We connect two ideas; complex numbers ('two dimensional' real numbers) and rotations of the plane, $\mathbf{R}^2$. We will further indicate another, similar, idea that connects certain 'three dimensional' complex numbers ('quaternions') with rotations in 3 dimensions, $\mathbf{R}^3$.

Note that we can identify a skew-symmetric 2×2 matrix A_z to each complex number $z = a + ib,\ i^2 = -1$, via

$$A_z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad z = a + ib$$

This is a 'good' representation of complex numbers as matrices since;

$$\begin{aligned} A_{z_1} + A_{z_2} &= A_{z_1+z_2} && \text{(preserves addition)} \\ A_{z_1}A_{z_2} &= A_{z_1 z_2} && \text{(preserves multiplication)} \\ A_{\bar{z}} &= A_z^T && \text{(conjugation becomes transpose)} \\ A_{1/z} &= A_z^{-1} && \text{(preserves inverses - check this!)} \end{aligned}$$

Note that under this identification the real numbers are diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ (so 1 corresponds to I_2), and the complex number i corresponds to the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Now we connect this to rotations. Recall that a complex number of modulus 1 can be written as

$$z = \cos\theta + i\sin\theta; \quad \text{and as a vector in the complex plane as } \mathbf{z} = (\cos\theta,\ \sin\theta)$$

so that the matrix corresponding to these numbers are

$$z = \cos\theta + i\sin\theta \longrightarrow A_z = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = R_\theta$$

(R_θ is counterclockwise rotation by θ.)Thus, we can identify complex numbers of modulus 1 with rotations of the plane $\mathbf{R}^2$. For example, i corresponds to rotation by $\pi/2$ radians (clockwise), 1 to no rotation ($\theta = 0$), and -1 corresponds to rotation by π (of course). So complex multiplication by a number z of modulus 1 becomes rotation of vectors;

$$|z| = 1 \iff zw = R_\theta \mathbf{w}$$

(complex multiplication on the left side, matrix multiplication on the right side in the second equality) In this way we can realize rotations in $\mathbf{R}^2$ using complex multiplication instead of using matrix multiplication. We now outline an approach to realizing rotations in $\mathbf{R}^3$ using multiplication of another type of number rather than using matrix multiplication. These numbers are called quaternions.

A **quaternion** (denoted by $\mathbf{H}$, for Hamilton who was one of the 'discoverers' of quaternions) is a number of the form

$$q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$$

where a, b, c, d are real numbers, and the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are 'numbers' that satisfy the following multiplication rules;

$$\begin{aligned} \mathbf{i}^2 &= \mathbf{j}^2 = \mathbf{k}^2 = -1 \\ \mathbf{ij} &= \mathbf{k}, \quad \mathbf{ji} = -\mathbf{k} \\ \mathbf{ki} &= \mathbf{j}, \quad \mathbf{ik} = -\mathbf{j} \\ \mathbf{jk} &= \mathbf{i}, \quad \mathbf{kj} = -\mathbf{i} \end{aligned}$$

Note that the last group of 6 products is exactly the same as if we were using the cross product with the standard unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in $\mathbf{R}^3$ (e.g., $\mathbf{i} \times \mathbf{j} = \mathbf{k}$).

As an example, if $p = 3 + 2\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}$ and $q = 4 + 6\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$, then

$$p + q = q + p = 7 + 8\mathbf{i} + 14\mathbf{j} + 16\mathbf{k}$$

and

$$pq = -111 + 24\mathbf{i} + 72\mathbf{j} + 35\mathbf{k}, \quad qp = -111 + 28\mathbf{i} + 24\mathbf{j} + 75\mathbf{k}$$

More generally, if $q = [a, \mathbf{v}]$, $p = [\alpha, \mathbf{w}]$, then

$$qp = [a\alpha - \mathbf{v} \cdot \mathbf{w}, \; a\mathbf{w} + \alpha\mathbf{v} + \mathbf{v} \times \mathbf{w}]$$

Note that quaternions 'extend' complex numbers in the sense that $\mathbf{C} \subset \mathbf{H}$, set $c = d = 0$, and hence $\mathbf{H}$ also extends $\mathbf{R}$; $\mathbf{R} \subset \mathbf{H}$.

