A Characterization of Bound States of Nonlinear Wave and Schrödinger Equations [1]

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Abstract: Bound states are functions on space-time that are localized in space uniformly in time and play an important role in the scattering theory of dispersive wave equations. We prove for a class of nonlinear wave and Schrödinger equations that bound state solutions are almost periodic in time. As an application we establish necessary conditions for the existence of bound states for these equations.

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1 Introduction

In this article we study solutions of nonlinear wave (NLW) and nonlinear Schrödinger (NLS) equations on space-time $\mathbb{R}^{n+1}$;

$$\partial_t^2 u - \Delta u + f(u) = 0 \qquad (NLW)$$

$$-i\partial_t u - \Delta u + f(u) = 0 \qquad (NLS)$$

Here $u : \mathbb{R}^n_x \times \mathbb{R}_t \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ is a nonlinear function. In the absence of the nonlinearity f, these equations are purely dispersive and all finite energy solutions spread out and decay in amplitude. More precisely, they decay in a local (spatial) norm to the zero solution. The nonlinearity, however, can act against dispersion with the result that some solutions may remain localized in space, or even blow-up (i.e., u or one of its derivatives becomes infinite in finite time). A general overview of the theory of these equations can be found in [S].

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The most natural, and simplest, space-time classification for solutions of these equations are bound states and scattering states. Bound states are solutions that are localized in space uniformly in time, while scattering states are solutions that decay locally in space with respect to time. Bound states represent particle-like solutions (such as solitons and solitary waves) while scattering states represent dispersive waves (radiation travelling to infinity). Not every solution is a bound state or a scattering state, but these types of solutions can be used to describe the large-time behavior of a general solution (scattering theory).

Linear wave or Schrödinger equations of the form $\partial_t^2 u + Hu = 0$ or $-i\partial_t u + Hu = 0$ where H is a self-adjoint operator, will possess bound states if H has non-empty discrete spectrum. Under certain assumptions on H every solution can be expressed as a superposition of bound states and scattering states (see for example [RS-III] or [Pe]). For integrable nonlinear equations (e.g., the sine-Gordon NLW; $\partial_t^2 u - \Delta u + \sin u = 0$, or the cubic NLS; $-i\partial_t u - \Delta u - 2|u|^2 u = 0$, both in dimension $n = 1$) one can use the method of inverse scattering to deduce the same, at least on a formal level [FT],[NMPZ].

For linear equations it is well known that in the large-time limit these modes - the bound states and scattering states - decouple due to dispersion and consequently the solution converges in a local (spatial) norm to the bound states while radiating energy to infinity. This is a conclusion of the so-called RAGE Theorem [R],[AG], [E] (see also [GI], [RS-III] or [Pe]), which states an equivalence between the space-time behaviour of solutions and the spectral subspaces of the linear operator generating the dynamics. For reference we state the RAGE Theorem for the Schrödinger equation (the linear wave equation can also be written in the form of the Schrödinger equation). Recall that the solution of the linear Schrödinger equation $i\partial_t \varphi(t) = H\varphi(t)$ for appropriate initial data φ can be written as $\varphi(t) = e^{-iHt}\varphi$.

RAGE Theorem

Suppose $H = -\Delta + V$ acting on $\mathcal{H} = L^2(\mathbb{R}^n)$ is a locally compact operator. Let $\mathcal{H}_d$ and $\mathcal{H}_c$ be the discrete and continuous spectral subspaces of H respectively, and $\varphi(t) = e^{-iHt}\varphi$. Then,

(a) *$\varphi \in \mathcal{H}_d$ if and only if for each $\varepsilon > 0$ there is an $R_\varepsilon > 0$ such that*

$$\sup_t \|\bar{\chi}_{R_\varepsilon}\varphi(t)\|_{L^2(\mathbb{R}^n)} < \epsilon$$

(b) *$\varphi \in \mathcal{H}_c$ if and only if for each $R > 0$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|\chi_R\varphi(t)\|_{L^2(\mathbb{R}^n)}\, dt = 0$$

Here, χ_R is the characteristic function of the ball of radius R in $\mathbb{R}^n$.

