Math Outside The Box

Complex numbers, Quaternions, and Rotations; Bringing Algebra and Geometry Together

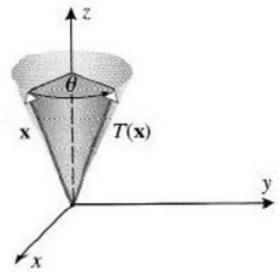
Presented by Dr Randall Pyke

Date: Feb 28

3:30 - 4:20pm K9509

An informal talk for first and second year students curious about mathematics

$$\begin{aligned} \mathbf{i}^2 &= \mathbf{j}^2 = \mathbf{k}^2 = -1 \\ \mathbf{ij} &= \mathbf{k}, \quad \mathbf{ji} = -\mathbf{k} \\ \mathbf{ki} &= \mathbf{j}, \quad \mathbf{ik} = -\mathbf{j} \\ \mathbf{jk} &= \mathbf{i}, \quad \mathbf{kj} = -\mathbf{i} \end{aligned}$$



$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Outline

PART I: Complex numbers and $\mathbf{R}^2$

- Complex numbers $\mathbf{C}$; extending the real numbers $\mathbf{R}$
- Rotations in $\mathbf{R}^2$ (and Rotations in $\mathbf{R}^3$)
- Complex numbers as vectors and rotations in $\mathbf{R}^2$

PART II: Quaternions and $\mathbf{R}^3$

- Extending $\mathbf{C}$; 'hyper-complex numbers' $\rightarrow$ quaternions
- Quaternions as vectors and rotations in $\mathbf{R}^3$
- Some calculations with quaternions
- Quaternions and vector analysis
  Quaternions in physics, computer graphics, ...

These slides, and more notes;
`www.sfu.ca/~rpyke --> Presentations --> Quaternions`

PART I: The story of the complex numbers (starting in the 16th Century)

The 'problems' with the real numbers **R**..... (why do we need 'more' numbers?)

Algebra: Extending (enlarging) numbers

Natural numbers $\mathbf{N} \to$ Rational numbers $\mathbf{Q} \to$ Real numbers $\mathbf{R}$

$$\mathbf{N} \subset \mathbf{Q} \subset \mathbf{R}$$

Roots of equation (solving equations);

(There's also 'topological' reasons for extending **Q** to **R**....)

Factorization Theorem:

Every polynomial $p(x) \in \mathbf{R}(x)$ *(i.e., with real coefficients) can be factored into linear terms and irreducible quadratic terms;*

$$p(x) = a l_1(x) l_2(x) \cdots l_k(x) q_1(x) q_2(x) \cdots q_m(x)$$

$l_i(x)$ linear, $q_j(x)$ irreducible quadratic;

$$q(x) = ax^2 + bx + c, \quad b^2 - 4ac < 0\,; \qquad \text{no roots}$$

We factor polynomials by finding roots;

$$\text{roots of } p(x) \ \longleftrightarrow \ \text{linear factor}$$

So, if there are no roots then are no (linear) factors.

An irreducible quadratic; $q(x) = x^2 + 1$ (cannot be factored)

$q(x) = x^2 + 1$

Introduce a new number, i, such that $q(i) = 0$;

$$q(x) = x^2 + 1 = (x + i)(x - i)$$

But of course, i is <u>not</u> a real number; let's call it an <u>imaginary</u> number.

So we "add" this new number i to the real numbers. And so all these other numbers are (automatically) added too;

$2i, \quad -i, \quad 1 + 2i, \ ...$ (using addition and multiplication)

Complex numbers (denoted by $\mathbf{C}$) are numbers of the form

$$z = a + \mathbf{i}b$$

where a, b are real numbers, and $\mathbf{i}$ is an 'imaginary' number that satisfies $\mathbf{i}^2 = -1 \quad (\mathbf{i} = \sqrt{-1})$
We call a the *real part* and b the *imaginary* part

<u>Fundamental Theorem of Algebra:</u>

Every polynomial $p(x) \in \mathbf{C}(x)$ (and hence in $\mathbf{R}(x)$) has a root, and hence can be completely factored (into linear terms).

Adding that one new number i has 'completed' the real numbers (in this algebraic sense). Now develop calculus in $\mathbf{C}$... ($f : \mathbf{C} \rightarrow \mathbf{C}$ Complex Variables; Math 322)

The Geometric Representation of Complex Numbers

(Wessel, Argand, Gauss,... 1799-1830)

The complex plane: Complex numbers as vectors

$$z = a + \mathbf{i}b; \leftrightarrow \vec{z} = (a, b); \quad \text{(vectors} \sim \text{points in } \mathbf{R}^2\text{)}$$

Vectors? The origins of **vector analysis**....

A vector is a quantity that describes direction and magnitude (e.g. displacement, velocity, orientation, ...)
Just numbers (quantitites) was inadequate for a description of Nature and geometry (Leibniz, 1679, ...)

Adding vectors; the parallelogram law
(beginning with ancient Greeks and common by early 18th century)

Complex algebra: Addition and multiplication of complex numbers

$$(2-3i)+(4+5i) \ =$$

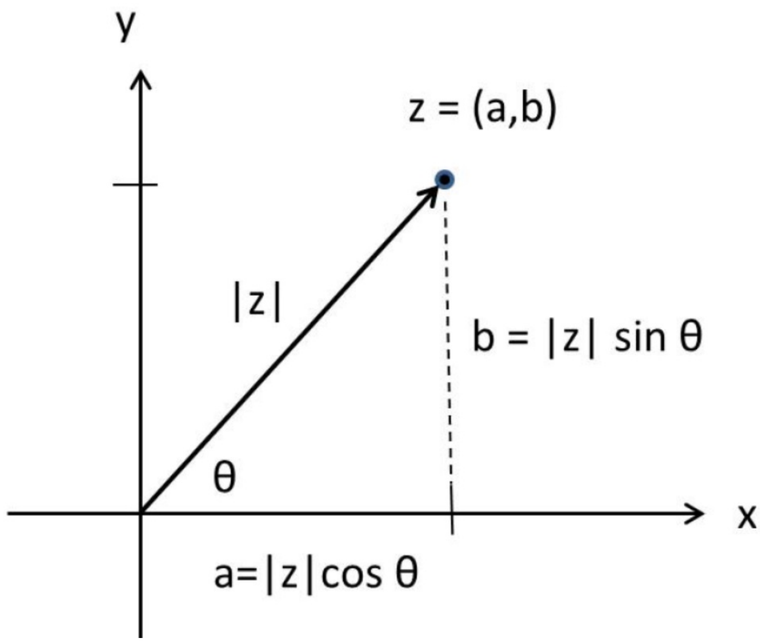
$$(2-3i)(4+5i) \ =$$

Addition of complex numbers in the complex plane;

Complex numbers behave just like vectors in $\mathbf{R}^2$!

$$z = a + \mathbf{i}b; \qquad \vec{z} = (a, b)$$

Polar form of complex numbers (better for multiplication of complex numbers...)



$$z = r(\cos\theta + i\sin\theta) \;=\; re^{i\theta}, \quad r = |z| = \sqrt{a^2 + b^2}$$

(θ is called the argument of z, $|z|$ is the modulus of z)

Example:

$$\begin{aligned} z &= 2 + 2\sqrt{3}\, i \\ &= 4e^{i\pi/3} \end{aligned}$$

Multiplication of complex numbers in polar form:

$$z = r(\cos\theta_1 + i\sin\theta_1), \qquad w = s(\cos\theta_2 + i\sin\theta_2),$$
$$r = |z|, \quad s = |w|\,; \quad \text{the 'lengths' of } z \text{ and } w$$

$$\begin{aligned} zw &= [r(\cos\theta_1 + i\sin\theta_1)]\,[s(\cos\theta_2 + i\sin\theta_2)] \\ &= rs\{[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2] \\ &+ \quad i[\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1]\} \\ &= rs(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \end{aligned}$$

(Remember this! The angle (argument) of the resulting product is the sum of the two angles)

Complex multiplication is a rotation!