We see that multiplication in $\mathbf{H}$ is not commutative (but it is associative! and that's the reason for the 'complicated' multiplication rules between the $\mathbf{i}, \mathbf{j}, \mathbf{k}$). However, when you restrict quaternions by setting $c = d = 0$, *those* quaternions (i.e., complex numbers) *do* satisfy commutativity. Incidentally, it is a theorem that you cannot extend $\mathbf{H}$ to a larger set of numbers that satisfy associativity and extend the 'usual' rules of algebra for $\mathbf{R}$ and $\mathbf{C}$, so in this sense, $\mathbf{H}$ is the *largest* set of (associative) numbers that contains $\mathbf{R}$ and $\mathbf{C}$.

Similarly as we did for complex numbers, we define the conjugate $\bar{q}$ of a quaternion $q = a + ib + ic + id$ to be $\bar{q} = a - ib - ic - id$. Then (check this!), the modulus (absolute value) of a quaternion q is $\sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$.

Now, just as complex numbers can be used to represent rotations in $\mathbf{R}^2$, quaternions can be used to represent rotations in $\mathbf{R}^3$ (and this was another reason for their 'invention'). You may have realized that rotations in $\mathbf{R}^3$ are much more complicated than rotations in $\mathbf{R}^2$, namely because of the extra freedom in choosing the axis of rotation in $\mathbf{R}^3$ (rotations in $\mathbf{R}^2$ have only one choice of the axis of rotation, the 'z -axis'). And so you may not be surprised to learn that the connection between quaternions as numbers (i.e., as represented above) and rotations in $\mathbf{R}^3$ is a much more complicated route than the indentification of numbers in $\mathbf{C}$ and rotations in $\mathbf{R}^2$ that was described above.

Rotations R in $\mathbf{R}^3$ are associated to quaternions in the following way. Suppose the axis of rotation is given by the unit vector $\mathbf{n} = (n_1, n_2, n_3)$, and the angle of rotation about this axis (given by the right hand rule) is θ. Then this rotation is represented by the quaternion $\mathbf{p}_R$;

$$p_R = \cos\frac{1}{2}\theta + n_1 \sin\frac{1}{2}\theta\,\mathbf{i} + n_2 \sin\frac{1}{2}\theta\,\mathbf{j} + n_3 \sin\frac{1}{2}\theta\,\mathbf{k}$$

which we will write more compactly as

$$p_R = [\cos\frac{1}{2}\theta,\ \mathbf{n}\sin\frac{1}{2}\theta]$$

Conversely, any unit quaternion (and these can always be written as $[\cos\frac{1}{2}\theta,\ \mathbf{n}\sin\frac{1}{2}\theta], |\mathbf{n}| = 1$), represents a rotation in $\mathbf{R}^3$ (given by $\mathbf{n}$ and θ).

Now we want to discuss how we can realize rotation of a vector $\mathbf{r}$ in $\mathbf{R}^3$ using quaternions (in particular, using multiplication of quaternions). We've just described how to identify rotations in $\mathbf{R}^3$ with quaternions, so now we have to describe how to identify vectors $\mathbf{r}$ in $\mathbf{R}^3$ with quaternions. To that end, given a unit vector $\mathbf{r} = (r_1, r_2, r_3) \in \mathbf{R}^3$, we define the quaternion $q_{\mathbf{r}} = [\cos\frac{1}{2}\theta,\ \mathbf{r}\sin\frac{1}{2}\theta]$ corresponding to the rotation around $\mathbf{r}$ by the angle $\theta = \pi$;

$$q_{\mathbf{r}} = 0 + r_1\,\mathbf{i} + r_2\,\mathbf{j} + r_3\,\mathbf{k} = [0,\ \mathbf{r}]$$

Quaternions of this form, with $a = 0$, are called *pure quaternions*.

Let $\bar{\mathbf{r}} = R\mathbf{r}$ be the rotation of $\mathbf{r}$ by R. Now, due to the rather complicated nature of rotations in $\mathbf{R}^3$ (and this is elucidated in the books by Altman and Vince), the quaternion that represents $\bar{\mathbf{r}}$ is given by the axis of rotation of the (pure) quaternion $q_{\bar{\mathbf{r}}}$;

$$q_{\bar{\mathbf{r}}} \;=\; [0,\ \bar{\mathbf{r}}] \;=\; p_R q_{\mathbf{r}} (p_R)^{-1} = p_R q_{\mathbf{r}} (p_R)^{-1}, \quad \text{where} \quad (p_R)^{-1} = [\cos\frac{1}{2}\theta,\ -\mathbf{n}\sin\frac{1}{2}\theta]$$