The localization described in part (a) is our definition of a bound state (while part (b) describes the behaviour of scattering states). Under the hypothesis of the Theorem $\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_c$ and the discrete spectrum $\sigma_d(H)$ of H consists of countably many isolated eigenvalues with finite multiplicity, perhaps with an accumulation point at the bottom of the essential spectrum $= \inf \sigma_c(H)$. We can therefore write $\sigma_d(H) = \{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ where we repeat an eigenvalue according to its multiplicity. Consequently, part (a) implies that bound states are almost periodic in time; $\varphi_b(t) = \sum_{\lambda_j \in \sigma_d(H)} c_j e^{-i\lambda_j t}\phi_j,\ H\phi_j = \lambda_j\phi_j,\ c_j = \langle \varphi, \phi_j\rangle$. With a slight strengthening of the assumptions on H (so that $\mathcal{H}_c = \mathcal{H}_{ac}$, the absolutely continuous spectral subspace of H), the convergence in part (b) holds pointwise in t, and so we see that solutions converge locally to bound state solutions; $\varphi(t) \stackrel{loc}{\to} \varphi_b(t) = e^{-iHt}\varphi_b$ as $t \to \infty$, where $\varphi = \varphi_b + \varphi_s,\ \varphi_b \in \mathcal{H}_d,\ \varphi_s \in \mathcal{H}_c$ (note that $\varphi(t) = e^{-iHt}\varphi_b + e^{-iHt}\varphi_s$). Here $\varphi(t) \stackrel{loc}{\to} \varphi_b(t)$ denotes convergence locally in space; $\lim_{t\to\infty} \|\chi_R(\varphi(t) - \varphi_b(t))\|_{L^2(\mathbb{R}^n)} = 0$.

For integrable nonlinear equations one can use the method of inverse scattering to formally establish an analogous result (see also [CVZ] and [DZ] which makes this rigorous in certain cases). That is, one can decompose the initial data into pieces corresponding to bound states and scattering states, with these two modes decoupling as $t \to \infty$.

This scenario is becoming a paradigm in the scattering theory of general nonlinear dispersive wave equations. That is, one expects that a general solution will converge locally to bound states as $t \to \infty$ while radiating energy, the convergence being driven by dispersion. Rigorous results corroborating this view include [SW1] and [BP] where it is proven that solutions of NLS initially near a soliton converge (locally) to a soliton (solitons are examples of bound states). When the equation possesses many bound states (modulo symmetries) the dynamics among them as the solution disperses is a fascinating problem [T], [SW2].

We see that bound states play an important role in describing the large-time behavior of solutions of nonlinear wave and Schrödinger equations. By definition, bound states share the property of being localized in space, uniformly in time. In this article we show that bound states also share a compactness property in time, namely, that they are almost periodic in time. This result extends part of the RAGE Theorem to nonlinear equations. As an application of this temporal property, we establish a necessary condition for the existence of bound state solutions. Proving existence or nonexistence of bound states for linear equations is equivalent to determining the spectrum of the linear operator H (via the RAGE Theorem). For integrable nonlinear equations one can use the formalism of inverse scattering to address this question. For general nonlinear equations with 'repulsive' nonlinearities one can prove apriori dispersive-like decay estimates (perhaps only for small solutions), which rule out the possibility of there being bound states (see [S] or [Ca] for a discussion and references). In this article we present new nonexistence results using virial relations and the almost periodicity of (the candidate) bound state solutions.

2 Statement of Results

To state our main result we introduce some notation and definitions. Both NLW and NLS can be written as evolution equations;

$$\partial_t \varphi = A\varphi + G(\varphi), \tag{2.1}$$

where in the case of NLW, $A = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix}$, $G(\varphi) = \begin{bmatrix} 0 \\ -f(u) \end{bmatrix}$, $\varphi = (u, \partial_t u)$, and where in the case of NLS, $A = i\Delta,\ G(\varphi) = -if(\varphi),\ \varphi = u$.

