2507 (but you should confirm this yourself).

Office hours: Wednesday Aug 8 12:30am - 2:30pm; Friday Aug 10 1:30pm - 4pm. (In both cases this could be longer if necessary (eg., we are busy). But let me know if you cannot show up until after these hours.). Both at the math department lounge (or my office which is just down the hallway). Note that term projects are due friday Aug 10 at 4pm. You may give them to me or email them to me.

`This information may be updated/corrected; please check the course webpage`

- Definitions: self-similarity, affine transformations, Contraction Mapping Principle, IFS, random sequence with probabilities $p_1, \ldots, p_k$, stable and unstable fixed points, stable sets, periodic orbits, prime period, ergodic orbits, time-series and histograms are, final-state diagram and bifurcation diagram, Julia and Mandelbrot sets, threshold radius $r(c)$ for Julia sets
- Explanations/Descriptions: chaos game (including proof why it works), how IFS draws a fractal, using log-log plots of $a(s)$ vs $1/s$ to estimate fractal dimension, consistent coverings, describing and explaining the structure of the final state diagram for the logistic equation,
- Calculations: determining the base k expansion $[x]_k$ of a number x, computing the fractal dimension D of a set and length exponent d of a curve, determining fixed points of an affine transformation $w = A + v$, determining the composition of affine transformations $w_{12} = w_1 \circ w_2$ (that is determining the matrix A_{12} and the shift vector v_{12} of w_{12}); calculating how long it would take a computer to draw a fractal via iterating an IFS or playing the chaos game with a certain random sequence, using graphical iteration to determine the orbits of a function $f(x)$, using symbolic dynamics to find periodic and ergodic points,
- Practice questions for Julia and Mandelbrot: Definitions of Julia sets and the Mandelbrot set for $q_c(z) = z^2 + c$, descriptions of the two methods to draw Julia sets (encirclements and chaos game), what is the basic method to draw the Mandelbrot set? (consider the orbit of 0), How can we determine some parts of the Mandelbrot set by considering cases when the Julia set has a stable period k orbit? (you should include some formulae here), If P_c has zero area is c necessarily in the Mandelbrot set?, If z is in P_c then the entire orbit of z must be in P_c and similarly for E_c,

Essay questions. (3 of the following 5 questions will be on the exam. You will be asked to write a 2 page essay on one of your choosing. The essay question will be worth 25% of the exam mark, which is approximately 45min of exam time.)

- Explain how one would produce the final state diagram of the logistic equations for parameter values between 0 and 4. Describe the features of the final state diagram and explain as many as you can by considering the shape of the compositions of the logistic equation. Explain what Universality is in this context.
- Write a brief 'instruction manual' that would help someone to create images of fractals using the VB program Fractal Pattern (that is available on the course cd). Imagine the person is trying to create certain new fractals, so you should describe how one should define the IFS to produce fractals with certain properties (eg., certain

type of self-similarity). You should also contrast the method of iterating an IFS and using the chaos game to draw the fractal. You can assume the person knows what a fractal is, what an IFS is and what the chaos game is, but they have not used any computer software to draw fractals.

(If you haven't used the Fractal Pattern program, I've posted a screen shot of the user interface on the course web page (index page) so you can see all the features of the program; you can load the parameters for the common fractals, or you can input the a, b, c, d, e, f parameters yourself, or input instead the two angles ψ, ϕ and reduction rates r, s as described in the text in section 5.2 and have the program compute the a, b, c, d, e, f, you can print the blue print of the IFS in addition to (and on top of) the image of the fractal, you can iterate the IFS, and it also can generate images using the chaos game (with random sequences of specified probabilities.))

- Give your view of the role chaos has played in science (and/or in other areas outside of mathematics). Begin with some historical remarks and definitions. Include some examples. Discuss any benefits, limitations, or disadvantages that the pursuit (search for) chaos in real systems may bring.
- Describe how fractals can be used to describe some objects in nature. Are natural objects fractals? Why is the 'language' of fractals more appropriate than 'classical' geometry in this context? What is complexity in this context? The related notion of self-similarity also arises in nature and art; describe this along with some examples. How can fractals be used to create music?
- Discuss Julia sets and the Mandelbrot set in the context of fractals and dynamics. For example, are these sets fractals in the sense we've defined (with affine transformations and IFS) or do they differ and in what ways? Discuss the differences in appearance of classical fractals (eg., Sierpinski, von Koch, the Fern, etc) and Julia sets. What variety can you find in fractals, and what variety can you find in Julia sets? Why do discrete dynamical systems appear in this context? How can one draw these objects? Are there practical limitations in these ways to draw these sets, and if so, what are they?

Formulae provided;

$$1 + r + r^2 + r^3 + \cdots = \sum_{i=0}^{\infty} r^i = \frac{1}{1-r}, \qquad r + r^2 + r^3 + r^4 + \cdots = \sum_{i=1}^{\infty} r^i = \frac{r}{1-r}$$

$$\log(x^n) = n\log x, \quad \log(xy) = \log x + \log y, \quad \log\left(\frac{x}{y}\right) = \log x - \log y, \quad \log 1 = 0$$

$$\log(10^x) = x, \quad 10^{\log y} = y, \quad \ln(e^x) = x, \quad e^{\ln y} = y$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{v}_1 = (a_1, b_1), \quad \mathbf{v}_2 = (a_2, b_2); \quad \|\mathbf{v}_1 - \mathbf{v}_2\|^2 = (a_1 - a_2)^2 + (b_1 - b_2)^2$$

$$z = a + ib, \quad z = r(\cos\theta + i\sin\theta) = re^{i\theta}, \quad r = |z| = \sqrt{a^2 + b^2}, \quad \theta = \tan\frac{b}{a}$$

$$(a+ib) + (x+iy) = (a+x) + i(b+y), \quad (a+ib)(x+iy) = (ax - by) + i(ay + xb), \quad e^{i\theta} = \cos\theta + i\sin\theta$$

$$h(x) = \sin^2\left(\frac{\pi x}{2}\right)$$