(and a stretching/compression)

Rotations in 2 dimensions

How to mathematically represent rotations?.....

Matrices and matrix algebra

A **matrix** is an array of numbers that represents a function (or transformation); $M : \mathbf{R}^n \to \mathbf{R}^n$

$$M\mathbf{v} \text{ or } M(\mathbf{v}) \quad = \quad \mathbf{w}$$

Example:

$$M = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}; \quad M\mathbf{v} = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix} = \mathbf{w}$$

M moves $\mathbf{v}$ to $\mathbf{w}$ (transformation of vectors....)

Some special transformations;

$D_k \,:\, \mathbf{R}^2 \to \mathbf{R}^2$ is stretching/compression by k
$R_\theta \,:\, \mathbf{R}^2 \to \mathbf{R}^2$ is rotation counterclockwise by angle θ;

D_k is represented by the matrix $A_k = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

R_θ is represented by the matrix $M_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Examples: $A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $M_{\pi/4} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

Check:

$$M_{\pi/4} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$M_{\pi/4} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$

We are interested in the <u>rotation matrices</u> here in this lecture

Representation of complex numbers as matrices

$$z = a + ib\,; \qquad A_z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

This is a 'good' representation of complex numbers as matrices since;

$$\begin{aligned} A_{z_1} + A_{z_2} &= A_{z_1+z_2} && \text{(preserves addition)} \\ A_{z_1} A_{z_2} &= A_{z_1 z_2} && \text{(preserves multiplication)} \end{aligned}$$

Examples;

$$\begin{aligned} 1 \;&;\; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ i \;&;\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ 2+3i \;&;\; \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \end{aligned}$$

$$z = \cos\theta + i\sin\theta = e^{i\theta} \;;\; \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = R_\theta \quad \text{(formerly } M_\theta\text{)}$$

The last example is an honest to goodness rotation matrix!
That is, we can think of unit complex numbers as rotations in $\mathbf{R}^2$

Summary: **Complex numbers as vectors and rotations in $\mathbf{R}^2$**

Identification of complex numbers with vectors in $\mathbf{R}^2$:

$$z = a + ib \quad \longrightarrow \quad \mathbf{v}_z = (a, b)$$
$$\mathbf{v} = (a, b) \quad \longrightarrow \quad z_{\mathbf{v}} = a + ib$$

Identification of (unit) complex numbers with rotations in $\mathbf{R}^2$:

$$w = \cos\theta + i\sin\theta = e^{i\theta} \quad \longleftrightarrow \quad A_w = R_\theta \; = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Representing rotation of vectors with complex multiplication:

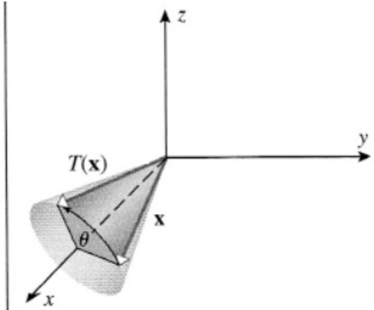
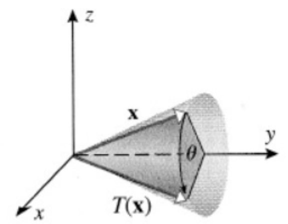
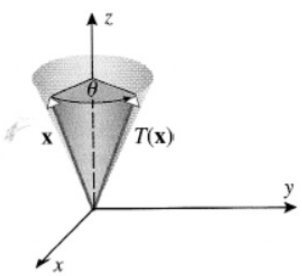
$$\mathbf{v} = (a, b), \quad R_\theta; \quad \tilde{\mathbf{v}} = R_\theta \mathbf{v} \quad \text{(rotate vector } \mathbf{v} \text{ to } \tilde{\mathbf{v}})$$
$$\tilde{\mathbf{v}} = \mathbf{v}_{wz}; \quad w = e^{i\theta} \;\text{(the rotation)},\; z = a + ib \;\text{(the vector)}$$

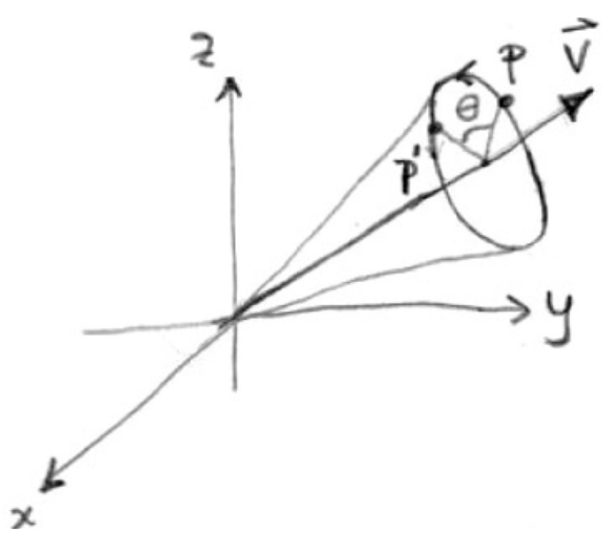
Multiplication of complex numbers (wz) corresponds to a rotation of the vector ($\tilde{\mathbf{v}} = \mathbf{v}_{wz}$)

Composition of rotations in $\mathbf{R}^2$ ($R_{\theta_2} \circ R_{\theta_1} = R_{\theta_2+\theta_1}$) corresponds to multiplication of complex numbers ($= A_{w_2 w_2}$)
(remember the polar form! $r = s = 1$)

PART II: **Quaternions and Rotations in 3 dimensions**

Rotations in $\mathbf{R}^3$ are very complicated!!

Rotation about the positive x-axis through an angle θ		$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Rotation about the positive y-axis through an angle θ		$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Rotation about the positive z-axis through an angle θ		$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$



Rotation by 30^o about the axis $\mathbf{v} = (\mathbf{0}, \mathbf{1}, \mathbf{1})$;

$$R = \begin{bmatrix} \cos\theta & -(1/\sqrt{2})\sin\theta & (1/\sqrt{2})\sin\theta \\ (1/\sqrt{2})\sin\theta & \frac{1}{2} + (1/\sqrt{2})\cos\theta & \frac{1}{2} - (1/\sqrt{2})\cos\theta \\ (1/\sqrt{2})\sin\theta & \frac{1}{2} - (1/\sqrt{2})\cos\theta & \frac{1}{2} + (1/\sqrt{2})\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0.8660 & -0.3536 & 0.3536 \\ 0.3536 & 0.9330 & 0.0670 \\ 0.3536 & 0.0670 & 0.9330 \end{bmatrix}$$

