Resonances, Stability, and Effective Stability in Hamiltonian Dynamical Systems

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I Introduction

These notes are intended to be an informal introduction for nonspecialists to some aspects of classical Hamiltonian systems. The main objects of study are integrable systems and perturbations there-of. Perturbations can lead to unstable motions and one of the aims of these notes is to understand how resonances of the underlying integrable system are responsible for these instabilities.

Perturbation Theory

We will discuss certain types of differential equations that arise in physics ("Hamiltonian equations"). By a finite-dimensional dynamical system we mean a set of ordinary differential equations (ODE's), and by an infinite-dimensional dynamical system we mean a set of partial differential equations (PDE's). In either case the dynamical system can be written in the form $\frac{d}{dt}u(t) = F(u(t))$, where $u(t) \in \mathbb{R}^n$, (ODE), [1] or $u(t) \in \mathcal{F}$, (PDE), where $\mathcal{F}$ is some (infinite-dimensional) space of functions[2] and $F : \mathbb{R}^n \to \mathbb{R}^n$ or $F : \mathcal{F} \to \mathcal{F}$ in the ODE or PDE case respectively. Let's call the variable t time. When we talk of the evolution of a dynamical system we are talking about the function $u(t)$ and how it changes in time. In these notes we are concerned with problems in classical mechanics so we will only treat ODE's (the "equations of motion"). Let me just mention, though, that many of the ideas and theorems we will discuss about ODE's can be carried over to PDE's (see, for example, references [19],[26],[32],[33],[S],[BFG],[FSW],[W]).

One of the main paradigms of mathematical physics is perturbation theory. Studying a dynamical system means studying the properties of the solutions, which are functions, of the set of differential equations that define the dynamical system. Typically, the equations that arise in physics ("real" dynamical systems) are difficult to solve. However, often they are close to an "ideal" one which we *can* solve (and this is why we call it ideal). The idea is to use the knowledge gained from having at hand solutions to the ideal problem to help understand the (similar) real problem. The difference between the real and ideal system is what we call the perturbation. Since the perturbation is in the equations of motion, which are infinitesimal, its effect is cumulative. If the perturbation is "small" we expect it to have a relatively small effect on the evolution of the underlying ideal system, at least at the start: the evolution of the real and ideal systems should be

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[1] $u(t) \in \mathbb{R}^n$ can also be written $u : \mathbb{R} \to \mathbb{R}^n$.

[2] $\mathcal{F} = \{f's : \mathbb{R}^m \to \mathbb{R}^n$ such that each function f has certain properties, for example, that it be integrable$\}$.

similar for some time until the perturbation affects the real system so much as to make it appear significantly different from the ideal one. The main objective of perturbation theory is to estimate this length of time and to quantify precisely in what respect the two systems differ.

Celestial Mechanics

The ODE's that arise in classical mechanics decribe the motion of particles as they move under the influence of one or more forces (e.g. gravitation, electric/magnetic) and, perhaps, subject to some constraints (for example, the particles may only be allowed to move along a certain curve or surface, or they may all be in a fixed position relative to one another so that they must move together as a whole[1]). In celestial mechanics the particles are celestial objects like planets, moons, comets, or stars. Here there are no rigid bodies (each planet etc., is treated like a particle with no internal motions) and the only force is gravitation (usually).

Ideal mechanical systems exhibit periodic (in time) or quasi-periodic (in time) motions, i.e., the solutions, $u(t)$, to the equations of motion are periodic or quasi-periodic. By a periodic function we mean that $u(t+T) = u(t)$ for all t and for some T (the period), which implies, via the theorem on Fourier series, that we can write u as $u(t) = \sum_{k\in\mathbb{Z}} u_k e^{i\omega kt}$ where $\omega \in \mathbb{R}$ ("frequency": $\omega = 2\pi/T$) and the $u_k \in \mathbb{R}$ ("Fourier coefficients": $u_k = \frac{1}{T}\int_{-T/2}^{T/2} u(t)e^{-i\omega kt}\,dt$). By a quasi-periodic function we mean that u can be written as $u(t) = \sum_{k\in\mathbb{Z}^n} u_k e^{i(\omega\cdot k)t}$ where $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ (frequency vector). Notice that a quasi-periodic function can also be realized as a function defined on the torus $\mathbb{T}^n$; $u(t) = U(t\omega)$ where $U : \mathbb{T}^n \to \mathbb{R}$, $(U = U(\theta_1, \ldots, \theta_n))$, and $t\omega = t\omega_1, \ldots, t\omega_n$. Then $u_k = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} U(\theta_1, \ldots, \theta_n)e^{-ik\cdot\theta}\,d^n\theta$. [2]

The significance of periodic and quasiperiodic solutions is straight forward: they are bounded for all time ($u(t)$ remains in a bounded subset of $\mathbb{R}^n$) and their motion is highly regular. For example, in the ideal solar system (where the planets do not influence each other but move just according to the gravitational force of the sun) the planets would never escape from the sun and the orbits would always be ellipses. [3]

In the real solar system the plantes *do* influence each other by their own gravitational fields. This is the perturbation. Since the planets are much smaller than the sun this perturbation is very weak compared to the interaction between the sun and the planets. So what about the real solar system? Will it too exhibit regular motions for all time or will a planet some day escape ? In other words, Is the solar system **stable**? But notice that stability here means for all time. This notion of stability is not a physically meaningful one because infinite times are never realized. What we should be asking is whether the solar system is **effectively stable**: whether it will exhibit regular motions for time scales that are relevant in this context (for example, the age of the universe).

KAM Theory and the Nekhoroshev Theorem

The approach taken in the perturbation theory of mechanical systems exploits the special structure (or geometry) of the equations of motion - their "Hamiltonian" structure (which will be described below). This structure singles-out a certain class of coordinate systems ("canonical

[1] A "rigid body".

[2] For $u(t) \in \mathbb{R}^n$, the above formulae are for each component of u.

[3] Actually, even in the ideal solar system there are a few exceptional orbits that can escape from the sun - those orbits that are straight lines radially away from the sun - plus we would have to arrange the initial positions of the planets so that they would not collide. But modulo these considerations the resulting motions would be regular.

coordinates"[4]) in which the equations have a special form. For any ideal system there is, from the class of canonical coordinates, a set of canonical coordinates, particular to the ideal system under consideration, in which the equations of motion assume a very simple form. When expressed in these coordinates the periodic and quasi-periodic behavior of the solutions are readily apparent. (In this formalism the ideal systems are referred to as "integrable" and the particular canonical coordinates are called "action- angle variables"; all of which will be described later on.)

Given a perturbed dynamical system (that is, one in which the equations of motion equal the equations of motion of an ideal system plus another, perturbation, term) which is (originally) expressed in canonical coordinates associated to the underlying ideal system (its action-angle variables), one looks for a new set of canonical coordinates that will put the perturbed equations into ideal form (so that these new coordinates will be action-angle variables for the perturbed system). Thus one looks for a transformation ("canonical transformation") from one set of canonical coordinates to another (actually, from one set of action-angle variables to another). If this can be done then one has shown that the perturbed system is actually an ideal one (but was just given in the "wrong" coordinates) and so has regular motions (periodic and quasi-periodic). That is, in this case one would say that perturbing the original ideal system still gives an ideal system - that the original ideal system is "robust". The bottom line would then be that most mechanical systems exhibit quite regular motions - because most mechanical systems are close to ideal ones. It was well known in the 19th century, though, that ideal mechanical systems were rare - most physical systems appeared much too complicated (on paper) to be of the ideal type (action-angle variables could not be found for them).

The approach to a general theory of the behavior of mechanical systems focussed on perturbations of ideal systems and how their regularity was affected by the perturbations. This was because the ideal systems were explicitly solvable and, really, this is one of the only ways to get at the general systems. Since every mechanical system can be written as an ideal system plus a perturbation (although the perturbation may be quite large), this approach is a way to understand *all* mechanical systems.

It was Henri Poincaré who, at the turn of the century, made a careful analysis of perturbation theory in celestial mechanics and concluded that inherent in the structure of any ideal mechanical system are mechanisms (**resonances**) for destroying that ideal structure whenever a perturbation is added. So the best one could do was to find new canonical coordinates that put the perturbed equations into as near an ideal form as possible. The questions then became: What is the long-time behavior of nearly ideal sytems? Do they appear regular for awhile before they become complicated, and if so how long does it take? Poincaré called these questions "the fundamental problem of dynamics". So overwhelming was Poincaré's analysis that progress on the problem was daunted by its apparent unsolvability. It wasn't until the 1950's when Andrei Kolmogorov proved that nearly ideal systems were still very regular. However, Kolmogorov reformulated the problem in such a way that it was possible to get a meaningful answer. It was the rigidity of the notion of stability (*all* solutions regular for *all* time) that made the problem so difficult to solve. Building on the insight provided by Kolmogorov's work, Nikolai Nekhoroshev (in 1977) proved a deep theorem about perturbed ideal mechanical systems again by first reformulating the notion of stability. For future reference, and as will become clearer later, I'll just mention here these new notions. Kolmogorov proved a "local" stability theorem which states that on a (very peculiar) *subset* of $\mathbb{R}^n$ the perturbed dynamical system behaves like an ideal one *for all time*, while Nekhoroshev proved a "global" stability theorem which states that on *all* of $\mathbb{R}^n$ the perturbed system behaves *nearly* ideal for a certain, calculable, *finite* time (which, remember, to us physically oriented people,

[4]Sometimes we will refer to coordinates as variables.

is a more meaningful concept than infinite times).

These two theorems, KAM (Kolmogorov-Arnold-Moser) and Nekhoroshev [1], are the cornerstones of modern perturbation theory of mechanical systems, but the techniques and ideas are so novel and deep that they have been influential in many other areas of mathematics and physics. In fact, there is a whole industry of "KAM techniques" in mathematics to handle the "small divisor problems" that seem to be endemic to perturbation and inverse function theory.

In the following section we describe some geometric notions about dynamical systems. Looking at a dynamical system "as a whole" (i.e., *all* solutions) we identify certain large-scale structures that impose regularity on the set of solutions. We also discuss the particular kinds of equations that arise in classical mechanics and the formalisms associated to them. This will allow us, in the subsequent sections, to formulate precisely the KAM and Nekhoroshev theorems.

Section III discusses some notions associated with KAM theory: resonances, invariant tori, stable tori, and diophantine inequalities. It ends with two versions of the KAM theorem.

Section IV is devoted to Nekhoroshev's theorem. To understand it we first have to understand the approach to perturbed mechanical systems via canonical transformations. We will then see how the resonances in the underlying ideal system manifest themselves in the perturbed system by "magnifying" the effect of the perturbation (without resonances the perturbation would hardly be noticable). By mapping-out the locations of the resonances we will be able to carefully estimate the evolution of the perturbed dynamics and thus to estimate how much it differs from the ideal one.

Contents

[1] The KAM theorem is stated on pp17 and Nekhoroshev's theorem is stated on pp27.

II Dynamical Systems

Ordinary Differential Equations

For us a dynamical system will be a first-order ODE on $\mathbb{R}^n$ (or more generally, on an n-dimensional manifold $\mathbf{M}^n$) :

$$\dot{x}(t) = F(x(t)) \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$, and $\dot{x} \equiv \frac{d}{dt}x$. Although equations of motion in physics are always of at least second-order (defined on $\mathbb{R}^m$, say), they can be written as a system of first-order equations, but now on a "larger" space ($\mathbb{R}^{2m}$, so the n above in the $\mathbb{R}^n$ is an even number). The space on which the first-order equations are defined is called the phase space. This is one manifestation of the Hamiltonian formulation of mechanics: it reduces Newton's and Lagrange's formulation of the equations of motion of a physical system, which are second-order (on "configuration space"), to a first-order system (on phase space).

The old (i.e., before Poincaré) approach to solving (1) was to find explicit formulae for individual solutions (different solutions are characterized by their different initial conditions). One drawback of this (other than it being very difficult to find explicit solutions) is that often the *qualitative behavior* of the solution is hidden: Is the solution periodic? What is its long-term behavior (i.e., after transients have died away)? Does the solution wander through out all of phase space or does it remain in some bounded region? Many physically meaningful questions can be answered from knowledge of the qualitative behavior of the solutions. In particular, questions of stability can be decided (stability here means that the solution stays in a bounded region of phase space for all time). But not everything can be answered and often an analytic formula is required. For example: How long does it take this solution to reach equilibrium? (the solution may wander through phase space for a while before settling down to remain in some particular region from then on).

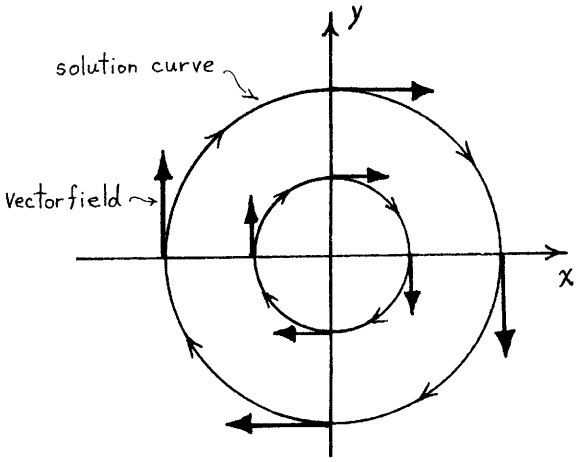
The new (i.e., after Poincaré) approach is to look at the solutions of (1) as a whole - the "geometry of phase space", by which I mean the geometry of the "flow" in the phase space associated to (1). What is "flow"? Well, treat (1) as defining a vectorfield on $\mathbb{R}^n$: at each $x \in \mathbb{R}^n$ there is a vector attached to that point, namely, the vector $F(x)$. [1] Then a solution of (1) is a curve (or "trajectory", or "orbit") in $\mathbb{R}^n$, denoted by $x(t)$, such that $\dot{x}(t) = F(x(t))$. If you think of $x(t)$ as describing the position at time t of a particle that is travelling along this curve, then the velocity, $\dot{x}(t)$, of the particle at the point $x(t)$ equals $F(x(t))$. (This is what we mean when we say "integrate a vectorfield": find a curve on the manifold whose velocity vector is everywhere equal to the vectorfield. In otherwords, we are solving an ODE). The collection of all these curves (one through every point $x \in \mathbb{R}^n$) is the flow of (1). (These notions are discussed in [1] and [7], for example.)