(Note that the inverse $(p_R)^{-1}$ of p_R is a rotation by amount $-\theta$.) So, to perform a rotation of a (unit) vector $\mathbf{r} \in \mathbf{R}^3$ by the (matrix) R using quaternion multiplication, one performs the triple quaternion multiplication above, and then extracts the vector $\bar{\mathbf{r}}$ from the resulting (pure) quaternion $q_{\bar{\mathbf{r}}}$. (Note that to realize the rotation of a *non-unit* vector we still use this method; first rotate the *normalized* vector, and then scale it back to the original size.)

Quaternions find use nowadays in the field of numerical mathematics and physics (fluid dynamics, for example), and other areas such as computer graphics (see Vince's book below). The interested reader may wish to pursue this by referring to some books on quaternions, such as the (excellent) one by Altman listed below.

Exercises

(1) The inverse of a non-zero quaternion is defined by

$$q^{-1} = \frac{\bar{q}}{|q|^2}$$

(i) Show that

$$qq^{-1} \;=\; q^{-1}q \;=\; 1$$

(note that q and q^{-1} commute!)
(ii) Find the inverse of $q = 1 + 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and confirm you answer (i.e., multiply them together and make sure the answer is 1!)

(2) Let $q = 2 - 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}, \quad p = 1 - 2\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$. Find the following;

$$|q|, \quad p+q, \quad 2q, \; 2q - 3p, \; qp, \quad pq, \quad \frac{q}{p} = pq^{-1} = p/q, \quad q^{-1}p = q\backslash p$$

(3) Let $\mathbf{v}$ be a unit vector in $\mathbf{R}^3$ and $\mathbf{r}$ another vector.
The (unit) quaternion $q_{\mathbf{v}} = [0, \mathbf{v}]$ represents both the vector $\mathbf{v}$ and the rotation about $\mathbf{v}$ by π. Let $\bar{\mathbf{r}}$ be rotation of $\mathbf{r}$ about this $\mathbf{v}$ by π.
Show that the vector $\bar{\mathbf{r}}$ in the pure quaternion $q_{\bar{\mathbf{r}}} = [0, \bar{\mathbf{r}}]$ defined by the conjugation

$$q_{\bar{\mathbf{r}}} = [0, \bar{\mathbf{r}}] = q_{\mathbf{v}} q_{\mathbf{r}} q_{\mathbf{v}}^{-1}$$

is the reflection of $\mathbf{r}$ through $\mathbf{v}$ in the plane $P = \text{span}\{\mathbf{v}, \mathbf{r}\}$, which is precisely the same as rotation of $\mathbf{r}$ about $\mathbf{v}$ by π. (Make a sketch!)

Answer:

$$\begin{aligned}
q_{\mathbf{v}} q_{\mathbf{r}} q_{\mathbf{v}}^{-1} &= [0, \mathbf{v}][0, \mathbf{r}][0, -\mathbf{v}] \\
& [0, \mathbf{v}][\mathbf{r} \cdot \mathbf{v}, \mathbf{v} \times \mathbf{r}] \\
&= [-\mathbf{v} \cdot (\mathbf{v} \times \mathbf{r}), \; (\mathbf{r} \cdot \mathbf{v})\mathbf{v} + \mathbf{v} \times (\mathbf{v} \times \mathbf{r})] \\
&= [0, \; (\mathbf{r} \cdot \mathbf{v})\mathbf{v} + \mathbf{v} \times (\mathbf{v} \times \mathbf{r})] \qquad \text{since } \mathbf{v} \times \mathbf{r} \text{ is } \perp \text{ to } \mathbf{v} \\
&= [0, \;\; \text{proj}_{\mathbf{v}}\mathbf{r} + \mathbf{v} \times (\mathbf{v} \times \mathbf{r})]
\end{aligned}$$

Now, $\mathbf{v} \times (\mathbf{v} \times \mathbf{r})$ lies in the plane P and is orthogonal to $\mathbf{v}$. Make a sketch of these vectors to see that the sum $\text{proj}_{\mathbf{v}}\mathbf{r} + \mathbf{v} \times (\mathbf{v} \times \mathbf{r})$ is just the reflection of $\mathbf{r}$ across the line $\text{span}_{\mathbf{v}}$.