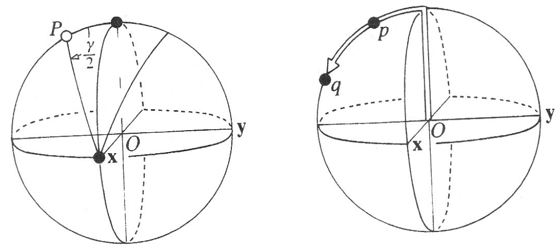
Vector formula for rotation $R(\theta\mathbf{n})$ of $\mathbf{r}$ by θ around (unit) vector $\mathbf{n}$;

$$R(\theta\mathbf{n})\mathbf{r} = \mathbf{r} + (\sin\theta)(\mathbf{n} \times \mathbf{r}) + 2(\sin^2\frac{\theta}{2})\mathbf{n} \times (\mathbf{n} \times \mathbf{r})$$

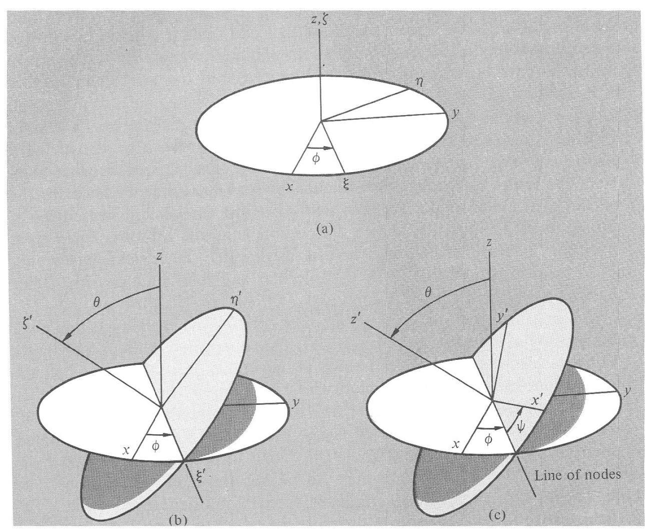
(So, don't necessarily need a matrix to represent a rotation)

Some facts about rotations in $\mathbf{R}^3$

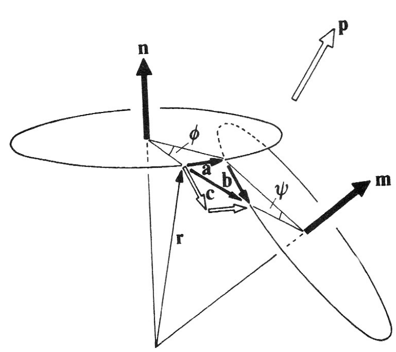
- Every motion of the sphere that keeps the centre fixed is a rotation about some axis through the centre.
  (That is, every rotation about a point is a rotation about an axis!)



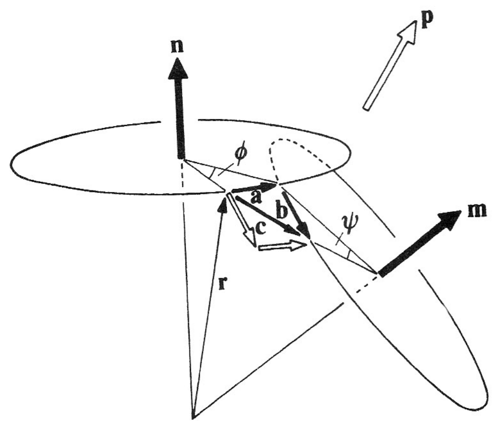
- Every rotation can be realized by a sequence of rotations about three orthogonal axis.



- Any sequence of rotations (about various axis through the origin) results in a rotation about an axis.



Composition of rotations results in a rotation



What is the relation between the two rotations (their axis $\mathbf{n_1}, \mathbf{n_2}$ and angles θ_1, θ_2) and the axis $\mathbf{n}_3$ and angle θ_3 of the resultant rotation? (Rodriques, 1847; Euler, Hamilton,....)

Answer;

$$\begin{aligned}
\cos\frac{\theta_3}{2} &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\mathbf{n}_1\cdot\mathbf{n}_2 \\
\sin\frac{\theta_3}{2}\mathbf{n}_3 &= \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\mathbf{n}_2 + \cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\mathbf{n}_1 + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\mathbf{n}_1\times\mathbf{n}_2
\end{aligned}$$

If we seek to represent rotations in $\mathbf{R}^3$ by (hyper-complex) numbers, a complicated multiplication rule is needed....

(since these numbers will represent rotations, and their products will represent the composition of rotations)

Quaternions: Extending C ?

Why? **C** is algebraically and topologically closed...

But what about vectors and rotations in $\mathbf{R}^3$? Are there 'numbers' that can represent those?

Will these new numbers still enjoy the properties of distribution, associativity, commutativity?....

Hamilton (1847): Is there a 3-dimensional version of complex numbers? Add another imaginary number **j**;

$$q = a + \mathbf{i}b + \mathbf{j}c, \quad a, b, c \in \mathbf{R}, \quad \mathbf{i}^2 = \mathbf{j}^2 = -1$$

But $\mathbf{ij} = ??$

!!! Add a fourth imaginary number **k**;

$$q = q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d, \quad a, b, c, d \in \mathbf{R}, \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 1$$

These numbers, *quaternions*, is the 'only' way to extend the complex numbers (Frobenius' Theorem, 1878). But there are octonions....

So, not only did the introduction of **j** and **k** lead to a 'good' extension of **C**, it was (later) realized that they (the quaternions) also represented rotations in $\mathbf{R}^3$

Quaternions

Hamilton's algebraic approach to quaternions:

Quaternions (denoted by $\mathbf{H}$) are numbers of the form

$$\begin{aligned} q &= a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d \\ &= [a,\ \mathbf{v}] \quad \text{where} \quad \mathbf{v} = \mathbf{i}b + \mathbf{j}c + \mathbf{k}d \ = \ (b, c, d), \quad \text{notation!} \end{aligned}$$

where a, b, c, d are real numbers, and the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are 'imaginary' numbers that satisfy the following multiplication rules;

$$\begin{aligned} \mathbf{i}^2 &= \mathbf{j}^2 = \mathbf{k}^2 = -1 \\ \mathbf{ij} &= \mathbf{k}, \quad \mathbf{ji} = -\mathbf{k} \\ \mathbf{ki} &= \mathbf{j}, \quad \mathbf{ik} = -\mathbf{j} \\ \mathbf{jk} &= \mathbf{i}, \quad \mathbf{kj} = -\mathbf{i} \end{aligned}$$

a is the *real part* and $\mathbf{v}$ is the *imaginary* part of the quaternion.
Note the similarity with the cross product ; $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, etc

Example

If $q = 4 + 6\mathbf{i} + 8\mathbf{j} + 9\mathbf{k}$ and $p = 3 + 2\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}$, then

$$qp = -111 + 28\mathbf{i} + 24\mathbf{j} + 75\mathbf{k}, \quad pq = -111 + 24\mathbf{i} + 72\mathbf{j} + 35\mathbf{k}$$

and

$$p + q = q + p = 7 + 8\mathbf{i} + 14\mathbf{j} + 16\mathbf{k}$$

More generally, if $q = [a, \mathbf{v}]$, $p = [\alpha, \mathbf{w}]$, then

$$qp = [a\alpha - \mathbf{v} \cdot \mathbf{w},\ a\mathbf{w} + \alpha\mathbf{v} + \mathbf{v} \times \mathbf{w}]$$

We see that $\mathbf{H}$ is not commutative!! (but it is associative)

Inverses and the non-cummutativity of quaternions

The <u>conjugate</u> of $q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ is

$$\bar{q} = a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d$$

The <u>norm</u> (or <u>modulus</u>) $|q|$ of q is

$$|q|^2 = a^2 + b^2 + c^2 + d^2$$

If $q \neq 0$, then the <u>inverse</u> of q is defined as,

$$q^{-1} = \frac{\bar{q}}{|q|^2}$$

in the sense that

$$q\,q^{-1} \;=\; q^{-1}q = 1$$

BUT, the expression $\dfrac{p}{q}$ is ambiguous because of the non-communtabilty of quaternion multiplication!

$$\frac{p}{q} \;=\; p\,q^{-1} \quad \text{OR} \quad q^{-1}p \qquad \text{(similar as for matrices.....)}$$

Stoke's / Cayley's notation;

$p\,q^{-1} = \frac{p}{q} \;=\; p/q \;=$

$q^{-1}p \;=\qquad q\backslash p \;=$

Tait's geometric approach to quaternions

For (Peter Guthrie) Tait, a quaternion q was a 'ratio' of two 3 dimensional vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^3$;

$$q \;=\; \text{``}\,\frac{\mathbf{v}}{\mathbf{w}}\,\text{''}$$

By which we mean

$$\mathbf{v} \;=\; q\,\mathbf{w}$$

That is, q 'changes' $\mathbf{w}$ into $\mathbf{v}$.