Example: The Harmonic oscillator; $\ddot{x}(t) = -x(t)$, $x \in \mathbb{R}$, $\ddot{x} \equiv \frac{d^2}{dt^2}x$ (Newton's equations).
Write as a first-order system on $\mathbb{R}^2$,

$$\left.\begin{array}{l} \dot{x} = y \\ \dot{y} = -x \end{array}\right\} \implies \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F(x,y) = \begin{pmatrix} y \\ -x \end{pmatrix}$$

(I'm suppresing the t-dependence of the functions x and y.)
The vectorfield for this dynamical system at the point (x, y) is the vector with components $(y, -x)$:

[1] We say $F(x) \in T_x\mathbb{R}^n$, the "tangent space" of $\mathbb{R}^n$ at x, which is just another $\mathbb{R}^n$.



In polar coordinates the equations of motion become: $\dot{r}(t) = 0,\ \dot{\theta}(t) = -1 \implies r(t) = r_o,\ \theta(t) = \theta_o - t.$ Solutions are circles traversed in the clockwise direction. (r_o, θ_o) is the initial condition that distinguishes different solutions.

The flow of (1) is the map $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ (which depends on t) and is defined by the formula; $\Phi_t(x_o) = x(t)$ where $x(0) = x_o$ (i.e., x_o is the point on the trajectory $x(t)$ at time 0). So the flow moves a point in $\mathbb{R}^n$ along the solution curve for (1) that passes through that point. In the example above, $\Phi_t(r_o, \theta_o) = (r_o, \theta_o - t) = (r(t), \theta(t))$.

Some "geometric" questions to ask about the system (1):

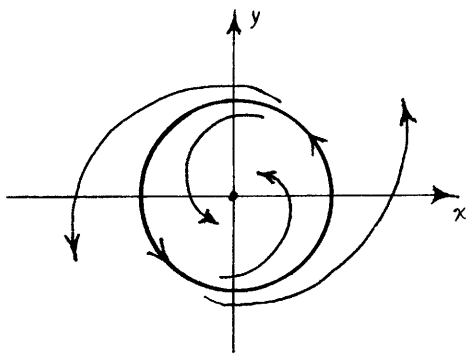
- are trajectories "trapped" in any region of phase space?
- are there invariant sets? i.e., is there a set $V \subset \mathbb{R}^n$ such that $\Phi_t(V) \subset V$ for all t? For example, are there any points $\underline{x}$ such that $\Phi_t(\underline{x}) = \underline{x}$ for all t? (these are called fixed points and are precisely the points where $F = 0$).

Periodic solutions are examples of invariant sets (these are the phase space trajectories of (1) that are closed curves). In the example above all circles are invariant sets.

A more interesting example:

$$\left.\begin{aligned} \dot{x} &= x - y - x(x^2 + y^2) \\ \dot{y} &= x + y - y(x^2 + y^2) \end{aligned}\right\} \implies \begin{aligned} \dot{r} &= r(r^2 - 1) \\ \dot{\theta} &= 1 \end{aligned}$$

Here we see that the circle $r = 1$ is an invariant set for this dynamical system (since $\dot{r} = 0$ there). For $r < 1$, $\dot{r} < 0$ so $r(t)$ is decreasing; for $r > 1$, $\dot{r} > 0$ so $r(t)$ is increasing, and $(0, 0)$ is a fixed point. Thus, a complete qualitative description of the flow of this system (i.e., *all* the solutions) is;



We conclude (effortlessly!) that there is only *one* periodic orbit and all other trajectories (solutions) either eventually approach the origin (if $r_o < 1$) or become monotonically unbounded (if $r_o > 1$).

Important observation: the circle $r = 1$ divides the phase space into two regions and the flow cannot cross from one region to the other.[1] So you see, if you can find an invariant surface in the phase space that bounds some region, then the flow inside that region will have to *remain* inside: the solutions of (1) that have initial conditions inside this region will be *bounded.*

[1] By the uniqueness of solutions to first- order ODE's, a trajectory cannot cross an invariant surface.

This is an important theme for us : *invariant surfaces (manifolds) in phase space act as a barrier to the phase flow. They give structure (regularity) to the flow, and hence the solutions will assume some regularity* (a complicated flow has few invariant surfaces; see [8],[9] for more discussion).

Now, if you wanted to ask about the example just given how long it takes a solution to approach the origin (if it does eventually), then you would have to do some (analytic) work. So let's say that, close to the origin $\dot{r} \approx -r, \ \Rightarrow \ r(t) = r_o e^{-t}$. To get within a distance d of the origin you would have to wait until $t = ln(r_o/d)$. This time $\nearrow \infty$ as $d \searrow 0$, of course, but it goes to infinity "very slowly" which means the trajectories are approaching the origin "very quickly" (even though they never get there). If $r_o > 1$, then writing $\dot{r} \approx r^3$ we get $r(t) = \sqrt{r_o{}^2/(1 - 2tr_o{}^2)}$ so that $r(t) = \infty$ when $t = 1/2r_o{}^2$. We say then that the flow is *incomplete*: it is not defined for all time.

This little bit of analytic work allows us to associate a time scale to the dynamical system : how long it takes to "reach equilibrium" or, the "relaxation time". These time scales are important in (real) physical applications. For example, if this time scale is very short in comparison to some relevant time scale associated to the system, then for all practical purposes the system will appear to be in equilibrium (i.e., nothing will seem to change). If on the other hand this time scale is relatively long then even though in the limit as $t \to \infty$ the system reaches equilibrium, it will take so long to do so that it may never *appear* to be in actual equilibrium. Physical systems often evolve towards an equilibrium state, but just how long it takes for them to get there is a profound problem in statistical mechanics; see [22],[S].

Structural Stability of Dynamical Systems

This is the geometric view of perturbation theory. A perturbed dynamical system has the form

$$\dot{x}(t) = F(x(t)) + \varepsilon f(x(t), t) \tag{2}$$

where ε is usually considered to be much smaller than 1. For $\varepsilon = 0$ we have the unperturbed system (1). Suppose we are able to solve (1), i.e., we know the geometry of its phase flow (e.g., fixed points, periodic orbits - in general, the invariant manifolds). Then we ask: How does the influence of the perturbation $\varepsilon f(x, t)$ affect this geometry? For example, does (2) have an invariant manifold (say, a periodic orbit) that is "near" to one of (1)? If so, then we say that this object is structurally stable under the perturbation f.[1] If the object is structurally stable for some (large) class of perturbations, then we may just say that it is structurally stable. It is important to remember that the notion of structural stability is associated to a class of functions (the perturbations).

Really, in physical applications we are only interested in studying structurally stable systems, or rather, we are interested in knowing what features of the phase flow of a dynamical system are structurally stable and which are structurally unstable. The reason being that we are never able to define a (physical) dynamical system *exactly* (for example, we don't know the exact masses of the planets) so whatever model we come up with we must be prepared to study how it changes under a perturbation. The uncertainties in our model will be incorporated into the $\varepsilon f(x, t)$ term. That's why our perturbation theory must consider *arbitrary* ("generic") perturbations f.

Hamiltonian Dynamical Systems

These are the (very special) dynamical systems that arise in classical mechanics. They are distinguished by the special form of the $F(x(t))$ term in (1). For a Hamiltonian system the phase space is always even dimensional (pairs of "velocity,position" coordinates) so we will call our phase

[1] Contrast this with the notion of stability with respect to initial conditions in a *fixed* dynamical system.

space simply $\mathbb{R}^{2n}$. The (Hamiltonian) equations of motion are prescribed by a function H (the Hamiltonian) defined on the phase space; $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$, and in special canonical coordinates $\{(p,q), p \in \mathbb{R}^n, q \in \mathbb{R}^n\}$ on $\mathbb{R}^{2n}$ take the form

$$\dot{p}(t) = -\partial H(p,q)/\partial q \quad ; \quad \dot{q}(t) = \partial H(p,q)/\partial p \tag{3}$$

or,

$$\begin{pmatrix} \dot{p}(t) \\ \dot{q}(t) \end{pmatrix} \quad = \quad J \nabla H(p(t), q(t)), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

or, in modern differential geometric language,

$$dH \quad = \quad i_{X_H} \omega$$

where X_H is the vectorfield associated to (3), ω is the "symplectic form" on phase space and $i_{X_H}\omega$ denotes the "interior product" (or "inner product") of X_H and ω (see [1],[2] for details).

For any dynamical system, an integral is a function h defined on phase space such that $(h \circ \Phi_t)(p,q) = h(p(t),q(t)) = h(t) = constant$, where Φ_t is the flow of the system. In other words, the trajectories $(p(t),q(t))$ lie on level surfaces of h: $h^{-1}(c)$, which is a submanifold of $\mathbb{R}^{2n}$ of codimension 1 if c is a regular value of h, i.e., $dh(x) \neq 0$ for all $x \in h^{-1}(c)$. We say that "h is conserved". For example, in certain 4-dimensional Hamiltonian systems (ones with rotational symmetry) angular momentum, given by the function $h(p,q) = p_1q_2 - p_2q_1$, is conserved (see [5] §9, [G] §3.2).

H is always an integral of a Hamiltonian system. Proof: calculate!

$$\begin{aligned} \frac{d}{dt}H(t) \;=\; \frac{d}{dt}H(p(t),q(t)) \;&=\; \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial q}\dot{q} \quad (chain rule) \\ &= -\frac{\partial H}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} \quad (via (3)) \\ &= 0 \end{aligned}$$

This means that "energy is conserved". Thus, the flow of a Hamiltonian system is restricted to $2n-1$ dimensional "energy surfaces" $H^{-1}(e)$, (e is the energy of the points (p,q) on the surface $H^{-1}(e)$). This will always be tacitly assumed: we will talk of the dynamics, always meaning the dynamics on an energy surface.

The existance of an integral means that the phase space has many invariant manifolds: the surfaces $h^{-1}(c)$ for appropriate $c's \in \mathbb{R}$. Consequently, we expect that the phase flow will be greatly restricted in the presence of integrals.

The more integrals there are the more restricted is the phase flow because the trajectories must stay on the level surfaces of all the integrals. Thus, if in addition to the energy integral H there is another integral g, then on the energy levels $H^{-1}(e)$ the flow must lie also on the level sets $g^{-1}(c)$ so the flow is restricted to a $2n-2$ dimensional submanifold of the phase space. We see then that in the presence of integrals we need only study the dynamics of a system of lower dimension (the flow on the level sets) which is generally easier the lower the dimension. We will see that if a Hamiltonian system has "sufficiently many" integrals then the dynamics are very simple. These are the so-called integrable systems and are the ideal systems mentioned at the start.

But first some more jargon;

The Poisson bracket $\{f,g\}$ of two functions f,g defined on phase space is given by the formula

$$\{f,g\} \;=\; \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}.$$

Notice that $\{f,g\}$ is a function on phase space and that $\{f,H\} = \frac{d}{dt}f$. In particular, $\dot{p} = \{p,H\}$ and $\dot{q} = \{q,H\}$. The Poisson bracket makes $C^\infty(\mathbb{R}^{2n})$ into a Lie algebra, where $C^\infty(\mathbb{R}^{2n})$ denotes the smooth functions on phase space. So: f is an integral $\Longleftrightarrow \{f,H\} = 0$. We say that f and g are in involution if $\{f,g\} = 0$.

Integrable Hamiltonian Systems

A Hamiltonian system is called integrable (or completely integrable) if there are n functions $\{f_1 = H, f_2, \ldots, f_n\}$ such that $\{f_i, H\} = 0$, $\{f_i, f_j\} = 0$ for $i,j = 1,\ldots,n$, and $\{df_1,\ldots,df_n\}$ are linearly independent almost everywhere ([1],pp 392-393). That is, there are n integrals in involution that are functionally independent (for a discussion of functional independence, see [LS pp175]).