The spaces of functions we will work with are the following. In the case of NLW, we let $X = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and in the case of NLS we let $X = H^1(\mathbb{R}^n)$. For $\mu > 0$, we let $X_\mu = H^{1+\mu}(\mathbb{R}^n) \times H^\mu(\mathbb{R}^n)$ or $X_\mu = H^{1+\mu}(\mathbb{R}^n)$ in the case of NLW or NLS respectively, where $H^\mu(\mathbb{R}^n)$ denotes the $L^2(\mathbb{R}^n)$ Sobolev space of order μ; $H^\mu(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \langle k \rangle^\mu \hat{u} \in L^2(\mathbb{R}^n)\}$, where $\hat{u}$ denotes the Fourier transform of u and $\langle k \rangle \equiv (1 + |k|^2)^{1/2}$. We denote the associated norms on these spaces by $\| \cdot \|_X^2 = \| \cdot \|_{H^1(\mathbb{R}^n)}^2 + \| \cdot \|_{L^2(\mathbb{R}^n)}^2$ or $\| \cdot \|_X^2 = \| \cdot \|_{H^1(\mathbb{R}^n)}^2$, and $\| \cdot \|_{X_\mu}^2 = \| \cdot \|_{H^{1+\mu}(\mathbb{R}^n)}^2 + \| \cdot \|_{H^\mu(\mathbb{R}^n)}^2$ or $\| \cdot \|_{X_\mu}^2 = \| \cdot \|_{H^{1+\mu}(\mathbb{R}^n)}^2$ in the case of NLW or NLS respectively.

We consider a solution $\varphi(x,t)$ of NLW or NLS as a map $\varphi : \mathbb{R} \to X$ which we will denote by $\varphi(t)$. For $\varphi \in X$, $\mathcal{O}(\varphi)$ will denote the orbit of φ; $\mathcal{O}(\varphi) = \{\varphi(t) \mid t \in \mathbb{R}, \varphi(0) = \varphi\}$, and $\mathcal{AP}(X)$ will denote the Banach space of almost periodic functions from $\mathbb{R}$ to X with norm $\|\varphi\|_{\mathcal{AP}(X)} = \sup_{t \in \mathbb{R}} \|\varphi(t)\|_X$ (see Definition 3.1 below).

Definition 2.1 *A function $\varphi : \mathbb{R} \to X$ is a* bound state *if for any $\varepsilon > 0$ there exists a ball $B_{R_\varepsilon} \subset \mathbb{R}^n$ of radius R_ε centered at 0 such that*

$$\sup_t \|\overline{\chi}_{R_\varepsilon} \varphi(t)\|_X < \varepsilon \tag{2.2}$$

where $\overline{\chi}_{R_\varepsilon}$ is the characteristic function of $\mathbb{R}^n \setminus B_{R_\varepsilon}$. A bound state solution of NLW or NLS *is a globally defined solution of NLW or NLS that is a bound state.*

Remark: Bound states as defined here are stationary. Both NLW and NLS possess a group of symmetry transformations which map one solution into another (e.g., the Lorentz group for NLW and the Galilean group for NLS). These symmetry transformation can transform a bound state into a 'travelling' bound state. Thus, some solutions may not be bound states in the sense of the Definition 2.1, but can be transformed into bound states by an appropriate symmetry transformation.

If $\varphi(t)$ is a globally defined solution of NLW or NLS with $\varphi(0) = \varphi$, we let $S\,(= S_\varphi) : \mathbb{R} \times \mathcal{O}(\varphi) \to \mathcal{O}(\varphi)$, $S(t) : \varphi_1 \mapsto \varphi_1(t) \equiv S(t)\varphi_1$, $\varphi_1 \in \mathcal{O}(\varphi)$, denote the flow of the dynamical system (2.1) along $\mathcal{O}(\varphi)$. Note that $\{S(t)\}_{t\in\mathbb{R}}$ is a group: $S(t_1+t_2) = S(t_1)S(t_2)$.

Our main result is the following.