How does one change one vector into another?

First, there is a scaling (length of $\mathbf{v}$ vs length of $\mathbf{w}$).
Then, there is a rotation.

How many parameters are needed to accomplish this (in 3 dimensions)?

Review

Complex numbers as vectors and rotations in $\mathbf{R}^2$

Identification of complex numbers with vectors in $\mathbf{R}^2$:

$$z = a + ib \quad \longrightarrow \quad \mathbf{v}_z = (a, b)$$
$$\mathbf{v} = (a, b) \quad \longrightarrow \quad z_{\mathbf{v}} = a + ib$$

Identification of (unit) complex numbers with rotations in $\mathbf{R}^2$:

$$w = \cos\theta + i\sin\theta = e^{i\theta} \quad \longleftrightarrow \quad A_w = R_\theta \; = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Representing rotation of vectors with complex multiplication:

$$\mathbf{v} = (a, b), \quad R_\theta; \quad \tilde{\mathbf{v}} = R_\theta \mathbf{v} \qquad \text{(rotate vector } \mathbf{v} \text{ to } \tilde{\mathbf{v}}\text{)}$$
$$\tilde{\mathbf{v}} = \mathbf{v}_{wz}; \quad w = e^{i\theta}, \; z = a + ib$$

Multiplication of complex numbers (wz) corresponds to a rotation of the vector ($\tilde{\mathbf{v}} = \mathbf{v}_{wz}$)

Composition of rotations in $\mathbf{R}^2$ ($R_{\theta_2} \circ R_{\theta_1} = R_{\theta_2+\theta_1}$) corresponds to multiplication of complex numbers ($= A_{w_2 w_2}$)

Quaternions as vectors and rotations in $\mathbf{R}^3$

Identification of (pure) quaternions with vectors in $\mathbf{R}^3$:

$$q = \mathbf{i}b + \mathbf{j}c + \mathbf{k}d \;=\; [0, \mathbf{v}] \quad\longrightarrow\quad \mathbf{v}_q = (b, c, d)$$
$$\mathbf{v} = (b, c, d) \quad\longrightarrow\quad q_{\mathbf{v}} = \mathbf{i}b + \mathbf{j}c + \mathbf{k}d = [0, \mathbf{v}]$$

Identification of (unit) quaternions with rotations $\mathbf{R}^3$:

$R = R(\mathbf{n}, \theta)$ is rotation about the unit vector $\mathbf{n} = (n_1, n_2, n_3)$ by angle θ. This rotation is represented by the unit quaternion p_R;

$$p_R = \cos\frac{\theta}{2} + n_1 \sin\frac{\theta}{2}\,\mathbf{i} + n_2 \sin\frac{\theta}{2}\,\mathbf{j} + n_3 \sin\frac{\theta}{2}\,\mathbf{k}$$

which we write more compactly as

$$p_R = [\cos\frac{\theta}{2},\; \mathbf{n}\sin\frac{\theta}{2}]$$

Conversely, any unit quaternion (and these can always be written as $[\cos\frac{1}{2}\theta,\; \mathbf{n}\sin\frac{1}{2}\theta],\quad \|\mathbf{n}\| = 1$), represents a rotation in $\mathbf{R}^3$ (given by $\mathbf{n}$ and θ).

Representing rotation of vectors with quaternion multiplication:

If $\bar{\mathbf{v}} = R\,\mathbf{v}$, then $q_{\bar{\mathbf{v}}}$ is given by;

$$q_{\bar{\mathbf{v}}} \;=\; [0,\; \bar{\mathbf{v}}] \;=\; (p_R)\, q_{\mathbf{v}}\, (p_R)^{-1}$$

where $(p_R)^{-1} = [\cos\frac{1}{2}\theta,\; -\mathbf{n}\sin\frac{1}{2}\theta]$.

$\star\star\star$ Composition of rotations in $\mathbf{R}^3$ corresponds to multiplication of quaternions; $\quad p_{R_1R_2} \;=\; (p_{R_1})\,(p_{R_2})$

Olinde Rodrigues' Paper of 1840 on Transformation Groups

JEREMY J. GRAY

Communicated by M. KLINE

In this article I wish to draw attention to a paper of OLINDE RODRIGUES which has been almost forgotten, and which is perhaps the first treatment of the subject of groups of motions. I shall proceed by discussing (1) The Contents, (2) The Context, and (3) The Significance of the paper.

1. The Contents of RODRIGUES's paper

In 1840 OLINDE RODRIGUES (1794–1851) published "*Des lois géométriques qui régissent les déplacements d'un système solide dans l'espace, et de la variation des coordonnées provenant de ces déplacements considérés indépendamment des causes qui peuvent les produire*" in LIOUVILLE's *Journal de Mathématiques*, Volume **5**, 380–440.

It is this paper which will be discussed. In it, as the title indicates, he studied the motions *(déplacements)* of a rigid body *(système solide)* in three dimensional space independently of any dynamical considerations. He began by giving a complete description of motions in synthetic terms, establishing successively that a body is fixed in space once three non-collinear points have been determined; that if two points are fixed the motion is a rotation about an axis through those points; that translations when composed give a translation which is independent of the order of composition and can be found by the '*loi du polygone des translations*' (p. 383) and that a translation is equal to an infinitesimal rotation about an axis perpendicular to the direction of the translation but situated at an infinite distance. RODRIGUES described the resultant as a rotation '*d'une amplitude infiniment petite autour d'un axe fixe infiniment éloigné et normal à la direction de cette translation*' (p. 381). This last observation allowed him to consider translations as a special class of infinitesimal rotations.