Let $c \in \mathbb{R}^n$ and $N_c = \{(p,q) \mid f_1(p,q) = c_1, \ldots, f_n(p,q) = c_n\}$. The Arnold-Liouville theorem ([1] pp393, [2] pp282, [4] pp110,114) states that if N_c is compact, its connected components are diffeomorphic to the n-torus $\mathbb{T}^n$ (if N_c is not compact then it is diffeomorphic to the "cylinder" $\mathbb{T}^{n-k} \times \mathbb{R}^k$, for some $k \leq n$). Furthermore, these tori are invariant manifolds of the flow (so the phase flow wraps around the tori) and on these tori the flow is very simple; it is just *translational*, i.e., if $\theta = (\theta_1,\ldots,\theta_n)$ are angular coordinates on $\mathbb{T}^n$, then in these coordinates the flow satisfies $\dot{\theta}_i = \omega_i$ for some $\omega_i \in \mathbb{R}$. [1] We call the $\omega_i's$ the frequencies of the flow on the torus (the frequencies generally depend upon which torus you are considering). Translational flow means that if you "unwrap" the torus to make $\mathbb{R}^n$ the trajectories are straight lines with slopes $\omega_1,\ldots,\omega_n$ and points on these trajectories move with constant velocity. In addition, there is a neighborhood (in phase space) around any such torus, call this neighborhood $\mathbf{D}$, where there are canonical coordinates $(I_1,\ldots,I_n,\theta_1\ldots,\theta_n) = (I,\theta)$ such that the level sets $I^{-1}(i)$, $i \in \mathbb{R}^n$, are n-tori $\mathbb{T}^n$ and θ are angular coordinates on these tori. In other words, $\mathbf{D} \approx \mathbf{P}^n \times \mathbb{T}^n$, where $\approx$ means diffeomorphic to and $\mathbf{P}^n$ is, say, an open ball in $\mathbb{R}^n$ (called the action space). The coordinates (I,θ) are called action-angle variables. Also, $I = I(f_1,\ldots,f_n)$ and $\boxed{H = H(I)}$. A consequence of this is that $\dot{I} = -\partial H/\partial\theta = 0$: the functions $I_1,\ldots,I_n$ are integrals (and are in involution). We define $\omega(I) \equiv \dot{\theta} = \partial H/\partial I$. Thus, *the action variables do not evolve and the frequencies are a function of the action only* (so the frequencies are constant on tori). Since $\omega(I) = \partial H/\partial I$, the properties of the frequencies are going to be determined by properties of H.

Kindly words of warning: even if a Hamiltonian system is integrable there may *not* be action-angle variables defined everywhere in phase space (for example, near a critical point of H) so phase space may *not* be simply $\mathbb{R}^n \times \mathbb{T}^n$ or $\mathbf{P}^n \times \mathbb{T}^n$. The existence of action-angle variables depends not only on properties of the Hamiltonian function H, but also on the topology of the phase space. However, for us we will just assume that our phase space *is* $\mathbf{P}^n \times \mathbb{T}^n$ - if your particular Hamiltonian system does not admit global action-angle variables then just interpret $\mathbf{P}^n \times \mathbb{T}^n$ to be a region of phase space that *does* admit them, then everything we say is only with respect to this region. (For more information on when action-angle variables can be defined, see the paper by J.J. Duistermaat, [20]).

[1] Here and through out we write $\dot{I}(t) \equiv \frac{d}{dt}I(t)$, $\dot{\theta}(t) \equiv \frac{d}{dt}\theta(t)$ where $(I(t),\theta(t))$ is a representation of the flow in (I,θ) coordinates.

Some consequences of integrability: all bounded motions are quasi-periodic (they may be periodic if there are enough relations among the frequencies), and (most) unbounded motions lie on surfaces with simple geometry. Thus, the phase flow of an integrable system is highly regular, in particular it is nothing like ergodic ("chaotic").

Perturbed Integrable Systems

Since many real-life Hamiltonian systems can be realized as perturbations of integrable ones (for example, the solar system) it is natural to ask whether this nice phase space geometry persists for the perturbed system. Are motions for the perturbed system (nearly) quasi-periodic? The *big* question of the day (~1900) was: Is a nearly integrable system (i.e., a perturbed integrable system) integrable? Or, more simply, how many integrals does a nearly integrable system have? Poincaré discovered that under "most" perturbations a nearly integrable system has *no* (analytic) integrals (see [14]). This seemed to suggest that Hamiltonian systems came in two types: integrable or completely nonintegrable (no integrals except for H). We've seen how the existence of integrals leads to the existence of invariant manifolds in phase space that inhibit the phase flow, so with the indication that nearly integrable systems have no integrals it was generally felt that the phase flow will wander through-out *all* of phase space (restricted to an energy surface) and that it would even be ergodic flow, which means the flow would uniformly sample all of the energy surface. This would make things simple because then you could apply the techniques of statistical mechanics to these systems and obtain information about *average* behavior (statistical mechanics assumes the ergodic hypothesis) - see [13] pp137, and [22].

But of course, there can be invariant manifolds in phase space even without the existence of any integrals, and in fact the KAM theorem constructs "local integrals" in phase space: it shows that there are many invariant manifolds in the phase space of nearly integrable systems that "come from" the invariant tori of the integrable system (for a precise discussion of "local integrability" see [30]).

Once computers were developed enough so they could readily integrate the equations of motion of nearly integrable systems, people were quite surprised by what they saw. For instance, in Princeton in the 1950's Fermi, Pasta and Ulam ([21]) numerically integrated a system of weakly-coupled harmonic oscillators. Without the coupling this system is integrable. They started the integration with a particular configuration (which would have remained stable without the coupling) and expected that it would decay into "irregular" behavior as all of the "modes" became excited - as prescribed by the principle of equipartition of energy (statistical mechanics- see [22]). But nothing of the sort happened. Instead, the motion became a little bit irregular as a few of the lower frequency modes became excited, but after a while the system returned to nearly the same configuration that it started with (but much sooner than that given by Poincaré's "Recurrence Theorem"- [2] pp71), and then began the cycle again, etc. So apparently the trajectory in phase space was not travelling through-out all of the energy level. It was as if there were some barriers, some invariant manifolds, preventing this; the trajectory seemed to be *diffusing* through phase space (this means the type of slow, irregular motion associated with a "random walk"). However, this experiment did not rule-out the possibility that the rate in which the higher frequency modes were being excited was very slow, too slow to be observed. We will see below, with Nekhoroshev's theorem, that nearly integrable systems can exhibit extremely slow diffusion in phase space.

III The KAM Theorem

Nearly Integrable Hamiltonian Systems

This is a theorem about the existence of invariant manifolds in the phase space of nearly integrable systems. Recall that, for us, an integrable Hamiltonian system has a Hamiltonian H of the form

$$H(I,\theta) = H_o(I)\,, \qquad (I,\theta) \in \mathbf{P}^n \times \mathbb{T}^n, \quad \mathbf{P}^n \subset \mathbb{R}^n. \tag{4}$$

That is, H depends only on the action variables. Consequently, the dynamics are given by

$$\dot{I}(t) = -\frac{\partial H}{\partial \theta} = -\frac{\partial H_o}{\partial \theta} = 0\,, \qquad \dot{\theta}(t) = \frac{\partial H}{\partial I} = \frac{\partial H_o}{\partial I} \equiv \omega_o(I).$$

$\omega_o(I)$ is the frequency vector of the (translational) flow of (4) on the torus $\mathbb{T}^n(I) \in \mathbf{P}^n \times \mathbb{T}^n$ labelled by the action coordinate I. (Remember that the tori $\mathbb{T}^n(I)$ are invariant manifolds for (4).) The frequencies depend only on I so that we can talk of the *frequencies in action space* $\mathbf{P}^n$. The properties of $\omega_o(I)$ play a crucial role in KAM theory (and perturbation theory in general). For example, they decide which tori will survive a (given) perturbation.

By a perturbed system we mean a nearly integrable Hamiltonian system whose Hamiltonian has the form

$$H(I,\theta) = H_o(I) + \varepsilon f(I,\theta) \tag{5}$$

and thus dynamics given by

$$\dot{I}(t) = -\frac{\partial H}{\partial \theta} = -\varepsilon\frac{\partial f}{\partial \theta}\,, \qquad \dot{\theta}(t) = \frac{\partial H}{\partial I} = \frac{\partial H_o}{\partial I} + \varepsilon\frac{\partial f}{\partial I} = \omega_o(I) + \varepsilon\frac{\partial f}{\partial I} \equiv \omega(I).$$

Here (I,θ) are action-angle variables for the integrable system H_o ($\varepsilon = 0$). In a nutshell, the KAM theorem says that "most" of the invariant tori of (4) survive the perturbation $\varepsilon f(I,\theta)$ for ε sufficiently small so that the phase space of the system (5) contains many invariant manifolds (tori).

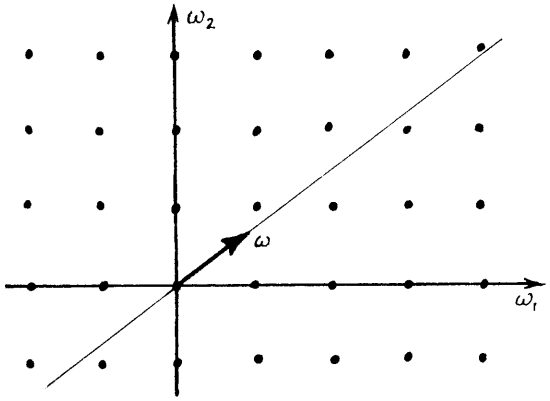
By "an invariant torus $\mathbb{T}^n(I_o)$ of (4) surviving the perturbation εf" (i.e.,that it is (structurally) stable) we mean: for all ε sufficiently small there is a (diffeomorphic image of a) torus, call it $\mathbb{T}^n_\varepsilon$, sitting inside of $\mathbf{P}^n \times \mathbb{T}^n$, that is invariant with respect to the flow of (5), upon which the flow (of (5)) is translational with frequencies $\omega_o(I_o)$ ("quasi-periodic with frequencies $\omega_o(I_o)$"), $\mathbb{T}^n_\varepsilon$ lies in a neighborhood U_ε of $\mathbb{T}^n(I_o)$ and $U_\varepsilon \longrightarrow \mathbb{T}^n(I_o)$ as $\varepsilon \searrow 0$ [1] (U_ε is typically a neighborhood of size $\mathcal{O}(\sqrt{\varepsilon})$ around $\mathbb{T}^n(I_o)$ in the cases we are considering ([4], pp 184)). In other words, if $\mathbb{T}^n(I_o)$ is a stable torus of (4), then by varying the initial conditions slightly we can find quasi-periodic orbits of (5) that have frequencies $\omega_o(I_o)$.

Resonant and Nonresonant Frequencies

Let's review some facts about frequency vectors $\omega \in \mathbb{R}^n$. We say that ω is nonresonant if $\omega_1, \ldots, \omega_n$ are linearly independent over the rationals, which is equivalent to saying that if $\omega \cdot k \equiv \omega_1 k_1 + \cdots + \omega_n k_n = 0$ for some $k \in \mathbb{Z}^n$, then $k = 0$. For example, if ω is nonresonant then the $\omega_i's$ must be distinct and at most one of them could be rational - the rest must be irrational numbers. ω is resonant if $\omega \cdot k = 0$ for some nonzero $k \in \mathbb{Z}^n$. More precisely, we say that ω satisfies a resonance of order N if $\omega \cdot k = 0$ for $|\,k\,| \equiv |\,k_1\,| + \cdots + |\,k_n\,| = N$. Notice that a resonant vector could

[1] More precisely, we say that there exists a family of diffeomorphisms $\Xi_\varepsilon : \mathbb{T}^n \longrightarrow \mathbf{P}^n \times \mathbb{T}^n$, such that $\Xi_0(\mathbb{T}^n) = \mathbb{T}^n(I_o)$, $\Xi_\varepsilon(\mathbb{T}^n)$ is close to $\mathbb{T}^n(I_o)$, and $\Xi_\varepsilon(\mathbb{T}^n)$ is an invariant manifold of (5).

have all irrational frequencies still. A quasi-resonant vector ω of size δ , $\delta > 0$, is one in which $|\omega \cdot k| \leq \delta$ for some nonzero $k \in \mathbb{Z}^n$. It is a well known fact that even if ω is nonresonant it still satisfies a quasi-resonance of arbitrarily small size, although the smaller you specify δ the larger (typically) must be $|k|$. For example, for $\omega's \in \mathbb{R}^2$ we can represent them as a vector in the plane:

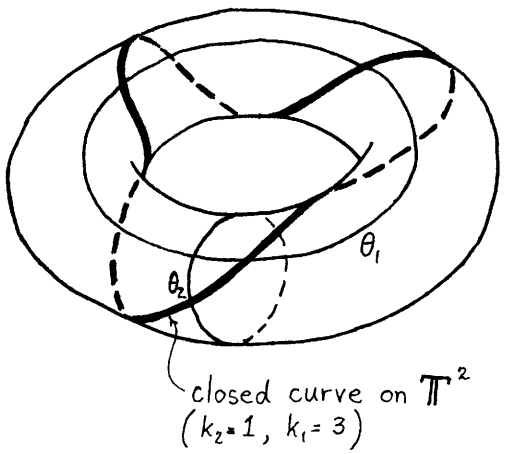


Here I've put dots at the lattice points of $\mathbb{Z}^2$ (the integer coordinates of $\mathbb{R}^2$). It's easy to see that ω is resonant when the line through ω passes through a lattice point. Of course ω has to be picked *extremely well* for this to happen so that typically, i.e., with probability one, ω is nonresonant. If ω is nonresonant the line through ω never passes through a lattice point, but you can see that it will (eventually) get arbitrarily close to one!

That is, if you give me any small number $\delta > 0$, I can find a lattice point (it may be quite far from the origin) by which the line through ω passes within a distance δ. This doesn't mean that a nonresonant ω *can't* satisfy a quasi-resonance that is close to the origin, of course it can, but your ω would have to be chosen more carefully so that it would be *less typical.* In other words, if you randomly choose a ω, then for sure it will be nonresonant, and the smaller quasi-resonances it satisfies would occur further from the origin. We can make these ideas precise using diophantine inequalities; ω satisfies a diophantine inequality with parameters $\gamma, \eta > 0$ if $|\omega \cdot k| \geq \gamma / |k|^\eta$ for *all* $k \in \mathbb{Z}^n, k \neq 0$. What this means is that even though ω may satisfy arbitrarily small quasi-resonances, they must occur far enough away from the origin: if $\tilde{k}$ is where this quasi-resonance occurs (so $|\omega \cdot \tilde{k}| < \delta$), then $|\tilde{k}| \geq (\gamma/\delta)^{1/\eta}$. If ω satisfies a diophantine inequality with parameters γ, η, then within any ball in $\mathbb{Z}^n$ centred at the origin ω satisfies no quasi-resonances (due to $k's$ from inside this ball) smaller than some certain number that we can calculate.