(4) Find an expression for $q_{\mathbf{v}} q_{\mathbf{r}} q_{\mathbf{v}}^{-1}$ for arbitrary quaternions, i.e., $q_{\mathbf{v}} = [a, \mathbf{v}], q_{\mathbf{r}} = [b, \mathbf{r}]$.

Answer:

$$q_{\mathbf{v}} q_{\mathbf{r}} q_{\mathbf{v}}^{-1} \quad =$$

$$
\begin{aligned}
&= \frac{1}{a^2+|\mathbf{v}|^2}[a,\mathbf{v}]\,[b,\mathbf{r}]\,[a,-\mathbf{v}] \\
&= \frac{1}{a^2+|\mathbf{v}|^2}[a,\mathbf{v}][ba+\mathbf{r}\cdot\mathbf{v},\ a\mathbf{r}-b\mathbf{v}+\mathbf{v}\times\mathbf{r}] \\
&= \frac{1}{a^2+|\mathbf{v}|^2}[a(\mathbf{r}\cdot\mathbf{v})+ba^2-a(\mathbf{v}\cdot\mathbf{r})+b\mathbf{v}\cdot\mathbf{v}-\mathbf{v}\cdot(\mathbf{v}\times\mathbf{r}), \\
&\qquad a^2\mathbf{r}-(ab)\mathbf{v}+a(\mathbf{v}\times\mathbf{r})+(ab)\mathbf{v}+(\mathbf{r}\cdot\mathbf{v})\mathbf{v}+a(\mathbf{v}\times\mathbf{r})-b(\mathbf{v}\times\mathbf{v})+\mathbf{v}\times(\mathbf{v}\times\mathbf{r})] \\
&= \frac{1}{a^2+|\mathbf{v}|^2}[\,ba^2+b\mathbf{v}\cdot\mathbf{v},\quad a^2\mathbf{r}+2a(\mathbf{v}\times\mathbf{r})+(\mathbf{r}\cdot\mathbf{v})\mathbf{v}+\mathbf{v}\times(\mathbf{v}\times\mathbf{r})]
\end{aligned}
$$

This agrees with the vector formula for rotation $R(\theta\mathbf{n})$ of $\mathbf{r}$ by θ around (unit) vector $\mathbf{n}$ (that is, when $a=\cos\frac{\theta}{2}, b=0, \mathbf{v}=\sin\frac{\theta}{2}\mathbf{n}$);

$$R(\theta\mathbf{n})\mathbf{r} \;=\; \mathbf{r}+(\sin\theta)(\mathbf{n}\times\mathbf{r})+2(\sin^2\frac{\theta}{2})\mathbf{n}\times(\mathbf{n}\times\mathbf{r})$$

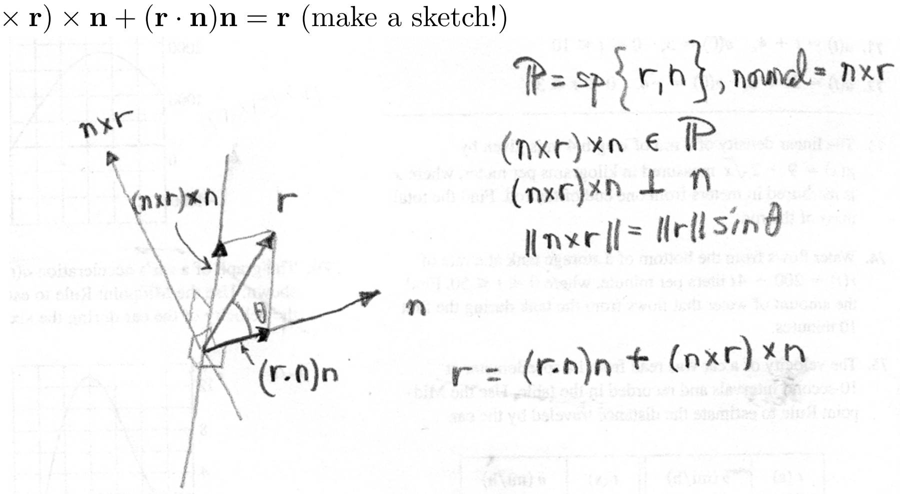
Our formula above reads

$$[0,\ \cos^2\frac{\theta}{2}\mathbf{r}+2\cos\frac{\theta}{2}\sin\frac{\theta}{2}(\mathbf{n}\times\mathbf{r})+\sin^2\frac{\theta}{2}(\mathbf{r}\cdot\mathbf{n})\mathbf{n}+\sin^2\frac{\theta}{2}\mathbf{n}\times(\mathbf{n}\times\mathbf{r})]$$