Archive for History of the Exact Sciences, Vol 21, 1980

Quaternion multiplication replicates rotation of vectors

Use quaternion multiplication to determine the vector $\bar{\mathbf{r}}$ *that is rotation of the vector* $\mathbf{r} = (1,1,1)$ *around the axis* $\mathbf{v} = (0,1,1)$ *by the angle* θ

Solution:

Let $\hat{\mathbf{r}} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $\hat{\mathbf{v}} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

$$\begin{aligned} p_R &= [\cos\frac{\theta}{2},\ \hat{\mathbf{r}}\sin\frac{\theta}{2}] \ = \ \cos\frac{\theta}{2} + 0\mathbf{i} + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\mathbf{j} + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\mathbf{k} \\ (p_R)^{-1} &= [\cos\frac{\theta}{2},\ -\hat{\mathbf{r}}\sin\frac{\theta}{2}] \ = \ \cos\frac{\theta}{2} + 0\mathbf{i} - \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\mathbf{j} - \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\mathbf{k} \\ q_{\hat{\mathbf{r}}} &= [0,\ \hat{\mathbf{r}}] \ = \ 0 + \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} \end{aligned}$$

$$\begin{aligned} p_R q_{\hat{\mathbf{r}}} &= -\frac{2}{\sqrt{6}}\sin\frac{\theta}{2} + \frac{1}{\sqrt{3}}\cos\frac{\theta}{2}\mathbf{i} + (\frac{1}{\sqrt{3}}\cos\frac{\theta}{2} + \frac{1}{\sqrt{6}}\sin\frac{\theta}{2})\mathbf{j} + (\frac{1}{\sqrt{3}}\cos\frac{\theta}{2} - \frac{1}{\sqrt{6}}\sin\frac{\theta}{2})\mathbf{k} \\ p_R q_{\hat{\mathbf{r}}}(p_R)^{-1} &= 0 + (\frac{1}{\sqrt{3}}\cos^2\frac{\theta}{2} - \frac{2}{\sqrt{12}}\sin^2\frac{\theta}{2})\mathbf{i} + (\frac{2}{\sqrt{12}}\sin^2\frac{\theta}{2} + \frac{1}{\sqrt{3}}\cos^2\frac{\theta}{2} + \frac{2}{\sqrt{6}}\cos\frac{\theta}{2}\sin\frac{\theta}{2})\mathbf{j} \\ &\qquad + (\frac{2}{\sqrt{12}}\sin^2\frac{\theta}{2} + \frac{1}{\sqrt{3}}\cos^2\frac{\theta}{2} - \frac{2}{\sqrt{6}}\cos\frac{\theta}{2}\sin\frac{\theta}{2})\mathbf{k} \\ &= 0 + \frac{1}{\sqrt{3}}\cos\theta\mathbf{i} + (\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\sin\theta)\mathbf{j} + (\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\sin\theta)\mathbf{k} \end{aligned}$$

So the (normalized) axis of rotation of the quaternion $p_R q_{\hat{\mathbf{r}}}(p_R)^{-1}$ is

$$\hat{\bar{\mathbf{r}}} = (\frac{1}{\sqrt{3}}\cos\theta,\ (\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\sin\theta),\ (\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\sin\theta))$$

We regain the rotated vector; $\bar{\mathbf{r}} = \sqrt{3}\,\hat{\bar{\mathbf{r}}} = (\cos\theta,\ 1 + \frac{1}{\sqrt{2}}\sin\theta,\ 1 - \frac{1}{\sqrt{2}}\sin\theta)$.

We verify this via matrix multiplication;

$$[T]\mathbf{r} = \begin{bmatrix} \cos\theta & -(1/\sqrt{2})\sin\theta & (1/\sqrt{2})\sin\theta \\ (1/\sqrt{2})\sin\theta & (1+\cos\theta)/2 & (1-\cos\theta)/2 \\ -(1/\sqrt{2})\sin\theta & (1-\cos\theta)/2 & (1+\cos\theta)/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \frac{1}{\sqrt{2}}\sin\theta + 1 \\ -\frac{1}{\sqrt{2}}\sin\theta + 1 \end{bmatrix}$$

Quaternion multiplication corresponds to composition of rotations

Let R_1 be rotation around the x-axis by 90^o, R_2 be rotation around the y-axis by 90^o, and $\mathbf{r} = (1, 1, 0)$.

(a) Find the matrix R of the composition R_2R_1 (that is, rotate about x-axis then *rotate about y-axis). Compute $R\mathbf{r}$.*

(b) Find the quaternions q_1 and q_2 associated to the rotations R_1 and R_2 respectively. Then compute q_2q_1.

(c) Compute $(q_2q_1)q_{\mathbf{r}}(q_2q_1)^{-1}$ and confirm that this agrees with your answer above for $R\mathbf{r}$.

(a) $R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \quad R\mathbf{r} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

(b) $q_1 = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}(1,0,0)], \quad q_2 = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}(0,1,0)], \quad q_2q_1 = [\frac{1}{2}, \frac{1}{2}(1,1,-1)]$

Note that $q_2q_1 = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\right]$ which corresponds to rotation around $\mathbf{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ by 120^o.

(c) Recall (see p.75 Altman) the formula for rotation $R(\theta\mathbf{n})$ of $\mathbf{r}$ by θ around (unit) vector $\mathbf{n}$;

$$R(\theta\mathbf{n})\mathbf{r} = \mathbf{r} + (\sin\theta)(\mathbf{n}\times\mathbf{r}) + 2(\sin^2\frac{\theta}{2})\mathbf{n}\times(\mathbf{n}\times\mathbf{r})$$

Taking $\theta = 120^o$ and $\mathbf{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, we compute

$$\begin{aligned} \mathbf{n}\times\mathbf{r} &= \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right) \\ \mathbf{n}\times(\mathbf{n}\times\mathbf{r}) &= \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}\right) \\ \mathbf{r} + (\sin\theta)(\mathbf{n}\times\mathbf{r}) + 2(\sin^2\frac{\theta}{2})\mathbf{n}\times(\mathbf{n}\times\mathbf{r}) &= (1,1,0) + (\frac{1}{2}, -\frac{1}{2}, 0) + (-\frac{1}{2}, -\frac{1}{2}, -1) = (1, 0, -1) \end{aligned}$$

This corroborates that our answer in (b) is the correct rotation quaternion. We verify;

$$\begin{aligned} q_2q_1 &= \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\right] \\ (q_2q_1)^{-1} &= \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\right] \\ q_{\mathbf{r}} &= [0, (1, 1, 0)] \\ (q_2q_1)q_{\mathbf{r}} &= [-1, (1, 0, 0)] \\ (q_2q_1)q_{\mathbf{r}}(q_2q_1)^{-1} &= [0, (1, 0, -1)] \end{aligned}$$

Miscellaneous calculations

(1) Let $\mathbf{v}$ be a unit vector in $\mathbf{R}^3$ and $\mathbf{r}$ another vector.

The (unit) quaternion $q_{\mathbf{v}} = [0, \mathbf{v}]$ represents both the vector $\mathbf{v}$ and the rotation about $\mathbf{v}$ by π. Let $\bar{\mathbf{r}}$ be rotation of $\mathbf{r}$ about this $\mathbf{v}$ by π.

Show that the vector $\bar{\mathbf{r}}$ in the pure quaternion $q_{\bar{\mathbf{r}}} = [0, \bar{\mathbf{r}}]$ defined by the conjugation

$$q_{\bar{\mathbf{r}}} = [0, \bar{\mathbf{r}}] = q_{\mathbf{v}} q_{\mathbf{r}} q_{\mathbf{v}}^{-1}$$

is the reflection of $\mathbf{r}$ through $\mathbf{v}$ in the plane $P = \text{span}\{\mathbf{v}, \mathbf{r}\}$, which is precisely the same as rotation of $\mathbf{r}$ about $\mathbf{v}$ by π. (Make a sketch!)