It is a fact that almost every $\omega \in \mathbb{R}^n$ (in the sense of Lebesgue measure) satisfies a diophantine inequality for some γ ($\gamma = \gamma(\omega)$) and if $\eta \geq n$ (the ones that don't are "algebraic numbers"). Let $\Omega(\gamma) = \{\omega \in \mathbb{R}^n$ such that ω satisfies a diophantine inequality with parameters $\gamma, n\}$. Then the above ideas can be made precise by saying that the measure of $\Omega(\gamma)^C$ is small and that it goes to zero as γ goes to zero (see [12] pp98).

Suppose $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$ is resonant. Then $k_1\omega_1 + k_2\omega_2 = 0$ for some $k_1, k_2 \in \mathbb{Z}, (\neq 0), \Longrightarrow$ $\omega_1/k_2 = -\omega_2/k_1$, and $\omega_1/\omega_2 = -k_2/k_1 \in \mathbb{Q}$. If (θ_1, θ_2) are angular coordinates on $\mathbb{T}^2$, denote by $\theta_1(t), \theta_2(t)$ the translational flow on $\mathbb{T}^2$ with frequency ω : $\theta_1(t) = \omega_1 t$, $\theta_2(t) = \omega_2 t$ (I take the initial condition to be the origin).



So the curve $c(t) = (\theta_1(t), \theta_2(t))$ on $\mathbb{T}^2$ wraps once around the θ_1-direction after $t = 2\pi/\omega_1$ and once around the θ_2-direction after $t = 2\pi/\omega_2$. If $\omega_1/k_2 = -\omega_2/k_1$, then $-k_2(2\pi/\omega_1) = k_1(2\pi/\omega_2)$ so that $c(t)$ wraps exactly k_2 times around the θ_1- direction and exactly k_1 times around the θ_2-direction after a time $T = k_2(2\pi/\omega_1)$.

That is, *c(t) is a closed curve on* $\mathbb{T}^2$: it is actually a circle. Notice that if $\omega \cdot k = 0$ then the larger $|\, k \,|$ is, the more times $c(t)$ wraps around the torus before closing on itself: flow of a high-order resonance samples more of the torus than flow of a lower-order resonance.[1] Similarily, if $\omega \in \mathbb{R}^n$ and $\omega \cdot k = 0$, then the translational flow with frequency ω actually lies on lower dimensional torus sitting in $\mathbb{T}^n$.

ω can satisfy more than one resonance. In fact, if $\omega \cdot k = 0$ then $\omega \cdot (mk) = 0$ for any $m \in \mathbb{Z}$. If $\omega \cdot k_1 = 0$ and $\omega \cdot k_2 = 0$, then $\omega \cdot (m_1 k_1 + m_2 k_2) = 0$ for any $m_1, m_2 \in \mathbb{Z}$. In general, if $\{k_1, \ldots, k_s\}$ are resonances for ω, then the lattice $\Lambda_\omega \subset \mathbb{Z}^n$ generated by these k'^s contains only resonances of ω (Λ_ω = all integral linear combinations of $k_1, \ldots, k_s$). You can convince yourself that ω can satisfy at most $n-1$ independent resonances.

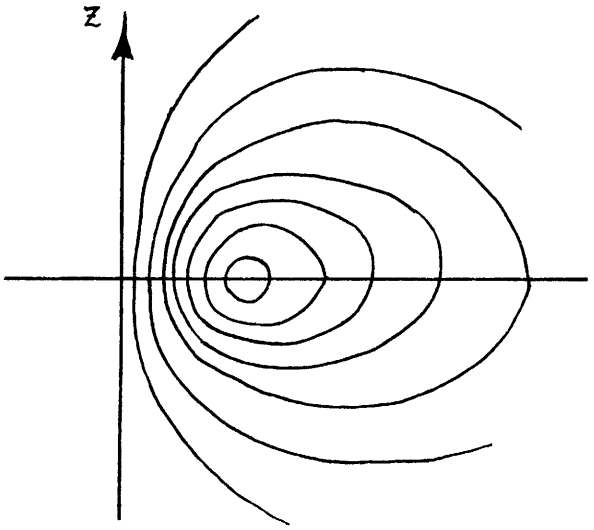
Also note the *important fact* that if $\omega \in \mathbb{R}^n$ is nonresonant, the translational flow on $\mathbb{T}^n$ with frequency ω covers $\mathbb{T}^n$ uniformly and densely (we say then that the flow on $\mathbb{T}^n$ is *ergodic*).

Perturbation of an Integrable System With Two Degrees of Freedom

Let's look at the case of a nearly integrable system with two degrees of freedom (this means that configuration space ("q-space") is two- dimensional):

$$H(I, \theta) = H_o(I) + \varepsilon f(I, \theta)\,, \qquad (I, \theta) \in \mathbf{P}^2 \times \mathbb{T}^2\,, \quad \mathbf{P}^2 \subset \mathbb{R}^2.$$

Here the energy surfaces are 3-dimensional (say, some open subset of $\mathbb{R}^3$) and they are foliated by 2-dimensional tori. Thus, through every point in the energy surface there passes one and only one such torus. Here's an example of a foliation of $\mathbb{R}^3$ by 2-tori:



Now rotate these "circles" around the z-axis.

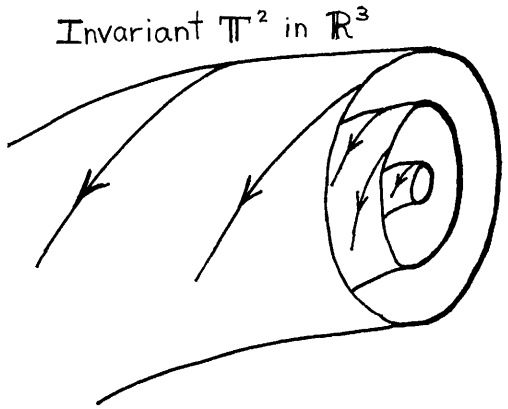
You can see that the coordinates identifying the tori (the I'^s) and the angular coordinates on each torus are coordinates for $\mathbb{R}^3$ (although in this case, when restricted to an energy surface, only *one* I coordinate is needed to lable the different tori (via the implicit function theorem applied to the equation $H(I) = e$) - so below I will use just one I coordinate).

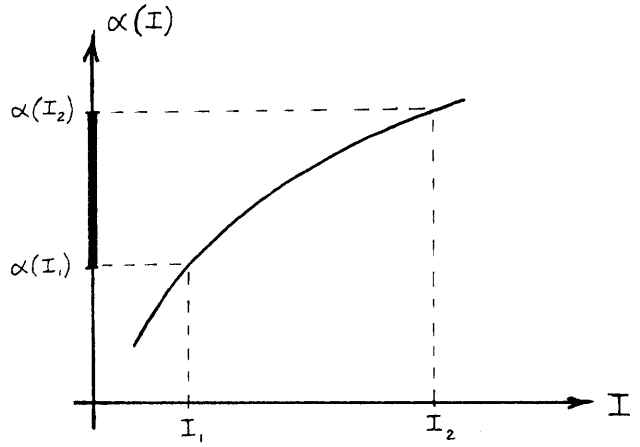
Following Arnold ([2] app.8) we will try to see what happens to these tori when you add a perturbation. We assume that the integrable Hamiltonian H_o is nondegenerate or isoenergetically nondegenerate. Nondegeneracy means the matrix $\partial^2 H_o / \partial I^2$ (the Hessian of H_o) is nondegenerate at all points (I, θ). Isoenergetic nondegeneracy means H_o is nondegenerate on energy levels (see for example, [2] pp402-403 for these definitions). Since $\omega_o(I) = \partial H_o / \partial I$, nondegeneracy implies that $\det(\partial \omega_o / \partial I) \neq 0$, i.e., the frequencies are functionally independent. Among the many consequences of this is the *important* property that, given any torus $\mathbb{T}^n(I_o)$ in phase space, there is a neighborhood

[1] This has relavence in the theory of averaging; see [2],[4].

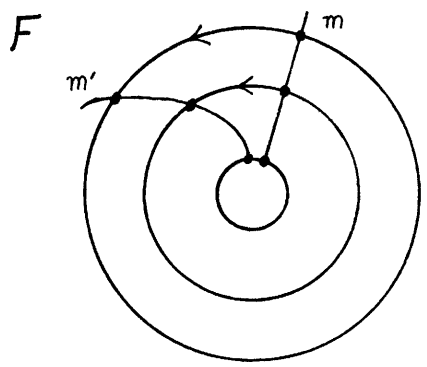
of $\omega_o(I_o)$ in $\mathbb{R}^n$ such that for any ω in this neighborhood there is a torus $\mathbb{T}^n(I_\omega)$ near to $\mathbb{T}^n(I_o)$ such that $\omega_o(I_\omega) = \omega$: nearby any torus with frequency ω there is a torus with any frequency close to ω. So if ω is not quite the right frequency we need, by adjusting our initial conditions slightly we can find a torus with precisely the frequencies we want.[1] Of course, the easy way to say all this is that in nondegenerate systems we can use the frequencies as coordinates in action space.

For a 2-degree of freedom system we have $\mathbb{T}^2$ sitting in $\mathbb{R}^3$ as shown in the diagram on the left below.





Call $\alpha(I) = \omega_1(I)/\omega_2(I)$. Then isoenergetic nondegeneracy implies that $\frac{d}{dI}\alpha(I) \neq 0$. Thus, $\alpha(I)$ has a graph that looks like that in the diagram on the right above. Take a (2- dimensional) plane and slice it through the family of tori that foliate the energy surface. These tori intersect the plane in circles. Define a map $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ via $F(m) = m'$ where m' is where the trajectory that starts at m returns to intersect the plane that cuts the torus ("first return map"). The circles where the invariant tori intersect the plane are invariant under F, and in fact F just rotates them, *but by different amounts* because $d\alpha/dI \neq 0$:



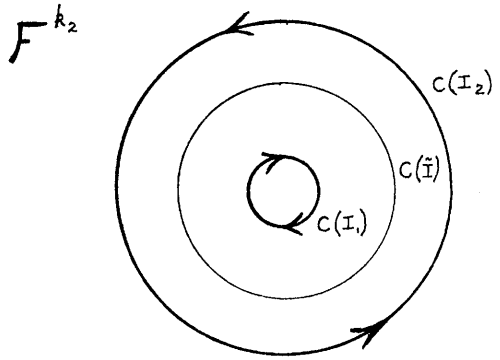
Now if $\alpha(\tilde{I}) \in \mathbb{Q}$, say $\alpha(\tilde{I}) = k_2/k_1$ for some $k_1, k_2 \in \mathbb{Z}$, then after the θ_1 coordinate on this torus makes k_2 complete cycles, the θ_2 coordinate makes k_1 complete cycles so that the trajectories on this torus are *periodic* (as we saw above). If we iterate the map F k_2-times ($F^{k_2} = F \circ \cdots \circ F$), then the circle $C(\tilde{I})$ in this plane that comes from the torus labeled by the coordinate $\tilde{I}$ *does not move under* F^{k_2}, i.e., $C(\tilde{I})$ is composed entirely of fixed points of F^{k_2}.

Of course, $\mathbb{T}^2(\tilde{I})$ is what we call a resonant torus. Observe that as we move from the torus labeled by the coordinate I_1 to that labeled by I_2, $\alpha(I)$ sweeps out an interval $[\alpha(I_1), \alpha(I_2)] \subset \mathbb{R}$. For every rational number in this interval there is a resonant torus, and since the rationals are dense in $\mathbb{R}$, *the resonant tori are dense in phase space.* This is a general feature of integrable, nondegenerate systems of any dimension:

[1] This property will be important in the construction of invariant tori for the perturbed system: if ε is sufficiently small then the perturbed Hamiltonian, H, is also nondegenerate. Then one can find initial conditions for the perturbed flow which has the same frequencies as the nonresonant unperturbed torus whose stability we are investigating.

If $\mathcal{T}(m) = \{$tori whose frequencies satisfy m-independent resonances$\}$, then $\mathcal{T}(m)$ is dense in phase space for each $m = 1, \ldots, n-1$. In particular, periodic orbits ($\mathcal{T}(n-1)$) are dense in phase space. But, in a similar way as for rational numbers, the $\mathcal{T}(m)$ each have zero measure. Thus, "most" invariant tori are nonresonant (the union of nonresonant tori has full measure), but arbitrarily nearby any nonresonant torus are tori with any prescribed number of resonances. ([2] pp402)

Continuing with the example above, let $\mathbb{T}^2(I_1)$ be a nonresonant torus lying inside of $\mathbb{T}^2(\tilde{I})$ and $\mathbb{T}^2(I_2)$ a nonresonant torus lying outside of $\mathbb{T}^2(\tilde{I})$. Denote by $C(I_1)$ the circle made by the intersection of $\mathbb{T}^2(I_1)$ with the plane, and similarily for $\mathbb{T}^2(I_2)$ and $\mathbb{T}^2(\tilde{I})$: $C(I_2)$ and $C(\tilde{I})$. Then F^{k_2} is a "twist map" that fixes $C(\tilde{I})$ and rotates $C(I_1)$ and $C(I_2)$ in opposite directions:



We jump ahead and use the KAM theorem to assure us that, if ε is sufficiently small and if $\omega(I_1)$ and $\omega(I_2)$ are "sufficiently nonresonant" (we can always find such nonresonant tori), $C(I_1)$ and $C(I_2)$ survive to become invariant circles C_1^ε and C_2^ε for $F_\varepsilon^{k_2}$, where F_ε is the return map for the perturbed flow.