Theorem 2.2 *Let $\varphi \in C(\mathbb{R}; X)$ be a bound state solution of NLW or NLS such that $\sup_t \|\varphi(t)\|_{X_\mu} < \infty$ for some $\mu > 0$. If $\{S(t)\}_{t\in\mathbb{R}}$ is equicontinuous on $\mathcal{O}(\varphi)$, then φ is an almost periodic function from $\mathbb{R}$ to X.*

The regularity condition $\sup_t \|\varphi(t)\|_{X_\mu} < \infty$, along with the hypothesis of localization (i.e., that $\varphi(t)$ is a bound state), assures that $\mathcal{O}(\varphi)$ is relatively compact in X (see Proposition 3.3). This compactness, combined with the equicontinuity of $\{S(t)\}_{t\in\mathbb{R}}$, guarantees the almost periodicity of $\varphi(t)$ (as we will show). Recall that $\{S(t)\}_{t\in\mathbb{R}}$ is equicontinuous if for any $\varepsilon > 0$ there is a δ such that if $\varphi_1, \varphi_2 \in \mathcal{O}(\varphi)$ with $\|\varphi_1 - \varphi_2\|_X < \delta$, then $\sup_{t\in\mathbb{R}} \|S(t)\varphi_1 - S(t)\varphi_2\|_X < \varepsilon$. In the linear case the flow will be equicontinuous on any orbit if the family $\{S(t)\}_{t\in\mathbb{R}}$ is bounded; $\|S(t)\varphi_1 - S(t)\varphi_2\|_X \le (\sup_t \|S(t)\|_{X\to X})(\|\varphi_1 - \varphi_2\|_X)$. In particular, this is true for the free linear wave and Schrödinger equations (i.e., $f(u) \equiv 0$), and for general linear equations under appropriate (mild) assumptions on the operator H. In the nonlinear case equicontinuity will have to be verified for each equation. We remark that without the hypothesis of equicontinuity the trajectory $\varphi(t)$, although relatively compact, may only be recurrent and not almost periodic (see for example Ch V.8 in [NS]).

Example

Consider the following assumptions on the nonlinearity f;

$$F(u) \ge -c|u|^2 - c|u|^{q+1} \quad \text{for} \;\; q < 1 + \frac{4}{n} \quad \text{in the case of NLW,}$$

$$F(u) \ge -c|u|^2 \quad \text{in the case of NLS,}$$

where $F' = f$. Furthermore, suppose $|f'(u)| \le c(1 + |u|^{p-1})$ where $1 < p < 1 + \frac{4}{n-2}$ ($1 < p < \infty$ if $n = 1, 2$). Then if $\varphi \in X_1$, there exists a global solution $\varphi(\cdot) \in C(\mathbb{R}; X_1)$ with $\varphi(0) = \varphi$ (see [S] Chapter 3). Applying Theorem 2.2, if the solution $\varphi(t)$ is a bound state such that $\sup_t \|\varphi(t)\|_{X_\mu} < \infty,\ \mu > 0$, and if $\{S_\varphi(t)\}_{t\in\mathbb{R}}$ is equicontinuous on $\mathcal{O}(\varphi)$, then $\varphi(t)$ is an almost periodic function from $\mathbb{R}$ to X.

Nonexistence of bound states

Here we combine the use of virial relations, as discussed in [Py], with the almost periodicity of bound states (Theorem 2.2) to prove nonexistence of bound states for certain equations.

A virial relation is an integral identity involving the solution of a differential equation. The identity can be derived directly from the equation itself or, if the equation can be formulated as a variational problem, from the action functional associated to the equation. A well known example from physics relates the time-average kinetic and potential energies of an n-particle system under the influence of central forces (usually referred to as *the* virial theorem; see for example [LL]). In mathematics virial relations have been used extensively, for example, in deriving necessary conditions for the existence of solutions of differential equations beginning with the work of Pohozaev [Po].