Answer:

$$\begin{aligned} q_{\mathbf{v}} q_{\mathbf{r}} q_{\mathbf{v}}^{-1} &= [0, \mathbf{v}][0, \mathbf{r}][0, -\mathbf{v}] \\ & [0, \mathbf{v}][\mathbf{r} \cdot \mathbf{v}, \mathbf{v} \times \mathbf{r}] \\ &= [-\mathbf{v} \cdot (\mathbf{v} \times \mathbf{r}),\ (\mathbf{r} \cdot \mathbf{v})\mathbf{v} + \mathbf{v} \times (\mathbf{v} \times \mathbf{r})] \\ &= [0,\ (\mathbf{r} \cdot \mathbf{v})\mathbf{v} + \mathbf{v} \times (\mathbf{v} \times \mathbf{r})] \qquad \text{since } \mathbf{v} \times \mathbf{r} \text{ is } \perp \text{ to } \mathbf{v} \\ &= [0,\ \ \text{proj}_{\mathbf{v}}\mathbf{r} + \mathbf{v} \times (\mathbf{v} \times \mathbf{r})] \end{aligned}$$

Now, $\mathbf{v} \times (\mathbf{v} \times \mathbf{r})$ lies in the plane P and is orthogonal to $\mathbf{v}$. Make a sketch of these vectors to see that the sum $\text{proj}_{\mathbf{v}}\mathbf{r} + \mathbf{v} \times (\mathbf{v} \times \mathbf{r})$ is just the reflection of $\mathbf{r}$ across the line span$_{\mathbf{v}}$.

(2) Find an expression for $q_{\mathbf{v}} q_{\mathbf{r}} q_{\mathbf{v}}^{-1}$ for arbitrary quaternions, i.e., $q_{\mathbf{v}} = [a, \mathbf{v}], q_{\mathbf{r}} = [b, \mathbf{r}]$.

Answer:

$$\begin{aligned}
q_{\mathbf{v}} q_{\mathbf{r}} q_{\mathbf{v}}^{-1} &= \\
&= \frac{1}{a^2 + |\mathbf{v}|^2} [a, \mathbf{v}]\, [b, \mathbf{r}]\, [a, -\mathbf{v}] \\
&= \frac{1}{a^2 + |\mathbf{v}|^2} [a, \mathbf{v}][ba + \mathbf{r} \cdot \mathbf{v},\ a\mathbf{r} - b\mathbf{v} + \mathbf{v} \times \mathbf{r}] \\
&= \frac{1}{a^2 + |\mathbf{v}|^2} [a(\mathbf{r} \cdot \mathbf{v}) + ba^2 - a(\mathbf{v} \cdot \mathbf{r}) + b\mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot (\mathbf{v} \times \mathbf{r}), \\
&\qquad a^2\mathbf{r} - (ab)\mathbf{v} + a(\mathbf{v} \times \mathbf{r}) + (ab)\mathbf{v} + (\mathbf{r} \cdot \mathbf{v})\mathbf{v} + a(\mathbf{v} \times \mathbf{r}) - b(\mathbf{v} \times \mathbf{v}) + \mathbf{v} \times (\mathbf{v} \times \mathbf{r})] \\
&= \frac{1}{a^2 + |\mathbf{v}|^2} [\, ba^2 + b\mathbf{v} \cdot \mathbf{v},\ \ a^2\mathbf{r} + 2a(\mathbf{v} \times \mathbf{r}) + (\mathbf{r} \cdot \mathbf{v})\mathbf{v} + \mathbf{v} \times (\mathbf{v} \times \mathbf{r})]
\end{aligned}$$

This agrees with the vector formula for rotation $R(\theta\mathbf{n})$ of $\mathbf{r}$ by θ around (unit) vector $\mathbf{n}$ (that is, when $a = \cos\frac{\theta}{2}, b = 0, \mathbf{v} = \sin\frac{\theta}{2}\mathbf{n}$);

$$R(\theta\mathbf{n})\mathbf{r} \ = \ \mathbf{r} + (\sin\theta)(\mathbf{n} \times \mathbf{r}) + 2(\sin^2\frac{\theta}{2})\mathbf{n} \times (\mathbf{n} \times \mathbf{r})$$

Our formula above reads

$$[0,\ \cos^2\frac{\theta}{2}\mathbf{r} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}(\mathbf{n} \times \mathbf{r}) + \sin^2\frac{\theta}{2}(\mathbf{r} \cdot \mathbf{n})\mathbf{n} + \sin^2\frac{\theta}{2}\mathbf{n} \times (\mathbf{n} \times \mathbf{r})]$$

Re-writing the vector part as;

$$\begin{gathered}
\cos^2\frac{\theta}{2}\mathbf{r} + \sin^2\frac{\theta}{2}((\mathbf{n} \times \mathbf{r}) \times \mathbf{n} + (\mathbf{r} \cdot \mathbf{n})\mathbf{n}) + \\
\sin\theta(\mathbf{n} \times \mathbf{r}) + 2\sin^2\frac{\theta}{2}\mathbf{n} \times (\mathbf{n} \times \mathbf{r})
\end{gathered}$$

Now use that
$(\mathbf{n} \times \mathbf{r}) \times \mathbf{n} + (\mathbf{r} \cdot \mathbf{n})\mathbf{n} = \mathbf{r}$

The struggle of vector analysis; 1850 - 1900

"*I believe that a struggle for existence is just commencing between the different methods and notations of multiple algebra, especially between the ideas of Grassman and of Hamilton*"
W. Gibbs, 1888.

Vectors/Matrices (linear algebra)	**Quaternions**
Gibbs (1839-1903 US)	Hamilton (1805-1865 Ireland)
Heaviside (1850-1925 England)	Tait (1831-1901 Scotland)
Grassman (1809-1877 Germany)	B. Peirce (1809-1880 US)
Maxwell (1831-1879 Scotland)	Maxwell
⋮	⋮

Some differences:

The 'vector product' $\times$ is not associative, but quaternion multiplication is (both are non-commutative).

Vector analysis with Quaternions

$$\mathbf{v}(t) = (v_1(t), v_2(t), v_3(t)) : \mathbf{R} \to \mathbf{R}^3,$$
$$q(t) = q_{\mathbf{v}(t)} = [0, \mathbf{v}(t)] = v_1(t)\mathbf{i} + v_2(t)\mathbf{j} + v_3(t)\mathbf{k} \; : \; \mathbf{R} \to \mathbf{H};$$

$$\text{then} \quad \frac{d}{dt} q_{\mathbf{v}(t)} = \dot{q}_{\mathbf{v}} = q_{\dot{\mathbf{v}}(t)}, \quad \dot{\mathbf{v}}(t) = (\dot{v}_1(t), \dot{v}_2(t), \dot{v}_3(t))$$
$$\frac{d^2}{dt^2} q_{\mathbf{v}(t)} = \ddot{q}_{\mathbf{v}} = q_{\ddot{\mathbf{v}}(t)}, \quad \ddot{\mathbf{v}}(t) = (\ddot{v}_1(t), \ddot{v}_2(t), \ddot{v}_3(t))$$

Cross product/ dot product: $\mathbf{v}, \mathbf{w} \in \mathbf{R}^3$; $q_{\mathbf{v}} = [0, \mathbf{v}]$, $q_{\mathbf{w}} = [0, \mathbf{w}]$

$$q_{\mathbf{v}\times\mathbf{w}} = [0, \mathbf{v} \times \mathbf{w}] = \frac{1}{2}(q_{\mathbf{v}} q_{\mathbf{w}} - q_{\mathbf{w}} q_{\mathbf{v}}) \equiv \frac{1}{2}[\, q_{\mathbf{v}} q_{\mathbf{w}} \,]$$
$$q_{\mathbf{v}\cdot\mathbf{w}} = [\mathbf{v} \cdot \mathbf{w}, \mathbf{0}] = -q_{\mathbf{v}} q_{\mathbf{w}} - \frac{1}{2}[q_{\mathbf{v}} q_{\mathbf{w}}]$$