Now, can we expect there to be a circle of fixed points for $F_\varepsilon^{k_2}$ near $C(\tilde{I})$? (i.e., that $\mathbb{T}^2(\tilde{I})$ survives the perturbation?) This kind of behavior, for a map (here, F^{k_2}) to have a whole curve of fixed points ($C(\tilde{I})$), is highly non-generic ("non-typical") in a precise sense. "Most" mappings of this form have only isolated fixed points, so that under a general perturbation ($F^{k_2} \rightarrow F_\varepsilon^{k_2}$), $F_\varepsilon^{k_2}$ will have only *finitely many fixed points near* $C(\tilde{I})$ ([2] pp404). This means that the resonant torus $\mathbb{T}^2(\tilde{I})$ is *destroyed by the perturbation.* But all is not lost. Poincaré proved in his "Last Geometric Theorem" ([2] app.9, [3] pp88) that the twist map $F_\varepsilon^{k_2}$ has $2kk_2$ fixed points near $C(\tilde{I})$ (here k is some integer greater than zero). These correspond to periodic orbits on $\mathbb{T}^2(\tilde{I})$ that *do* survive the perturbation - the so-called "stable periodic orbits" (which have special significance in the quantization of the classical system; see [17],[25] §15.6, [27]).

For more discussion about this description of the destruction of resonant tori and the resulting complicated dynamics, see [3] pp89-93, [4] pp186-190, [12], and [17].

KAM Theorem

The KAM theorem is an *existence* theorem, it says which tori survive under a perturbation. It says nothing about which tori are destroyed. Proofs of the KAM theorem proceed as follows (see for example, [6] §36, or [23] pp605-620). If ω satisfies the diophantine inequality $\mid \omega \cdot k \mid \geq \gamma(\varepsilon) / \mid k \mid^n$ $\forall k \in \mathbb{Z}^n$, where $\gamma(\varepsilon)$ is prescribed by the particular perturbation, one can construct (using an iterative, "linearized" scheme that rapidly converges to) a family of quasi-periodic solutions of the perturbed system that have the frequency ω, and that this family defines an invariant manifold that is diffeomorphic to $\mathbb{T}^n$. Furthermore, this invariant manifold is close to the unperturbed one that carries the frequency ω.

$\gamma(\varepsilon)$ is a calculable function that $\searrow 0$ as $\varepsilon \searrow 0$ ([23] pp605), so although we can't say for sure which tori will be destroyed by the perturbation, we may be able to say something about

the "robustness" of an (unperturbed) torus. First notice that if ω is resonant then it will not satisfy any diophantine inequality (i.e., for any γ). It is consistent, therefore, to say that *resonant tori are destroyed no matter how small the perturbation is.* Now, as the strength of the (particular) perturbation grows (i.e., εf as ε increases from zero), the function $\gamma(\varepsilon)$ increases which means the set $\Omega(\gamma(\varepsilon))$ of frequencies satisfying a diophantine inequality with parameters $\gamma(\varepsilon), n$ gets smaller (see pp12). So there are less and less candidate stable tori as the strength of the perturbation increases.

A nonresonant frequency ω is "near" to resonance when the minimum γ of the set of diophantine inequalities with parameters γ, n that it satisfies, is small. We conclude then that those nonresonant tori whose frequencies are close to resonance *may* become unstable first (as ε increases), and that those tori whose frequencies are further from resonance (i.e., the minimum γ is larger) are the most robust (they will survive a stronger perturbation). [1] See ref.'s [17] pp33-34, [25] §9.8, and [27] pp129-131.

For a fixed ε, $\Omega(\gamma(\varepsilon))$ is a closed set of large measure, by which we mean that the measure of $\Omega(\gamma(\varepsilon))^C$ is small (see pp12). However, $\Omega(\gamma(\varepsilon))^C$ is *open and dense.* Define the set $\mathbf{P}^n_\varepsilon \subset \mathbf{P}^n$ to be those $I's \in \mathbf{P}^n$ such that $\omega(I) \in \Omega(\gamma(\varepsilon))$. Then $\mathbf{P}^n_\varepsilon \times \mathbb{T}^n \subset \mathbf{P}^n \times \mathbb{T}^n$ are *stable* tori (the perturbed system has an invariant torus close to each torus in $\mathbf{P}^n_\varepsilon \times \mathbb{T}^n$) and form a closed set of large measure in phase space ($\mathbf{P}^n \times \mathbb{T}^n$). [2] Furthermore, $\mathbf{P}^n_\varepsilon \times \mathbb{T}^n$ has *no interior* (like a Cantor set) which means every torus in this set lies arbitrarily close to one outside of this set. The set of tori that contain the candidate unstable tori, $(\mathbf{P}^n_\varepsilon)^C \times \mathbb{T}^n$ (where $(\mathbf{P}^n_\varepsilon)^C$ contains those $I's \in \mathbf{P}^n$ such that $\omega(I) \in \Omega(\gamma(\varepsilon))^C$), is open and dense in phase space. What this all means is that you can never decide whether, for the perturbed system, a particular initial condition will give quasi-periodic (bounded) motion (i.e., that the trajectory that begins at this initial condition lies on a (perturbed) stable torus of the unperturbed system) or nonquasi-periodic (and perhaps unbounded) motion, because you will never be able to pin point its initial condition *exactly*, which is what is required since $\mathbf{P}^n_\varepsilon \times \mathbb{T}^n$ has no interior.

We say, in this situation, that the issue of stability is undecidable. So, although the KAM theorem does describe in detail the features of the perturbed system, in the end it does not (*can not*) answer the question which motivated its development: are motions of a nearly integrable system stable? An exception is the case where $n = 2$ (two degrees of freedom). Here the stable tori $\mathbf{P}^2_\varepsilon \times \mathbb{T}^2$ confine the trajectories that lie between them so that although the initial condition may not lie on an invariant torus, the trajectory will be trapped between two of them forever (and so will be bounded). For $n > 2$ this is not the case: the invariant tori of the perturbed system do not confine the space in between them so that a trajectory can wander out to infinity (if it wanted to); see [2] pp407, and [4] pp 189.

The behavior of the trajectories that wander through out phase space is the content of Nekhoroshev's theorem. Here there is no such sensitivity to initial conditions. Roughly speaking, Nekhoroshev's theorem states that no matter where the trajectory of a perturbed system starts it can not wander away very far over time scales that are exponentially long in the parameter ε ($T \sim e^{\frac{1}{\varepsilon}}$). So, although a torus of the unperturbed system may be destroyed by the perturbation, these trajectories still live near where the torus used to be for a very long time (this is what we call the "life time" of the (unstable) torus). Clearly, this is a more relevant result for physical systems.

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[1] In two dimensions the least resonant frequency is the one where $\omega_1/\omega_2 = \frac{\sqrt{5}-1}{2}$ = "the golden mean".

[2] For the perturbed dynamics we actually mean diffeomorphic images of $\mathbf{P}^n_\varepsilon \times \mathbb{T}^n$ and $(\mathbf{P}^n_\varepsilon)^C \times \mathbb{T}^n$ in $\mathbf{P}^n \times \mathbb{T}^n$ - see the footnote on pp11.

We will discuss this in more detail in the next section, but let's end this section here with a couple versions of the KAM theorem.

Theorem (KAM) ([2] pp405, [4] pp183) *If the unperturbed system is nondegenerate or isoenergetically nondegenerate, then for a sufficiently small Hamiltonian perturbation most nonresonant invariant tori do not vanish but are only slightly deformed, so that in the phase space of the perturbed system there are invariant tori densely filled with quasi-periodic phase curves winding around them, with a number of independent frequencies equal to the number of degrees of freedom. These invariant tori form a majority in the sense that the measure of the compliment of their union is small when the perturbation is small. In the case of isoenergetic nondegeneracy the invariant tori form a majority on each level manifold of the energy* •

Theorem (KAM) ([13] pp123) *If*

(i) $H(I,\theta)$ *is analytic in a complex neighborhood of* $\mathbf{P}^n \times \mathbb{T}^n$;
(ii) $H_o(I)$ *is strictly non-isochronous, i.e.,* $\det(\partial^2 H_o/\partial^2 I) \geq d > 0$ *for* $I \in \mathbf{P}^n$;
(iii) $\varepsilon \leq \varepsilon_o$ *where* ε_o *depends on the Hamiltonian,*

then there exists a canonical transformation $\mathcal{C}_\varepsilon$: $(I,\theta) \longrightarrow (I',\theta')$, *an integrable Hamiltonian* $h'_\varepsilon(I')$, *both* $\mathcal{C}_\varepsilon$ *and* h'_ε *being of class* C^∞ *in* $\mathbf{P}^n \times \mathbb{T}^n$, *and a set* $\mathbf{P}^n_\varepsilon \subset \mathbf{P}^n$ *of large Lebesgue measure, precisely* $\mathrm{meas}(\mathbf{P}^n/\mathbf{P}^n_\varepsilon) \longrightarrow 0$ *for* $\varepsilon \to 0$, *such that the new Hamiltonian* $H' = H \circ \mathcal{C}$ *satisfies the relation*

$$H'_\varepsilon(I',\theta') \overset{\mathbf{P}^n_\varepsilon}{=} h'_\varepsilon(I')$$

where $\overset{\mathbf{P}^n_\varepsilon}{=}$ *denotes equality restricted to* $\mathbf{P}^n_\varepsilon$ •

Remarks

The set $\mathbf{P}^n_\varepsilon$ in this theorem is the same set as I discussed above.

See ref. [4] pp185 for a discussion of what can be done in the case when H_o is degenerate (but the perturbation removes the degeneracy).

Appendix 8 of [2] discusses applications of the KAM theorem and its generalizations.

IV Perturbation Theory and Nekhoroshev's Theorem

Perturbation Theory (again)

Consider the nearly integrable Hamiltonian of the form

$$H(I,\theta) = H_o(I) + \varepsilon f(I,\theta) \tag{6}$$

with dynamics given by

$$\begin{aligned} \dot{I}(t) &= -\frac{\partial H}{\partial \theta} = -\varepsilon \frac{\partial f}{\partial \theta}, \\ \dot{\theta}(t) &= \frac{\partial H}{\partial I} = \frac{\partial H_o}{\partial I} + \varepsilon \frac{\partial f}{\partial I} = \omega_o(I) + \varepsilon \frac{\partial f}{\partial I} \equiv \omega(I). \end{aligned} \tag{7}$$

Here $I \in \mathbf{P}^n \subset \mathbb{R}^n, \theta \in \mathbb{T}^n$ are action-angle variables for the integrable system with Hamiltonian H_o.

For the (unperturbed) integrable system $H(I,\theta) = H_o(I)$, the evolution of the action variables are trivial: $\dot{I}(t) = 0$, i.e., $I(t) = I(0)$. For the perturbed system this may no longer be the case ($\dot{I}(t)$ may no longer be zero; see eq.(7)). We are interested in questions of stability: Do the trajectories (solutions) of the perturbed system (7) wander all over phase space or do they remain in a bounded region (like the trajectories of the unperturbed system)? To answer this question notice that it is enough to consider only the evolution of the action variables I because the angle variables θ are bounded (are cyclic). Our objective will be to estimate the velocity $\dot{I}(t)$ of the action variables under the perturbed dynamics (7) which will allow us to estimate how far $I(t)$ travels from the initial point $I(0)$ during a certain period of time.[1]

We see from eq.(7) that the fastest evolution $I(t)$ could have is $\mathcal{O}(\varepsilon)$. Evolution of this order will be called fast motion.

Our zeroth- order estimate for the evolution of the action variable of the system (7) is

$$\mid I_j(t) - I_j(0) \mid = \varepsilon \mid \int_0^t \frac{\partial f}{\partial \theta_j}(I(s),\theta(s))\, ds \mid \leq \varepsilon t \| \frac{\partial f}{\partial \theta_j} \|_\infty \tag{8}$$

so that $\mid I_j(t) - I_j(0) \mid \lesssim 1$ for $t \lesssim \varepsilon^{-1}$. Notice that, for large t, $\int_0^t \partial f/\partial \theta_j$ may be small so that (8) may be a poor estimate in the long run.

For a general dynamical system $\dot{x}(t) = F(x(t)) + \varepsilon f(x(t),t)$, a priori there is no reason to expect an estimate any better than $\mid x_\varepsilon(t) - x_o(t) \mid \lesssim \varepsilon$ for $t \lesssim 1$, where $x_o(t)$ is a solution for the unperturbed equation and $x_\varepsilon(t)$ is a solution of the perturbed equation with $x_o(0) = x_\varepsilon(0)$. For *Hamiltonian* dynamical systems however, as we will see, their remarkable geometry can allow us to make *much* better estimates.