Let's consider NLW, and suppose $\varphi(t)$ is a bound state solution. Assuming the hypotheses of Theorem 2.2 are satisfied, $\varphi(t)$ is almost periodic in time. As described in [Py], by combining the dilation and gauge transformations, one obtains the following virial relation valid for any $c \in \mathbb{R}$;

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathbb{R}^n} \left\{(\frac{1}{2} - c)(\partial_t\varphi)^2 + (c + \frac{2-n}{2n}) \mid \nabla\varphi \mid^2\right\} = \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathbb{R}^n} \{F(\varphi) - c\varphi f(\varphi)\} \tag{2.3}$$

We assume that $\varphi \in \mathcal{AP}_q$ for some $q \in [2, \infty]$ where $\mathcal{AP}_q \equiv \{\varphi \in \mathcal{AP}(H^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n))$ such that $\partial_t\varphi$ exists in the strong sense as a uniformly continuous map $\mathbb{R} \to L^2(\mathbb{R}^n)\}$. Here $\mathcal{AP}(\mathcal{B})$ denotes the set of almost periodic functions from $\mathbb{R}$ to the space $\mathcal{B}$ (see Definition 3.1 below). We also assume that $f \in C(\mathbb{R}, \mathbb{R})$ with $f(0) = 0$ and that either $\mid f(z) \mid \leq c(\mid z \mid + \mid z \mid^{q/2})$, or f satisfies a Lipschitz condition at the origin.

If for some $c \in [\frac{n-2}{2n}, \frac{1}{2}]$, the inequality

$$F(z) - czf(z) \leq 0 \tag{2.4}$$

holds, then both sides of Equation (2.3) must be zero. In the case $c > \frac{n-2}{2n}$ this then implies that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathbb{R}^n} |\nabla\varphi|^2 \;=\; 0 \;=\; \lim_{T\to\infty} \frac{1}{T}\int_0^T \|\nabla\varphi\|_2^2. \tag{2.5}$$

Parseval's relation for Hilbert space-valued almost periodic functions ([LZ] pp 31) states that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|\nabla\varphi\|_2^2 = \sum_{k\in\mathbb{N}} \|\phi_k\|_2^2 \tag{2.6}$$

where $\sum_{k\in\mathbb{N}} \phi_k\, e^{i\lambda_k t}$ is the Fourier series associated to $\nabla\varphi(t)$ (see Definition 3.1 below). By the uniqueness of these series it follows from equations (2.5) and (2.6) that $\nabla\varphi(t) = 0$ (in $L^2(\mathbb{R}^n)$) for all t which implies that, since $\varphi(t) \in L^2(\mathbb{R}^n)$, $\varphi(t) = 0$ for all t. Thus, if f satisfies (2.4) with $c \in (\frac{n-2}{2n}, \frac{1}{2}]$, then this NLW has no (non-trivial) bound states.

Remark: The virial relation (2.3) contains no boundary terms (in space and in time). The occurrence of boundary terms themselves do not exclude the use of virial relations, but one then one has to control these terms. A virial relation for NLW or NLS will have no boundary terms if the solution has vanishing or periodic boundary conditions. Vanishing at spatial infinity is assured by assuming the solution is integrable in x. Only dispersive solutions will have vanishing boundary conditions in time, so to obtain virial relations without boundary terms for bound states one must assume some kind of compactness property in time of the solution. For example, the existence of the ergodic mean

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi(t)\, dt$$

for $\varphi(t)$ and the other terms appearing in the action functional defining the equation, allows us to obtain virial relations without boundary terms in time. One important property of almost periodic functions is that they do have ergodic means.

3 Proof of Theorem 2.2

We state the definition of almost periodic functions that we will use (see for example [LZ]).

Definition 3.1 *A strongly continuous function* $\varphi : \mathbb{R} \to X$ *is* almost periodic *if the set of translates of* φ, $\mathcal{T}(\varphi) = \{\varphi^h(t) \equiv \varphi(t+h);\ h \in \mathbb{R}\}$, *has compact closure in* $C_b(\mathbb{R};\ X)$ *where* $C_b(\mathbb{R};\ X)$ *denotes the metric space of bounded strongly continuous functions from* $\mathbb{R}$ *to* X *equipped with the supremum norm:* $\|\varphi\|_{C_b} \equiv \sup_t \|\varphi(t)\|_X$. *The set of almost periodic functions from* $\mathbb{R}$ *to* X *will be denoted by* $\mathcal{AP}(X)$.