Newton's equations; $\mathbf{F} = m\mathbf{a}$;

$$q_{\mathbf{F}} = m q_{\mathbf{a}} = m \ddot{q}_{\mathbf{v}}$$

Rigid body dynamics; $\frac{d\mathbf{L}}{dt} = \tau = \mathbf{r} \times \mathbf{F}$;

$$\dot{q}_{\mathbf{L}} = q_{\tau} = \frac{1}{2}[q_{\mathbf{r}} q_{\mathbf{F}}]$$

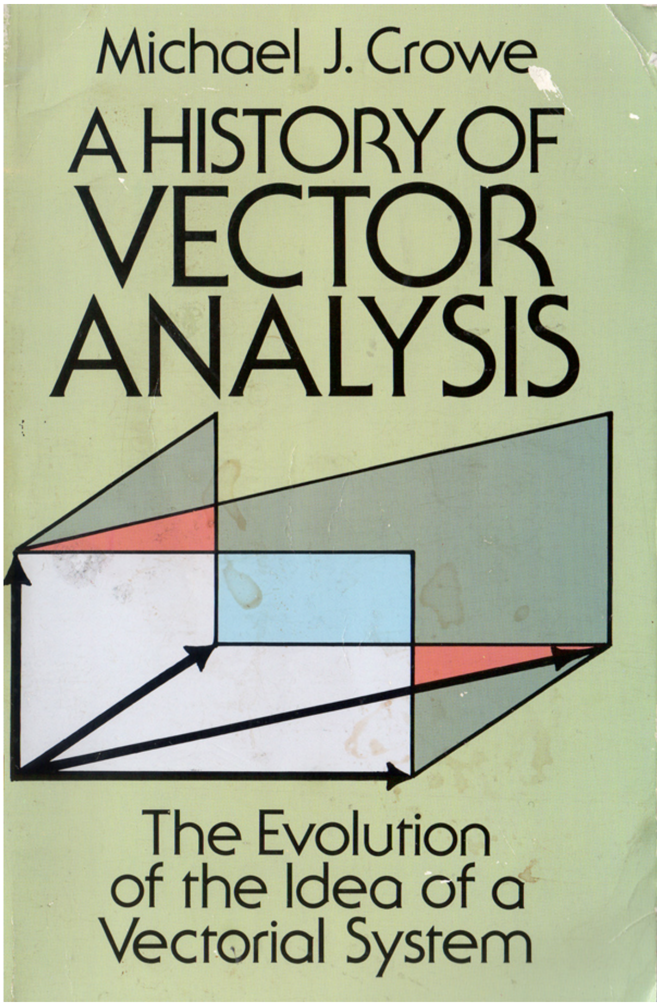
Divergence/Curl: $\mathbf{q}(x, y, z) = [0, \; \mathbf{V}(x, y, z)]$; quaternion field
Quaternion 'divergence' operator; $Q_{\nabla} = [\, 0, \; \nabla \,]$

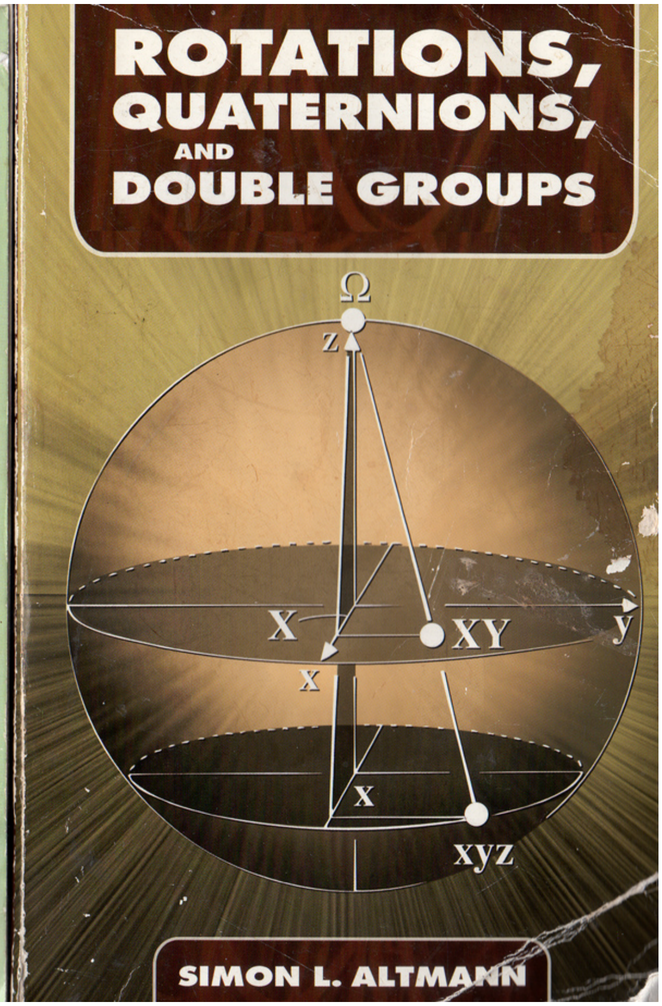
$$Q_{\nabla}\, \mathbf{q} = [-\nabla \cdot \mathbf{V}, \; \nabla \times \mathbf{V}\,]$$

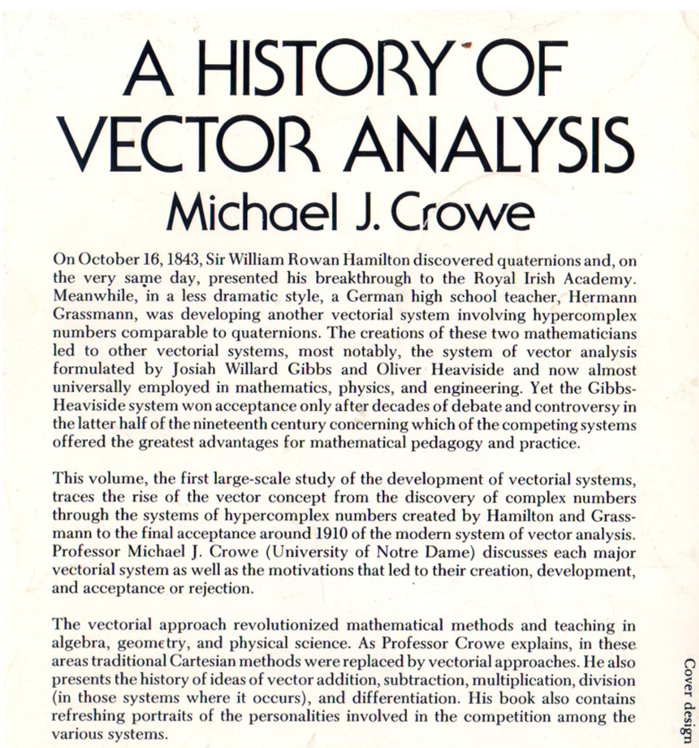
Maxwell's equations: $\mathbf{q}_{\mathbf{E}} = [0, \mathbf{E}]$, $\mathbf{q}_{\mathbf{B}} = [0, \mathbf{B}]$, $\mathbf{q}_{\rho} = [\rho, 0]$;

$$Q_{\nabla}\, \mathbf{q}_{\mathbf{E}} = -\mathbf{q}_{\rho} - \dot{\mathbf{q}}_{\mathbf{B}}$$

And more..... quaternionic Julia and Mandelbrot sets!