Re-writing the vector part as;

$$
\begin{aligned}
\cos^2\frac{\theta}{2}\mathbf{r}+\sin^2\frac{\theta}{2}((\mathbf{n}\times\mathbf{r})\times\mathbf{n}+(\mathbf{r}\cdot\mathbf{n})\mathbf{n})+ \\
\sin\theta(\mathbf{n}\times\mathbf{r})+2\sin^2\frac{\theta}{2}\mathbf{n}\times(\mathbf{n}\times\mathbf{r})
\end{aligned}
$$

Now use that
$(\mathbf{n}\times\mathbf{r})\times\mathbf{n}+(\mathbf{r}\cdot\mathbf{n})\mathbf{n}=\mathbf{r}$ (make a sketch!)



(5) Use quaternion multiplication to determine the vector $\bar{\mathbf{r}}$ that is rotation of the vector $\mathbf{r} = (1,1,1)$ around the axis $\mathbf{v} = (0,1,1)$ by the angle θ.

Answer:

Let $\hat{\mathbf{r}} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $\quad \hat{\mathbf{v}} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

$$\begin{aligned}
p_R &= [\cos\frac{\theta}{2},\ \hat{\mathbf{r}}\sin\frac{\theta}{2}] \ = \ \cos\frac{\theta}{2} + 0\mathbf{i} + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\mathbf{j} + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\mathbf{k} \\
(p_R)^{-1} &= [\cos\frac{\theta}{2},\ -\hat{\mathbf{r}}\sin\frac{\theta}{2}] \ = \ \cos\frac{\theta}{2} + 0\mathbf{i} - \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\mathbf{j} - \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\mathbf{k} \\
q_{\hat{\mathbf{r}}} &= [0,\ \hat{\mathbf{r}}] \ = \ 0 + \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
p_R q_{\hat{\mathbf{r}}} &= -\frac{2}{\sqrt{6}}\sin\frac{\theta}{2} + \frac{1}{\sqrt{3}}\cos\frac{\theta}{2}\mathbf{i} + (\frac{1}{\sqrt{3}}\cos\frac{\theta}{2} + \frac{1}{\sqrt{6}}\sin\frac{\theta}{2})\mathbf{j} + (\frac{1}{\sqrt{3}}\cos\frac{\theta}{2} - \frac{1}{\sqrt{6}}\sin\frac{\theta}{2})\mathbf{k} \\
p_R q_{\hat{\mathbf{r}}}(p_R)^{-1} &= 0 + (\frac{1}{\sqrt{3}}\cos^2\frac{\theta}{2} - \frac{2}{\sqrt{12}}\sin^2\frac{\theta}{2})\mathbf{i} + (\frac{2}{\sqrt{12}}\sin^2\frac{\theta}{2} + \frac{1}{\sqrt{3}}\cos^2\frac{\theta}{2} + \frac{2}{\sqrt{6}}\cos\frac{\theta}{2}\sin\frac{\theta}{2})\mathbf{j} \\
&\qquad + (\frac{2}{\sqrt{12}}\sin^2\frac{\theta}{2} + \frac{1}{\sqrt{3}}\cos^2\frac{\theta}{2} - \frac{2}{\sqrt{6}}\cos\frac{\theta}{2}\sin\frac{\theta}{2})\mathbf{k} \\
&= 0 + \frac{1}{\sqrt{3}}\cos\theta\mathbf{i} + (\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\sin\theta)\mathbf{j} + (\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\sin\theta)\mathbf{k}
\end{aligned}$$

So the (normalized) axis of rotation of the quaternion $p_R q_{\hat{\mathbf{r}}}(p_R)^{-1}$ is

$$\hat{\bar{\mathbf{r}}} = (\frac{1}{\sqrt{3}}\cos\theta,\ (\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\sin\theta),\ (\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\sin\theta))$$

We regain the rotated vector; $\bar{\mathbf{r}} = \sqrt{3}\,\hat{\bar{\mathbf{r}}} = (\cos\theta,\ 1 + \frac{1}{\sqrt{2}}\sin\theta,\ 1 - \frac{1}{\sqrt{2}}\sin\theta)$.