Remark: Almost periodic functions admit an elegant representation, from which their many properties may be derived [Co],[LZ]. For example, we can associate a Fourier series to any $\varphi \in \mathcal{AP}(X)$;

$$\varphi(t) \ \sim \ \sum_{k\in\mathbb{N}} \phi_k \, e^{i\lambda_k t}, \quad \lambda_k \in \mathbb{R},\ \phi_k \in X \tag{3.7}$$

where $\phi_k = \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\lambda_k t}\varphi(t)\,dt$. λ_k are the *frequencies* of $\varphi(t)$. The question of convergence of this Fourier series is just as delicate as for periodic functions. In general though, by re-summing the Fourier series of an almost periodic function one can find trigonometric polynomials that approximate the function arbitrarily well, uniformly in time. More precisely, for any $\varepsilon > 0$ there exists a trigonometric polynomial $p_\varepsilon(t) = a_0 + a_1 e^{i\lambda_{k_1} t} + \cdots + a_m e^{i\lambda_{k_m} t}$, $a_i \in X$, such that $\sup_{t\in\mathbb{R}} \|\varphi(t) - p_\varepsilon(t)\|_X < \varepsilon$. Some special classes of almost periodic functions are the periodic functions; $\lambda_k = k\lambda$; here there is one independent frequency, and the quasi-periodic functions; $\lambda_k = \vec{k}\cdot\vec{\lambda}$, where $\vec{k} \in \mathbb{N}^n$ and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, with $\lambda_1, \ldots, \lambda_n$ (linearly) independent over $\mathbb{Q}$; here there are n independent frequencies. In general, an almost periodic function has a countably infinite number of independent frequencies.

Geometrically, φ is almost periodic with l independent frequencies if $\varphi(x,t) = \gamma(x, \vec{\lambda}t)$ for some $\gamma : \mathbb{T}^l \to X$ where $\mathbb{T}^l$ is the l-torus $\mathbb{T}^l = \Pi_{k=1}^l S^1$. We call γ the *generating function* of φ and $\vec{\lambda}$ the *frequency* of φ. In general, almost periodic functions φ can be characterized by generating functions γ that are defined on the infinite dimensional torus $\mathbb{T}^\infty$. Then, there is a dense embedding $\Gamma : \mathbb{R} \to \mathbb{T}^\infty$ such that $\varphi(t) = \gamma(\Gamma(t))$. If the image of this embedding in $\mathbb{T}^\infty$ sits in a finite-dimensional torus of dimension l, then $\varphi(x)$ is quasi-periodic with l

independent frequencies.

Proposition 3.2 *If $\varphi \in \mathcal{AP}(X)$ then φ is a bound state.*

An important property of almost periodic functions is that the closure of their orbits in the phase space X are compact [LZ]. From this Proposition 3.2 follows. Conversely, to prove that a bound state is almost periodic we first show that the closure of its orbit is compact under a regularity assumption. For this we require a convenient characterization of compact subsets of the spaces X.

Proposition 3.3 *A bounded subset $K \subset X$ has compact closure in X if for any $\varepsilon > 0$ there exists an $R_\varepsilon > 0$ such that $\sup_{\varphi \in K} \|\overline{\chi}_{R_\varepsilon}\varphi\|_X < \varepsilon$ and if for some $\mu > 0$, $\sup_{\varphi\in K} \|\varphi\|_{X_\mu} < \infty$.*