A HISTORY OF VECTOR ANALYSIS

Michael J. Crowe

On October 16, 1843, Sir William Rowan Hamilton discovered quaternions and, on the very same day, presented his breakthrough to the Royal Irish Academy. Meanwhile, in a less dramatic style, a German high school teacher, Hermann Grassmann, was developing another vectorial system involving hypercomplex numbers comparable to quaternions. The creations of these two mathematicians led to other vectorial systems, most notably, the system of vector analysis formulated by Josiah Willard Gibbs and Oliver Heaviside and now almost universally employed in mathematics, physics, and engineering. Yet the Gibbs-Heaviside system won acceptance only after decades of debate and controversy in the latter half of the nineteenth century concerning which of the competing systems offered the greatest advantages for mathematical pedagogy and practice.

This volume, the first large-scale study of the development of vectorial systems, traces the rise of the vector concept from the discovery of complex numbers through the systems of hypercomplex numbers created by Hamilton and Grassmann to the final acceptance around 1910 of the modern system of vector analysis. Professor Michael J. Crowe (University of Notre Dame) discusses each major vectorial system as well as the motivations that led to their creation, development, and acceptance or rejection.

The vectorial approach revolutionized mathematical methods and teaching in algebra, geometry, and physical science. As Professor Crowe explains, in these areas traditional Cartesian methods were replaced by vectorial approaches. He also presents the history of ideas of vector addition, subtraction, multiplication, division (in those systems where it occurs), and differentiation. His book also contains refreshing portraits of the personalities involved in the competition among the various systems.

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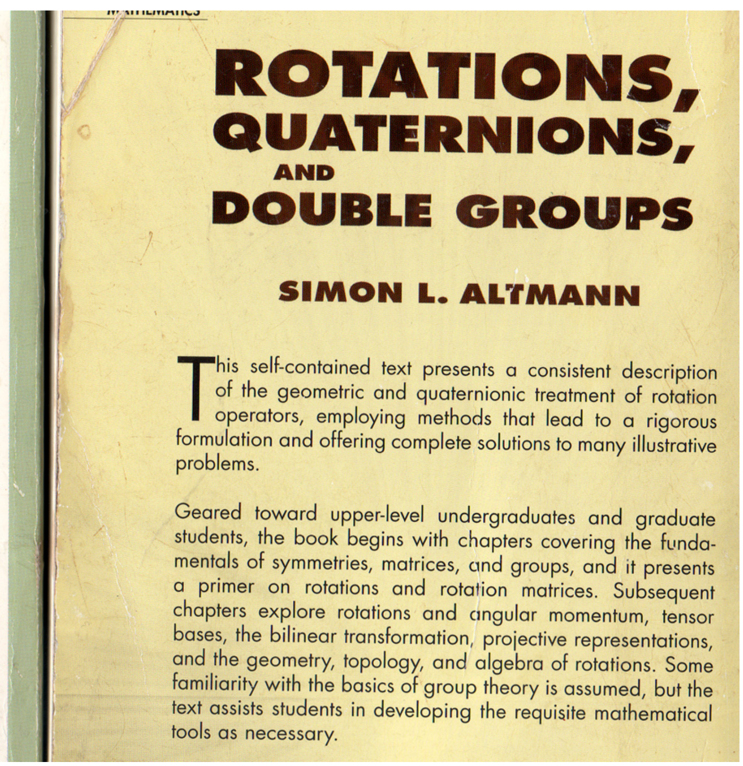

ROTATIONS, QUATERNIONS, AND DOUBLE GROUPS

SIMON L. ALTMANN

This self-contained text presents a consistent description of the geometric and quaternionic treatment of rotation operators, employing methods that lead to a rigorous formulation and offering complete solutions to many illustrative problems.

Geared toward upper-level undergraduates and graduate students, the book begins with chapters covering the fundamentals of symmetries, matrices, and groups, and it presents a primer on rotations and rotation matrices. Subsequent chapters explore rotations and angular momentum, tensor bases, the bilinear transformation, projective representations, and the geometry, topology, and algebra of rotations. Some familiarity with the basics of group theory is assumed, but the text assists students in developing the requisite mathematical tools as necessary.

AN ELEMENTARY TREATISE

ON

QUATERNIONS

BY

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Optimal wing hinge position for fast ascent in a model fly

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It was thought that the wing hinge position can be tuned to stabilize an uncontrolled fly. However here, our Floquet stability analysis shows that the hinge position has a weak dependence on the flight stability. As long as the hinge position is within the fly's body length, both hovering and ascending flight are unstable. Instead, there is an optimal hinge position, h^*, at which the ascending speed is maximized. h^* is approximately half way between the centre of mass and the top of the body. We show that the optimal h^* is associated with the anti-resonance of the body–wing coupling, and is independent of the stroke amplitude. At h^*, the torque due to wing inertia nearly cancels the torque due to aerodynamic lift, minimizing the body oscillation thus maximizing the upward force. Our analysis using a simplified model of two coupled masses further predicts, $h^* = (m_t/2m_w)(g/\omega^2)$. These results suggest that the ascending speed, in addition to energetics and stability, is a trait that insects are likely

2.1. *Three-dimensional dynamic flight simulation*

To simulate three-dimensional free flight with flapping wings, we solve the Newton–Euler equations for the coupled wing–body system (Chang & Wang 2014). The insect model consists of $(n+1)$ rigid bodies, where n is the number of wings on the body. Each wing is modelled as an ellipsoid connected to the body, also an ellipsoid, through a ball joint that allows for three degrees of freedom in rotation. The body kinematics are given by its position $\boldsymbol{r}^b$, linear velocity $\boldsymbol{v}^b$, body orientation quaternion $[q^b]$ and angular velocity $\boldsymbol{\omega}^b$. In our current implementation we use quaternions to represent the body and wing orientations. This has the advantage of avoiding gimbal lock and simplifies the algebra. For the results the quaternions are converted to Euler angles, which are easier to understand, as they refer to the rotations about body axes.

The Newton–Euler equations governing the body dynamics are

$$m^b \boldsymbol{a}^b = m^b \boldsymbol{g} - \sum_{i=1}^{n} \boldsymbol{F}_i^c, \tag{2.1}$$

$$\boldsymbol{I}^b \boldsymbol{\beta}^b = -\boldsymbol{\omega}^b \times (\boldsymbol{I}^b \boldsymbol{\omega}^b) - \sum_{i=1}^{n} \boldsymbol{\tau}_i^c - \sum_{i=1}^{n} \boldsymbol{r}_i^b \times \boldsymbol{\tau}_i^c. \tag{2.2}$$

$$\dot{\boldsymbol{r}}^b = [q]^b(\boldsymbol{v}^b), \tag{2.7}$$

$$\dot{\boldsymbol{v}}^b = \boldsymbol{a}^b - \boldsymbol{\omega}^b \times \boldsymbol{v}^b, \tag{2.8}$$

$$[\dot{q}]^b = \tfrac{1}{2}[q]^b \cdot [\omega]^b, \tag{2.9}$$

$$\dot{\boldsymbol{\omega}}^b = \boldsymbol{\beta}^b. \tag{2.10}$$

References:

Rotations, Quaternions, and Double Groups, by Simon L. Altman, Dover Publications (inexpensive paperback!)

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The Princeton Companion to Mathematics, Section III.76. A nice discussion on quaternions. And this (big!) book is accessible online through the SFU library.

Mathematical Thought from Ancient to Modern Times, Chapter 32, by Morris Kline.

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