Consider the evolution (7) in a neighborhood $U(I_o)$ of the torus labelled by the action coordinate I_o (i.e., the torus $\mathbb{T}^n(I_o)$). Following [13] and [14], our approach will be to look for a canonical transformation $\mathcal{C}_r$ defined on $U(I_o)$ which takes (I,θ) coordinates to (I',θ') coordinates, is ε-close to the identity transformation;[2]

$$\mathcal{C}_r : U(I_o) \longrightarrow U(I_o); \quad \mathcal{C}_r(I,\theta) = (I',\theta') = (I'(I,\theta),\theta'(I,\theta)),$$

[1] This is what we mean by "effective stability": $I(t)$ will remain close to $I(0)$ for $t \leq T$ where T is the "time scale of effective stability".

[2] We specify $\mathcal{C}_r$ to be ε-close to the identity transformation because the perturbed dynamics are ε-close to the unperturbed dynamics (whose trajectories lie on tori $\mathbb{T}^n(I)$); see eq. (7). Also, $\mathcal{C}_r$ being a *canonical* transformation means that the dynamics in the new coordinates are given by: $\dot{I}' = -\partial H/\partial \theta'$, $\dot{\theta}' = \partial H/\partial I'$, whence the estimates below follow.

$$| (I,\theta) - (I',\theta') | \lesssim \varepsilon,$$

and is such that our Hamiltonian H will be in normal form to order $r+1$ in the new coordinates;

$$H \circ \mathcal{C}_r(I,\theta) \;=\; H(I',\theta') \;=\; H'_o(I') + \varepsilon^{r+1} f'(I',\theta').\ ^{1}$$

(See also ref. [6] §30.) Then since $| I'(t) - I(t) | \lesssim \varepsilon$ and $| I'(t) - I'(0) | \leq t\varepsilon^{r+1}\|\partial f'/\partial\theta\|_\infty$, we have the (better) estimate

$$| I(t) - I(0) | \sim \varepsilon \;\; for \;\; t \lesssim \varepsilon^{-r}.$$

(These will be referred to as "polynomial estimates".)

Let $r = 1$. How do we find $\mathcal{C}_1$? We look for a generating function S_1 of the form

$$S_1(I',\theta) \;=\; I' \cdot \theta + \varepsilon W_1(I',\theta).$$

Then,

$$\begin{aligned} I &= \frac{\partial S_1}{\partial \theta}(I',\theta) \;=\; I' + \varepsilon \frac{\partial W_1}{\partial \theta}(I',\theta) \\ \theta' &= \frac{\partial S_1}{\partial I'}(I',\theta) \;=\; \theta + \varepsilon \frac{\partial W_1}{\partial I'}(I',\theta). \end{aligned} \tag{9}$$

Expand H_o and f in a Taylor series about the point (I',θ');

$$\begin{aligned} H_o(I) &= H_o(I(I',\theta')) = H_o(I') \;+\; \frac{\partial H_o}{\partial I}(I')[I - I'] \;+ \cdots \\ &= H_o(I') \;+\; \varepsilon\omega_o(I') \cdot \frac{\partial W_1}{\partial \theta}(I',\theta) \;+\; \mathcal{O}(\varepsilon^2) \\ &= H_o(I') \;+\; \varepsilon\omega_o(I') \cdot \frac{\partial W_1}{\partial \theta}(I',\theta') \;+\; \mathcal{O}(\varepsilon^2), \end{aligned}$$

$$\begin{aligned} f(I,\theta) &= f(I(I',\theta'),\theta(I',\theta')) \\ &= f(I',\theta') \;+\; \frac{\partial f}{\partial I}(I',\theta')[I - I'] \;+\; \frac{\partial f}{\partial \theta}(I',\theta')[\theta - \theta'] \;+\; \mathcal{O}(\varepsilon^2) \\ &= f(I',\theta') \;+\; \mathcal{O}(\varepsilon). \end{aligned}$$

Substitute this into (6)

$$H(I',\theta') \;=\; H_o(I') \;+\; \varepsilon[\omega_o(I') \cdot \frac{\partial W_1}{\partial \theta}(I',\theta') \;+\; f(I',\theta')] \;+\; \mathcal{O}(\varepsilon^2). \tag{10}$$

To put (6) into normal form to order 2, we require the terms in the square brackets in (10) to be independent of the angle variable θ', i.e., we need that

$$\omega_o(I') \cdot \frac{\partial W_1}{\partial \theta}(I',\theta') \;+\; f(I',\theta') \;=\; \Psi_1(I'). \tag{11}$$

We call (11) the fundamental equation of perturbation theory. It is a linearized Hamiltonian-Jacobi equation that is to be solved for the unknown functions W_1 and Ψ_1.

[1] Of course, we have $H(I',\theta') = H_o(I') + \varepsilon f(I',\theta')$ to begin with.

Notice that in equation (11) expressions like $\omega_o(I')$ and $f(I',\theta')$ mean the known functions ω_o and f evaluated at I',θ'. If we can solve (11) then we have a formula[1] for $S_1(I',\theta)$. If the Hessian $\partial^2 S_1/\partial I'\partial\theta$ is nondegenerate we can solve the equation $I = \frac{\partial S_1}{\partial\theta}(I',\theta)$ implicitly for I': $I' = I'(I,\theta)$. This, along with the formula $\theta' = \frac{\partial S_1}{\partial I'}(I',\theta)$, will define the transformation $\mathcal{C}_1$.

Since θ' are angular coordinates (${\theta'}_j + 2\pi = {\theta'}_j$, see (9)), and we're assuming all functions to be smooth, we can express f and W_1 by their Fourier series

$$f(I',\theta') \;=\; \sum_{k\in\mathbb{Z}^n} f_k(I')e^{ik\cdot\theta'}, \qquad W_1(I',\theta') \;=\; \sum_{k\in\mathbb{Z}^n} W_{1,k}(I')e^{ik\cdot\theta'}.$$

Now we solve (11) by equating Fourier coefficients,

$$\begin{array}{ll} k=0: & \Psi_1(I') = f_0(I') \\ k\neq 0: & (i\omega_o(I')\cdot k)W_{1,k}(I') + f_k(I') = \; 0. \end{array}$$

The second equation requires that

$$W_{1,k}(I') \;=\; i\frac{f_k(I')}{\omega_o(I')\cdot k}. \tag{12}$$

If $\omega_o(I')\cdot k = 0$ then we will need that $f_k(I') = 0$ too. For nondegenerate systems $\omega_o(I')\cdot k$ will vanish for some k on a dense set of $I's$.[2] This means that if we wish to solve (11), the Fourier coefficients, $f_k(I')$, of f will have to be zero for every $k \in \mathbb{Z}^n$ that characterizes a resonance of $\omega_o(I')$ in $U(I_o)$. I don't think this means that *all* the Fourier coefficients of f will have to vanish, but so many will that f would be of a very special form - too special, in fact. f would be "nongeneric" (non-typical; recall the discussion about structural stability on pp7). This is the content of Poincaré's nonexistence theorem ([14]). If we wanted to prove that H was integrable then we would at least have to carry-out the above procedure for each order r. But there are ways out: we will "work around" the resonances (see below). Also notice that even if $\omega_o(I')\cdot k$ never vanishes for any $k \in \mathbb{Z}^n$ (i.e., that $\omega_o(I')$ is nonresonant), it could become arbitrarily small so that the convergence of $\sum_k W_{1,k}$ is far from assured. In fact, *any* $\omega \in \mathbb{R}^n$ satisfies infinitely many, arbitrarily small, quasi-resonances (see pp11,12).

I'll end this discussion with an observation. If H *can* be put into a higher-order normal form (or if H *is* in fact integrable), then there *is* a canonical transformation from $(I,\theta) \longrightarrow (I',\theta')$ and this transformation will have a generating function S (at least locally; see [2] pp258, [4] pp22, [5] §45). Thus, if the above procedure to find the generating function for this canonical transformation fails then it *truely* fails: there *does not* exist such a transformation, and H *can not* be put into normal form.

Isochronous Systems

We begin our approach to this problem by first studying the simpler situation where $\omega_o(I) = \omega$ = *constant* through-out phase space,

$$H(I,\theta) \;=\; \omega\cdot I \;+\; \varepsilon f(I,\theta). \tag{13}$$

[1] Notice the difference between a *formula* and a *function*. In the Taylor expansions above we are treating H_o and f as formulas.

[2] That is, for a dense set of $I's \in U(I_o)$, the frequencies $\omega_o(I')$ will be resonant; see the bottom of pp14.

Here the underlying (degenerate) integrable system is the isochronous system $H_o(I) = \omega \cdot I$.

Let's consider the case of a (perturbation of an) isochronous system that arises near an elliptic equilibrium point of a Hamiltonian system ([1] pp489-503, [2] §23,app.7, [G] ch.6).

Suppose $(0,0)$ is an equilibrium point of the system ((3), pp8) i.e., that $dH(0,0) = (\frac{\partial H}{\partial p}(0,0), \frac{\partial H}{\partial q}(0,0)) = 0$, and that the "infinitesimally symplectic" $2n \times 2n$ matrix $\mathcal{H}$ $(= J\, d^2H)$,

$$\mathcal{H}_{i,j} \quad = \quad \begin{pmatrix} -\frac{\partial^2 H(0,0)}{\partial p_j \partial q_i} & -\frac{\partial^2 H(0,0)}{\partial q_j \partial q_i} \\ \frac{\partial^2 H(0,0)}{\partial p_j \partial p_i} & \frac{\partial^2 H(0,0)}{\partial q_j \partial p_i} \end{pmatrix} \qquad i,j = 1,\ldots,n; \quad i = row, j = col$$

has purely imaginary eigenvalues $\{i\omega_1, \ldots, i\omega_n, -i\omega_1, \ldots, -i\omega_n\}$ where $\{\omega_1, \ldots, \omega_n\}$ are distinct.[1] Then, after going to "normal mode" coordinates: $(p,q) \longrightarrow (P,Q)$ (this can be done by a linear canonical transformation[2]), and rescaling the P, Q ($P_j \to \sqrt{\omega_j} P_j$, $Q_j \to Q_j/\sqrt{\omega_j}$), we can, via Taylor expansion, write the Hamiltonian as

$$H(P,Q) \;=\; \frac{1}{2}\sum_{j=1}^{n} \omega_j (Q_j^2 + P_j^2) \;+\; \sum_{s=1}^{\infty} H_s(P,Q),$$

where the functions H_s are homogeneous polynomials of degree $s+2$ in P, Q. Introduce the rescaled variables $P'_j = \varepsilon^{-1} P_j$, $Q'_j = \varepsilon^{-1} Q_j$ ("blowing-up") and write

$$H'(P',Q') \;=\; \varepsilon^{-2} H(\varepsilon P', \varepsilon Q') \;=\; \frac{1}{2}\sum_{j=1}^{n} \omega_j ({Q'}_j^2 + {P'}_j^2) \;+\; \sum_{s=1}^{\infty} \varepsilon^s H'_s(P',Q') \tag{14}$$

(so we've made explicit the "natural perturbation parameter", namely, the distance from the origin). Define canonical coordinates (I,θ) [3] via the formulae

$$P'_j \;=\; \sqrt{I_j}\cos\theta_j, \quad Q'_j \;=\; \sqrt{I_j}\sin\theta_j, \qquad j = 1,\ldots,n.$$

Then (14) becomes

$$H'(I,\theta) \;=\; \omega \cdot I \;+\; \sum_{s=1}^{\infty} \varepsilon^s H'_s(I,\theta), \tag{15}$$

where now the functions H'_s have only finitely many Fourier components (in θ) since they are polynomials in $\cos\theta$ and $\sin\theta$.

To put (15) into normal form to order ε^2 we need to solve (via the fundamental equation (11))

$$\omega \cdot \frac{\partial W_1}{\partial \theta}(I',\theta') + \; H'_1(I',\theta') \;=\; \Psi_1(I').$$

This has a (formal) solution given by

$$\Psi_1(I') \;=\; H'_{1,0}(I'),$$

---

[1] Thus, the "linearization" of eq. (3) about the point $(0,0)$ is : $\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \mathcal{H} \begin{pmatrix} p \\ q \end{pmatrix}$.

[2] Normal mode coordinates diagonalize $\mathcal{H}$ so that the linearized equations are uncoupled harmonic oscillators: $H = \frac{1}{2}\sum_j (Q_j^2 + \omega_j^2 P_j^2)$.

[3] (I,θ) are action-angle variables for the harmonic oscillator $H = \frac{1}{2}\sum_j \omega_j ({Q'}_j^2 + {P'}_j^2)$.

$$W_1(I',\theta') \;=\; \sum_{k\neq 0} i\frac{H'_{1,k}(I')}{\omega\cdot k}e^{ik\cdot\theta'}. \tag{16}$$

(Here, $H'_{1,k}(I')$ is the k^{th}-Fourier coefficient of $H'_1(I',\theta')$.)

Notice that the sum for W_1 in (16) is *finite*. If ω is *nonresonant*, then, since this sum converges, it defines a bona-fide canonical transformation. Our hamiltonian is in normal form to order ε^2 in these coordinates,

$$H'(I',\theta') = \omega\cdot I' + \varepsilon H'_{1,0}(I') + \varepsilon^2 f'(I',\theta').$$

(Now the integrable part is $H'_o(I') = \omega\cdot I' + \varepsilon H'_{1,0}(I')$, which is isochronous to order ε.)

Resonant vectors are dense in $\mathbb{R}^n$, so what to do in that case? This does not present a problem and in fact it helps us understand just how resonances are responsible for instabilities.