We verify this via matrix multiplication;

$$[T]\mathbf{r} = \begin{bmatrix} \cos\theta & -(1/\sqrt{2})\sin\theta & (1/\sqrt{2})\sin\theta \\ (1/\sqrt{2})\sin\theta & (1+\cos\theta)/2 & (1-\cos\theta)/2 \\ -(1/\sqrt{2})\sin\theta & (1-\cos\theta)/2 & (1+\cos\theta)/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \frac{1}{\sqrt{2}}\sin\theta + 1 \\ -\frac{1}{\sqrt{2}}\sin\theta + 1 \end{bmatrix}$$

(6) (Quaternion multiplication corresponds to composition of rotations.)
Let R_1 be rotation around the x-axis by 90^o, R_2 be rotation around the y-axis by 90^o, and $\mathbf{r} = (1,1,0)$.

(a) Find the matrix R of the composition R_2R_1 (that is, rotate about x-axis *then* rotate about y-axis). Compute $R\mathbf{r}$.

(b) Find the quaternions q_1 and q_2 associated to the rotations R_1 and R_2 respectively. Then compute q_2q_1.

(c) Compute $(q_2q_1)q_{\mathbf{r}}(q_2q_1)^{-1}$ and confirm that this agrees with your answer above for $R\mathbf{r}$.

Answer:

(a) $R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \quad R\mathbf{r} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(b) $q_1 = [\frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}(1,0,0)], \quad q_2 = [\frac{1}{\sqrt{2}}, \ \frac{1}{\sqrt{2}}(0,1,0)], \quad q_2 q_1 = [\frac{1}{2}, \frac{1}{2}(1,1,-1)]$

Note that $q_2 q_1 = \left[\frac{1}{2}, \ \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\right]$ which corresponds to rotation around $\mathbf{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ by 120^o.

(c) Recall (see p.75 Altman) the formula for rotation $R(\theta\mathbf{n})$ of $\mathbf{r}$ by θ around (unit) vector $\mathbf{n}$;

$$R(\theta\mathbf{n})\mathbf{r} \;=\; \mathbf{r} + (\sin\theta)(\mathbf{n}\times\mathbf{r}) + 2(\sin^2\frac{\theta}{2})\mathbf{n}\times(\mathbf{n}\times\mathbf{r})$$

Taking $\theta = 120^o$ and $\mathbf{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, we compute

$$\begin{aligned} \mathbf{n}\times\mathbf{r} &= \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right) \\ \mathbf{n}\times(\mathbf{n}\times\mathbf{r}) &= \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}\right) \\ \mathbf{r} + (\sin\theta)(\mathbf{n}\times\mathbf{r}) + 2(\sin^2\frac{\theta}{2})\mathbf{n}\times(\mathbf{n}\times\mathbf{r}) &= (1,1,0) + (\frac{1}{2}, -\frac{1}{2}, 0) + (-\frac{1}{2}, -\frac{1}{2}, -1) \;=\; (1,0,-1) \end{aligned}$$

This corroborates that our answer in (b) is the correct rotation quaternion. We verify;

$$\begin{aligned} q_2 q_1 &= \left[\frac{1}{2}, \ \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\right] \\ (q_2 q_1)^{-1} &= \left[\frac{1}{2}, \ \frac{\sqrt{3}}{2}\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right] \\ q_{\mathbf{r}} &= [0, (1,1,0)] \\ (q_2 q_1) q_{\mathbf{r}} &= [-1, (1,0,0)] \\ (q_2 q_1) q_{\mathbf{r}} (q_2 q_1)^{-1} &= [0, (1,0,-1)] \end{aligned}$$

References:

Rotations, Quaternions, and Double Groups, by Simon L. Altman, Dover Publications (inexpensive paperback!)

A History of Vector Analysis, by Michael J. Crowe, Dover Publications.

The Princeton Companion to Mathematics, Section III.76. A nice discussion on quaternions. And this (big!) book is accessible online through the SFU library.

Mathematical Thought from Ancient to Modern Times, Chapter 32, by Morris Kline.

Quaternions for Computer Graphics, by John Vince. This is a description of the use of quaternions as used in computer graphics. Note; this is a different approach and description than the 'mathematical' view I describe here or that which is espoused in Altman's book.