Proof: We show that K is totally bounded in X. Let $\varepsilon > 0$ and find R_ε so that $\sup_{\varphi\in K} \|\overline{\chi}_{R_\varepsilon}\varphi\|_X < \varepsilon$. Let $R > R_\varepsilon$. For $\varphi \in K$ let $\tilde{\varphi} \equiv \chi_R\varphi$ and $\overline{\varphi} \equiv \overline{\chi}_R\varphi$ where χ_R is a smooth function such that $\chi_R \equiv 1$ on B_{R_ε} and $\chi_R \equiv 0$ on B_R^C, and $\overline{\chi}_R = 1 - \chi_R$. We set $\tilde{K} = \{\tilde{\varphi} \,|\, \varphi \in K\}$. For any subset $\Omega \subset \mathbb{R}^n$ let $X(\Omega)$ denote the corresponding Sobolev spaces defined on Ω (as defined above; so for example $X(\mathbb{R}^n) = X$). Since $\tilde{K}$ is bounded in $X_\mu(B_R)$, $\mu > 0$, and every $\tilde{\varphi} \in \tilde{K}$ is supported in B_R, by the Rellich-Kondrachov Theorem, $\tilde{K}$ has compact closure in $X(B_R)$. Therefore, there exists an ε-net for $\tilde{K}$ in $X(B_R)$, i.e., a finite set $\{\phi_1, \ldots, \phi_N\} \subset X(B_R)$ such that for any $\tilde{\varphi} \in \tilde{K}$, there is a $k \in \{1, \ldots, N\}$ such that $\|\tilde{\varphi} - \phi_k\|_{X(B_R)} = \|\tilde{\varphi} - \phi_k\|_X < \varepsilon$ (here we extend $\tilde{\varphi}$ and ϕ_k to all of $\mathbb{R}^n$ with $\tilde{\varphi} \equiv 0$ and $\phi_k \equiv 0$ on $\mathbb{R}^n \setminus B_R$). Since $\varphi = \tilde{\varphi} + \overline{\varphi}$, we have that $\|\varphi - \phi_k\|_X \leq \|\tilde{\varphi} - \phi_k\|_X + \|\overline{\varphi}\|_X < 2\varepsilon$. Thus, $\{\phi_1, \ldots, \phi_N\}$ is a 2ε-net for K in X $\square$

A necessary and sufficient condition that K have compact closure in X is that K be uniformly localized in space and momentum, i.e., that for any $\varepsilon > 0$ there is an R_ε such that $\sup_{\varphi\in K} \|\overline{\chi}_{R_\varepsilon}\varphi\|_X < \varepsilon$ and $\sup_{\hat{\varphi}\in K} \|\overline{\chi}_{R_\varepsilon}\hat{\varphi}\|_X < \varepsilon$ (see for example [RS-IV] or [GI]). The condition $\sup_{\varphi\in K} \|\varphi\|_{X_\mu} < \infty$ for some $\mu > 0$ guarantees uniform localization in momentum, but it is not a necessary condition so this hypothesis could be weakened (and hence the regularity hypothesis in Theorem 2.2 could be weakened).

The idea of the proof of Theorem 2.2 is to go from compactness of $\mathcal{O}(\varphi)$ in X to compactness of $\mathcal{T}(\varphi)$ in $C_b(\mathbb{R};\, X)$ via the equicontinuity of $\{S(t)\}_{t\in\mathbb{R}}$.

Proof of Theorem 2.2: Let $\{\varphi^{h_j}\} \equiv \{\varphi(t+h_j)\}$ be any sequence from $\mathcal{T}(\varphi)$. By Proposition 3.3, $\overline{\mathcal{O}(\varphi)}$ is compact in X so there exists a subsequence $\{h_j'\}$ of $\{h_j\}$ such that $\{\varphi(h_j')\}$ is a Cauchy sequence in X. Now consider the subsequence $\{\varphi^{h_j'}\}$ of $\{\varphi^{h_j}\}$. Let $\varepsilon > 0$. Since we are assuming that the flow $S(t)$ on $\overline{\mathcal{O}(\varphi)}$ is equicontinuous, there is a $\delta > 0$ such that $\|S(t)\varphi_1 - S(t)\varphi_2\|_X < \varepsilon$ for all t if $\varphi_1, \varphi_2 \in \overline{\mathcal{O}(\varphi)}$ and $\|\varphi_1 - \varphi_2\|_X < \delta$. Because $\{\varphi(h_j')\}$ is a Cauchy sequence, we can find an N_δ such that $\|\varphi(h_k') - \varphi(h_l')\|_X < \delta$ if $k, l > N_\delta$. Therefore, $\|\varphi(t+h_k') - \varphi(t+h_l')\|_X = \|S(t)\varphi(h_k') - S(t)\varphi(h_l')\|_X < \varepsilon$ for all t if $k, l > N_\delta$ which implies that $\{\varphi^{h_j'}\}$ is a Cauchy sequence in $C_b(\mathbb{R};\, X)$. Thus, $\overline{\mathcal{T}(\varphi)}$ is sequentially compact, and hence compact, in $C_b(\mathbb{R}; X)$ $\square$

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