So suppose we have the Hamiltonian given by (15) where ω is a resonant frequency and let $\Lambda_\omega \subset \mathbb{Z}^n$ be the resonant module (or lattice) of ω ($\Lambda_\omega = \{k\in\mathbb{Z}^n \;|\; \omega\cdot k = 0\}$; see pp13). Λ_ω is a submodule ("subspace") of $\mathbb{Z}^n$ of dimension $l < n$ where l is the number of independent resonances satisfied by ω. In this case we can not get rid of all the angle variables [1] so our fundamental equation is

$$\omega\cdot\frac{\partial W_1}{\partial\theta}(I',\theta') \;+\; H'_1(I',\theta') \;=\; \Psi_1(I',\theta'),$$

which is solved by

$$\Psi_1(I',\theta') \;=\; \sum_{k\in\Lambda_\omega} H'_{1,k}(I')e^{ik\cdot\theta'}$$

$$W_1(I',\theta') \;=\; \sum_{k\notin\Lambda_\omega} i\frac{H'_{1,k}(I')}{\omega\cdot k}e^{ik\cdot\theta'}. \tag{17}$$

Our normal form to order ε^2 modulo Λ_ω is then

$$H'(I',\theta') \;=\; \omega\cdot I' \;+\; \varepsilon\sum_{k\in\Lambda_\omega} H'_{1,k}(I')e^{ik\cdot\theta'} \;+\; \varepsilon^2 f'(I',\theta',)$$

with the evolution of I' given by

$$\dot I'(t) \;=\; -\frac{\partial H'}{\partial\theta'} \;=\; -i\varepsilon\sum_{k\in\Lambda_\omega} kH'_{1,k}(I')e^{ik\cdot\theta'} + \mathcal{O}(\varepsilon^2). \tag{18}$$

Thus, $I'(t)$ can only have fast motion in the direction of vectors $k\in\Lambda_\omega$. [2] By iterating this procedure we will see that the remainder term causes a drift in $I'(t)$ that is exponentially small in ε so **motion of $I'(t)$ is essentially determined by the resonances**.

To apply this procedure again we look for a generating function of the form

$$S_2(I'',\theta') = \omega\cdot I'' + \varepsilon^2 W_2(I'',\theta').$$

[1] That is, we can not solve equation (11) when there are non-zero $k's$ such that $\omega\cdot k = 0$ and f is an arbitrary (smooth) function.

[2] Formula (18) also shows us how to construct "approximate Integrals": if $\alpha\in\mathbb{R}^n$ is such that α is orthogonal to Λ_ω, then the function $g(I') = \alpha\cdot I'$ satisfies $\dot g(I'(t)) = \{g, H'\} = \mathcal{O}(\varepsilon^2)$.

Going through the same calculations as before, we will be faced with a fundamental equation of the form

$$\omega \cdot \frac{\partial W_2}{\partial \theta'}(I'', \theta'') + f'(I'', \theta'') = \Psi_2(I'')$$

or,

$$\omega \cdot \frac{\partial W_2}{\partial \theta'}(I'', \theta'') + f'(I'', \theta'') = \Psi_2(I'', \theta'')$$

in the nonresonant or resonant cases respectively. Here the function $f'(I'', \theta'')$ is a polynomial in the H_s and their first-order derivatives and so has only finitely-many Fourier components. We can solve for W_2 and Ψ_2 by equating Fourier coefficients just as before. Ultimately, this defines a canonical transformation $\mathcal{C}_2$: $(I', \theta') \longrightarrow (I'', \theta'') \ = \ \mathcal{C}_2(I', \theta')$. Then, $H \circ (\mathcal{C}_2 \circ \mathcal{C}_1)$ [1] is in normal form to order ε^3;

$$H \circ (\mathcal{C}_2 \circ \mathcal{C}_1) = \omega \cdot I'' + h_2(I'') + \varepsilon^3 f''(I'', \theta'')$$

or,

$$H \circ (\mathcal{C}_2 \circ \mathcal{C}_1) = \omega \cdot I'' + h_2^{\Lambda_\omega}(I'', \theta'') + \varepsilon^3 f''(I'', \theta''),$$

where $h_2^{\Lambda_\omega}$ has non-zero Fourier components only for those $k_s' \in \Lambda_\omega$.

If we iterate this procedure infinitely-many times we get a *formal* power series (in the I_j) for the canonical transformation $\mathcal{C}_\infty \equiv \lim\ r \to \infty\ (\mathcal{C}_r \circ \cdots \circ \mathcal{C}_1)$ and $h_\infty, h_\infty^{\Lambda_\omega}$ such that the Hamiltonian is in integrable, or integrable modulo Λ_ω, form;

$$H \circ \mathcal{C}^\infty \ = \ \omega \cdot I^\infty \ + \ h_\infty(I^\infty),$$

$$H \circ \mathcal{C}^\infty \ = \ \omega \cdot I^\infty \ + \ h_\infty^{\Lambda_\omega}(I^\infty, \theta^\infty).$$

These formal series, however, usually diverge.[2]

Now suppose f is just an arbitrary (smooth) function so that now the series in (16) or (17) for W_1 (depending on whether ω is nonresonant or resonant) can have infinitely many terms. [3] Then we still have to deal with the small denominators $\omega \cdot k$. Carl Siegel observed that if the numerators, call them $f_k(I')$ for simplicity, vanished sufficiently rapidly (as $k \longrightarrow \infty$) and if we had control on the size of the small denominators, then there might be a chance that the series for W converged. Vanishing of the Fourier coefficients $f_k(I')$ is related to the smoothness of $f(I', \theta')$ in θ' (this follows from properties of the Fourier transform), while control over the small denominators is precisely the diophantine inequalities for ω (pp12).

So assume ω is nonresonant and satisfies a diophantine inequality with parameters γ, n; or, if ω is resonant, that ω satisfies such an inequality for those $k \notin \Lambda_\omega$, [4] and the perturbation $f(I, \theta)$ is analytic in a complex strip $\mid Im\theta \mid < \rho$. Then the $f_k(I')$ will decay exponentially fast so the terms $\frac{f_k(I')}{\omega \cdot k}$ will also decay (exponentially) fast [5] making the series convergent.

Thus, *for Hamiltonians of the form (13), if ω satisfies a diophantine inequality outside of its resonant module and if f is smooth enough, then H can be put into (Birkhoff) normal form modulo Λ_ω to any order.*

---

[1] Notice that $\mathcal{C}_2 \circ \mathcal{C}_1$ is ε- close to the identity.

[2] The above construction of the $\mathcal{C}_r$ is Birkoff's Theorem and the normal forms are called Birkhoff normal forms; see[4] pp111-114; 255-258, [12] pp92-93, [14] pp8, and [28] pp30-34.

[3] Each term is finite, though.

[4] See [14] pp11. We can just say the $\Lambda_\omega = \emptyset$ if ω is nonresonant.

[5] The denominator $\omega \cdot k$ can vanish not faster than $\sim k^{-n}$ while the numerator vanishes like $\sim e^{-|k|\rho}$. This shows, too, that we can even do with polynomial decay of the f_k as long as it beats the decay of the denominator: we need that f be "sufficiently smooth".

There are still some frequency vectors ω that we haven't yet dealt with. These are the ω'^s that do not satisfy any diophantine inequality, or at least not a *prescribed* diophantine inequality, outside of its resonant module. In this case we define a "quasi-resonant module" $\Lambda_\omega^{q-r} \supset \Lambda_\omega$ to be the module generated by those $k'^s \in \mathbb{Z}^n$ for which $\mid \omega \cdot k \mid < \gamma / \mid k \mid^n$ for some prescribed γ.[1] In practice, one is only concerned about k'^s such that $\mid k \mid < M$ (see the discussion of Nekhoroshev's theorem below) in which case *every* $\omega \in \mathbb{R}^n$ can be made to satisfy some diophantine inequality outside of some Λ_ω^{q-r} so we can carry out the normal form reduction modulo Λ_ω^{q-r} (see [GDFGS] pp184).

Another use of these quasi-resonant modules is that they allow us to obtain more precise estimates. For example, suppose that outside the resonant module Λ_ω there are some $k'^s, \mid k \mid < M$, that produce a quasi-resonance (so that $\omega \cdot k$ is a small denominator). Then the main contribution to the error (or remainder) term in the normal form modulo Λ_ω will come from these k'^s. If we include them in Λ_ω and look at the resulting normal form, the error term becomes smaller and we have a more accurate idea of what are the fast motions (cf. eq.(18)).

We consider now how to estimate remainder term.

For simplicity consider the case where ω is nonresonant. If H is in normal form to order $r+1$

$$H(I', \theta') \;=\; \omega \cdot I' \;+\; h(I') \;+\; \varepsilon^{r+1} f_r(I', \theta'),$$

then we immediately get the estimate $\mid \dot{I'} \mid \leq C_r \varepsilon^{r+1}$. We can put H into normal form to any order we want so (and this is the basic idea) let's choose $r = r(\varepsilon)$. Now it is crucial to control the r-dependence of C_r. For instance, if we can show that the $C_r'^s$ satisfy an estimate like $C_r \leq r^{rm}$, $m > 0$, then $\mid \dot{I'} \mid \leq r^{rm} \varepsilon^{r+1}$. Now take $r(\varepsilon) = (\frac{1}{\varepsilon e})^{1/m}$. Then $\mid \dot{I'} \mid \leq \varepsilon e^{(-\alpha/\varepsilon^{1/m})}$ where $\alpha = 1/e$. Remember that our canonical transformation was ε-close to the identity, so that in the original coordinates we have the estimate

$$\mid I(t) - I(0) \mid \lesssim \varepsilon \quad for \quad t \lesssim e^{\alpha/\varepsilon^{1/m}}.$$

These are the celebrated exponential estimates of Nekhoroshev. To be assured of the estimates on C_r as above, conditions like analyticity of H in a complex neighborhood of $\mathbf{P}^n \times \mathbb{T}^n$ are sufficient (see [13] pp129).

Nekhoroshev's Theorem

Let's gather our accomplishments to this point. We've seen that there are ways to handle resonances and small denominators ("quasi-resonances") in perturbation theory for isochronous systems via diophantine inequalities and the use of normal forms modulo a resonance lattice. To treat the general system $H(I, \theta) = H_o(I) + \varepsilon f(I, \theta)$ we will use these techniques according to what properties the frequency vector $\omega_o(I)$ has. We will divide the action space up into regions according to the resonant or quasi-resonant properties of $\omega_o(I)$ in these regions, apply the normal form reduction appropriate to each region, and then piece them all together to get an estimate of the evolution of $I(t)$ for *all* of phase space. The "geometry of resonances" in action space is the heart (but not soul!) of Nekhoroshev's theorem (Nekhoroshev's paper [29] is good reading about this).

[1] This is a kind of "micro-local analysis" of the function $\omega_o(I)$, where the Λ_ω^{q-r} is something like the "wave front set".

So now let us consider the general nearly integrable system

$$H(I,\theta) \;=\; H_o(I) \;+\; \varepsilon f(I,\theta). \tag{19}$$

We want to obtain estimates on the evolution of $I(t)$. Before, in the isochronous case, the frequency vector $\omega_o(I) = \frac{\partial H_o}{\partial I}$ was constant on all of phase space and we were able to find a canonical transformation that put (19) into normal form modulo some lattice Λ_ω. Now the frequency vector is *not* constant. At any *particular point* I_o in action space we can put (19) into normal form according to the properties of $\omega_o(I_o)$, but apriori this normal form is not valid in any *neighborhood* of I_o, which is what we want, because $I(t)$ is moving. [1] For nondegenerate systems the properties of $\omega_o(I)$ are always changing as I changes (the frequencies can be used as coordinates in action space, remember). [2] However, the *quickly* varying (resonant) properties of $\omega_o(I)$ (i.e., those coming from infinitesimal or nearly infinitesimal changes in I) are due to the properties of $\omega_o(I)\cdot k$ for those k'^s in a neighborhood of infinity in $\mathbb{Z}^n$! We saw this already on pp12: a nonresonant $\omega \in \mathbb{R}^n$ satisfies arbitrarily small quasi-resonances but the smallest ones occur for those k'^s that are very large. In that case, if you restrict yourself to consider only those k'^s that lie inside some ball in $\mathbb{Z}^n$ centred at the origin, then an ω that satisfies some diophantine inequality will have only *finitely* many quasi-resonances and these quasi-resonances are bounded from below (do not become arbitrarily small).

I claim that, by choosing an appropriate $M = M(I_o) > 0$, I can be assured that there is a neighborhood $\Delta(I_o, M) \subset \mathbf{P}^n$ of I_o such that $\omega_o(I)$ shares the same properties (as far as normal forms are concerned) as $\omega_o(I_o)$ in $\Delta(I_o, M)$ when restricted to those k'^s with $|k| < M$. Then the normal form modulo $\Lambda_{\omega_o(I_o)}$ is valid in the region $\Delta(I_o, M)$.

We can see this more clearly in the case of $\omega \in \mathbb{R}^2$ (cf. the diagram on pp12). Suppose the line through ω intersects a lattice point inside of the ball $B(M)$ of radius M centred at the origin. Then you can see that we can change ω slightly (i.e., rotate it slightly back and forth) and not intersect any other lattice point inside of $B(M)$. [3] The amount we can change ω will depend on the size of the ball. Thus, by restricting to those k such that $|k| \leq M$, $\omega_o(I)$ will have no other resonances in a neighborhood of I_o.

In the nonresonant case, where the line through ω does not intersect a lattice point inside of $B(M)$, we can change ω slightly and still not intersect a lattice point inside of $B(M)$. Suppose, further, that this ω satisfies a diophantine inequality with parameters γ, n inside of $B(M)$. Then if we change ω slightly it *still* satisfies a diophantine inequality inside of $B(M)$ but with, perhaps, a different γ, say, $\gamma/2$. [4] Here then, we see that by restricting to those $|k| \leq M$, $\omega_o(I)$ will, in a neighborhood of I_o, satisfy a similiar diophantine inequality as $\omega_o(I_o)$ does.

For the other ω, the "algebraic numbers" that don't satisfy any diophantine inequality on *all* of $\mathbb{Z}^n$, they *will* satisfy a diophantine inequality on any ball $B(M)$ in $\mathbb{Z}^n$.

"Shifting ω slightly" in the description just given translates into; "moving from I_o to a nearby I and seeing how $\omega_o(I)$ changes from $\omega_o(I_o)$". Remember that in nondegenerate systems the frequencies can be used as coordinates in action space (pp14) so that if we know the neighborhood of I_o in action space then we know the neighborhood of $\omega_o(I_o)$ in frequency space, i.e., we know exactly how the frequencies change with I.

The bottom line of all this is that, by restricting to those $|k| < M$ for appropriate M, the resonant and quasi-resonant properties of $\omega_o(I_o)$ will be shared by those $\omega_o(I)$ in a certain neigh-

[1] We want to put (19) into normal form in some neighborhood of I_o that contains $I(t)$ for some length of time.

[2] To show you how delicate this issue is, recall that those I'^s in which $\omega_o(I)$ is resonant is dense in $\mathbf{P}^n$.

[3] Of course, the line will intersect *many* lattice points *outside* of $B(M)$.

[4] This can also be seen analytically by looking at the formula $|\omega\cdot k| \leq \frac{\gamma}{|k|^n}$, $|k| < M$ and then changing ω slightly.

borhood of I_o. Thus, our frequency is "constant" in these regions, at least as far as the appearance of the normal form is concerned.[5]

Notice that if the Fourier coefficients of order greater than M of the perturbation f vanished, then we would only be considering those $k's$ with $\mid k \mid < M$ anyways[1] and the above normal form would be perfectly legitimate in $\Delta(I_o, M)$. So let's just truncate the Fourier series for f at order M (an "ultraviolet cut-off" [2]), call this function f^M, and calculate the normal form with f^M as the perturbation instead of f. The price we pay by doing this is that the remainder term in the normal form using f^M is no longer accurate for the original f: there are now contributions to the remainder term that come from the neglected f'^s_k. But one can show that, for an appropriately chosen M, this contribution is not any bigger than what the error term associated to an f with finitely many Fourier coefficients is already. [3] Thus, our normal forms valid inside of $\Delta(I_o, M)$ are: $H(I', \theta') = H_o(I') + \varepsilon f_o(I') + \varepsilon^2 f'(I', \theta')$ and $H(I', \theta') = H_o(I') + \varepsilon \sum_{k \in \Lambda_\omega} i f^M_k(I') e^{ik\cdot\theta'} + \varepsilon^2 f'(I', \theta')$ in the nonresonant and resonant case respectively.

Introducing an ultraviolet cut-off also results in *resonance-free* regions in $\mathbf{P}^n$. [4]

Now we have control on $\dot{I}(t)$ inside of $\Delta(I_o, M)$. By iterating the normal form algorithm, and choosing M appropriately, we can show that the remainder term is exponentially small in ε. Thus, $I(t)$ would essentially stay inside of $\Delta(I_o, M)$ if there were no fast motions (if there were no resonances of order less than M there). If there are fast motions then we know in what directions they are (cf. eq.(18)) so we know where $I(t)$ will exit the region $\Delta(I_o, M)$ and we can also estimate how long it will take to do so. How long depends not only on how fast the fast motions are, but also on the size of $\Delta(I_o, M)$ which may be limited by the size of the domain of definition of the canonical transformation $\mathcal{C}^r$ that puts the Hamiltonian into normal form. Typically, as the order of the normal form increases the size of the domain of the canonical transformation shrinks so we must make a trade-off between making the remainder term very small (larger r) and the domain $\Delta(I_o, M)$ large (smaller r). Part of the analytical work in the proof of Nekhoroshev's theorem is finding this optimal r (see [GDFGS] pp186-188, [13] pp134, for example).

If $I(t)$ is in a nonresonant region its velocity is exponentially small so that it stays in this region for a long time. The regions $\Delta(I_o, M)$ are chosen so that if $I(t)$ is initially in a resonant region, it will leave that region and eventually enter a nonresonant region[5] (i.e., a region where there are no resonances of order less than or equal to some M) where it will essentially stop. To obtain global estimates on the evolution of $I(t)$ (i.e., for exponentially long times), one has to keep track of the sizes of the resonant regions Δ. Thus, the statement of Nekhoroshev's theorem contains two numbers that characterize the distance $I(t)$ has travelled from $I(0)$ and the time taken to travel there.

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[5] In equations (16) and (17) we will have $\omega_o(I)$ instead of ω, and the resulting series for W_1 will be defined on $\Delta(I_o, M)$.

[1] See eq.(12)

[2] The idea of introducing an ultraviolet cut-off is standard technique in KAM theory; see [12] pp105, [15] pp5, [23] pp586-587, [29] pp21-22.

[3] If $r = 1$ then we can show that the higher-order harmonics of f contribute only to $\mathcal{O}(\varepsilon^2)$ in the normal form, which is the size of the remainder (see the references given in footnote (2)).

[4] There are only finitely many $k \in \mathbb{Z}^n$ with $|k| < M$.

[5] This is where the "convexity" property of H_o is used in Nekhoroshev's theorem.

Here now are two versions of Nekhoroshev's theorem:

Theorem (Nekhoroshev) ([29] pp4) *Consider the nearly integrable Hamiltonian*

$$H(I,\theta) \;=\; H_o(I) \;+\; \varepsilon f(I,\theta) \tag{20}$$

where $\varepsilon << 1$. *Suppose that* H_o *satisfies certain "steepness conditions". Then there are positive constants* a, b *and* ε_o *with the following property. Let* $0 < \varepsilon < \varepsilon_o$. *Then for every solution* $I(t), \theta(t)$ *of the system (20)* (i.e., eq. (7)),

$$\mid I(t) - I(0) \mid \;<\; \varepsilon^b$$

for all $t \in [0, T]$, *where*

$$T \;=\; \frac{1}{\varepsilon} \exp(\frac{1}{\varepsilon^a}).$$

The constants a, b *depend on* H_o; ε_o *depends only on* H_o *and on the two parameters in* f *that estimate, respectively, the largest quantity and the rate of decrease of the coefficients in the Fourier series for* f *in terms of* θ. *This estimate is proven under the assumption that* H *is analytic. If we only require* H *to be smooth, then the estimate is not exponential, but a polynomial estimate whose exponent is larger the more derivatives* H *has.* •

Remarks

a and $b \;\searrow 0$ as n (the number of degrees of freedom) increases to infinity.

"Steepness" is a kind of convexity and is shown by Nekhoroshev to be generic.

Theorem (Nekhoroshev/Pöschel) ([31] pp190) *In the same notation as above, suppose* H_o *is* l, m *quasi-convex and* $|||f||| \leq \varepsilon_o$. *Then for every orbit with initial condition* (I_o, θ_o) *in* $\mathbf{P}^n \times \mathbb{T}^n$ *one has*

$$\mid I(t) - I(0) \mid \leq \; R_o(\frac{\varepsilon}{\varepsilon_o})^a \qquad for \qquad \mid t \mid \leq T_o \exp(s_o(\frac{\varepsilon_o}{\varepsilon})^a)$$

except when $\|\omega(I_o)\| \leq m r_o$ *in which case* $|I(t) - I(0)| \leq r_o$ *for all* t. *Here,* $a = 1/2n$ •

Remarks

T_o depends on ε in some implicit way.

H is assumed to be real analytic on a complex neighborhood of $\mathbf{P}^n \times \mathbb{T}^n$. $|||f|||$ is an exponentially-weighted norm on the Fourier coefficients of f.

H_o is quasi-convex if at every point $I \in \mathbf{P}^n$ at least one of the inequalities

$$\mid \omega(I) \cdot \xi \mid \;>\; l\|\xi\| \quad , \quad \left\langle (\partial^2 H_o / \partial^2 I)\xi, \xi \right\rangle \;\geq\; m\|\xi\|^2$$

holds for each $\xi \in \mathbb{R}^n$ and for some l, m ("l, m quasi-convex").

Notice the case when $\omega(I_o)$ is sufficiently small; you have perpetual stability (i.e., this orbit remains close to the unperturbed one for all time).

Pöschel discusses how the different resonant properties in different regions of phase space produce different "life-times" and that, generally, regions of stronger resonances (i.e., $|k|$ is smaller) exhibit longer life-times for the destroyed orbits (this is related to the remark made just above).

Geometry of Resonances

To give you some idea of how to cover the action space by the domains Δ I mentioned above, and a flavour of the proof of Nekhoroshev's theorem, I will give some definitions. Details can be found in the references (particularily [12],[15],[23],[29],[31]). Note that ref.([31]) gives a different construction than the others and that given here.

As before, Λ_ω will denote the resonant lattice of ω. For nondegenerate systems (of which the Hamiltonians in the above theorems are) we know that we can use the frequencies to lable the tori, that is, we can use frequencies as coordinates on $\mathbf{P}^n$ via the formula

$$\mathbf{P}^n \ni I \longrightarrow \omega_o(I) \in \mathcal{F}_\omega \subset \mathbb{R}^n .$$

This mapping has an inverse $\hat{}: \mathcal{F}_\omega \to \mathbf{P}^n$, $\hat{\omega} = I_\omega$, where $\omega_o(I_\omega) = \omega$. $\hat{}$ is called the frequency map. It is easier to look at the geometry of resonances in frequency space $\mathcal{F}_\omega$ and then use the frequency map to put that structure on $\mathbf{P}^n$ (and hence phase space).

Take $k \in \mathbb{Z}^n$ and define the set $\Omega(k) = \{\omega \in \mathbb{R}^n \mid \omega \cdot k = 0\}$. $\Omega(k)$ is just the plane $k\perp$ in $\mathbb{R}^n$ that is perpindicular (in $\mathbb{R}^n$) to k. In general, $\Omega(k_1, \ldots, k_s)= \{\omega \in \mathbb{R}^n \mid \omega \cdot k_j = 0 \;\; j = 1, \ldots, s < n\}$ is $\Pi(k_1, \ldots, k_s)\perp$ where $\Pi(k_1, \ldots, k_s)$ = span of $\{k_1, \ldots, k_s\}$ as vectors in $\mathbb{R}^n$. So, the $\Omega(k_1, \ldots, k_s)$ are planes of various codimensions in $\mathbb{R}^n$. We can also define, for any sublattice Λ of $\mathbb{Z}^n$, the set $\Omega(\Lambda)=\{\omega \in \mathbb{R}^n \mid \omega \cdot k = 0$ for all $k \in \Lambda\}$. This will be a collection of planes in $\mathbb{R}^n$. As Λ varies over all possible sublattices Λ of $\mathbb{Z}^n$, the $\Omega(\Lambda)$ will cover all the resonant frequencies in $\mathbb{R}^n$. Notice that since a resonant frequency ω can satisfy many different resonances, ω may lie in more that one $\Omega(\Lambda)$.[1] Since resonant frequencies are dense in $\mathbb{R}^n$, the collection $\Omega = \bigcup_{\Lambda \in \mathbb{Z}^n} \Omega(\Lambda)$ will be a dense set of planes in $\mathbb{R}^n$. [2]

Also used in the proofs are the resonant zones $\Omega^\delta(\Lambda) =\{\omega \in \mathbb{R}^n \mid \omega \cdot k_j \leq \delta$ for $j = 1, \ldots, s\}$ where $\{k_1, \ldots, k_s\}$ is a basis for Λ. What these $\Omega^\delta(\Lambda)$ are, I think, are "pancakes" lying along the plane $\Omega(\Lambda)$ which are thicker near the origin and thinner further away from the origin, approaching $\Omega(\Lambda)$ as $|\omega| \to \infty$. It is in these resonant zones where the fast motions occur.

Resonant surfaces $\mathcal{R}(\Lambda)$ in action space are defined by $\mathcal{R}(\Lambda) =\{I \in \mathbf{P}^n \mid \omega(I) \in \Omega(\Lambda)\}$. Then $\mathcal{R}(\Lambda)$ is just $\hat{\Omega}(\Lambda)$, and since $\hat{}$ is a diffeomorphism, $\mathcal{R}(\Lambda)$ is a collection of curves and surfaces in $\mathbf{P}^n$. The resonant block associated to the resonant surface $\mathcal{R}(\Lambda)$ is the difference between the resonance zone $\mathcal{R}^\delta(\Lambda)$ of $\mathcal{R}(\Lambda)$ and the union of the resonance zones of all resonant surfaces not containing $\mathcal{R}(\Lambda)$.[3] The multiplicity of a zone or block is the dimension of the associated lattice Λ.

Now we can describe the evolution of $I(t)$. In the words of Nekhoroshev ([29] pp25): "We can choose the resonance width of the zones used to construct the set of blocks so that if a point $I(t)$ leaves each zone close to the corresponding block, then it passes into a block of lower multiplicity than that of this zone. For this reason, as $I(t)$ moves from its initial position $I(0)$, it passes from one block of lower multiplicity until it reaches a nonresonance block. The speed in this block is (essentially) zero, and so it is impossible for $I(t)$ to move further from $I(0)$."

[1] It will lie in the intersection of them, which is also a plane.
[2] For $\mathbb{R}^2$, Ω is the collection of straight lines throught the origin with rational slope.
[3] Here the zone "width" δ is fixed.

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