Nonlinear Wave Equations: Constraints on Periods and Exponential Bounds for Periodic Solutions.*

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Abstract

We show there is an upper bound to the allowed frequencies of time periodic solutions of a class of nonlinear wave equations: if φ is a $2\pi/\omega$ -periodic solution then $\omega^2 \leq f'(0)$, where f is the nonlinearity. We also prove that

$$\int_0^{2\pi/\omega} \int_{\mathrm{R}^N} e^{2\alpha|x|} \Big(|\varphi(x,t)|^2 + |\nabla\varphi(x,t)|^2 \Big) d^N x \, dt \; < \; \infty$$

for all $\alpha^2 \; < \; f'(0) - \lfloor \sqrt{\frac{f'(0)}{\omega^2}} \rfloor^2 \omega^2$ where $\lfloor a \rfloor$ denotes the integer part of a.

1 Introduction

In this article we study periodic solutions of the nonlinear wave equation (NLW)

$$\partial_t^2 \varphi - \Delta\varphi + f(\varphi) = 0 \tag{1.1}$$

where $\varphi : \; \mathrm{R}_x^N \times \mathrm{R}_t \to \mathrm{R}, \;\; f : \; \mathrm{R} \to \mathrm{R}$ with $f(0) = 0$, and $\partial_t^2 = \partial^2/\partial t^2$, $\Delta = \sum_{i=1}^N \partial^2/\partial {x_i}^2$. By a periodic solution we understand solutions that are periodic in time t, and L^2 in x. This

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notion extends, on the one hand, the concepts of bound states of the Schrödinger equation and standing waves of linear wave equations and, on the other hand, the concept of periodic solutions of dynamical systems (equation (1.1) can be viewed as an infinite dimensional Hamiltonian system). Both concepts are among the simplest and most basic in the fields mentioned.

To state our results we introduce some notation. Let S^1_ω denote the circle of radius ω^{-1}. The class of solutions we consider is the following set:

$$\mathcal{D}_\omega \equiv \Big\{\varphi \in H^1(\mathrm{R}^N \times S^1_\omega) \quad ; \quad \text{if } \psi \text{ is any of } \varphi, \partial_t\varphi \text{ or } x\cdot\nabla\varphi, \text{ then } \|\psi\|_{L^\infty(\mathrm{R}^N\times S^1_\omega)} < \infty$$
$$\text{and } \lim_{|x|\to\infty} |\,\psi(x,t)\,| = 0 \text{ uniformly in t} \Big\}. \tag{1.2}$$

(This class of solutions can, probably, be enlarged.) Here $H^1(\Omega)$ stands for the Sobolev space of order 1 for functions on Ω.

Our main result is a characterization of two fundamental properties of periodic solutions: their frequencies and their spatial localization. Consequently, we find that the spatial and temporal properties of periodic solutions are related. More precisely, we prove the following theorems.

Theorem 1.1 *Suppose $f \in C^3(\mathrm{R},\mathrm{R})$. Let φ be a nontrivial $2\pi/\omega$-periodic solution of NLW on $\mathcal{D}_\omega$. Then $\omega^2 \le f'(0)$.*

Theorem 1.2 *Suppose $f \in C^3(\mathrm{R},\mathrm{R})$ and that $f'(0) \neq m^2\omega^2,\ \ m \in \mathrm{Z}$. Let φ be a $2\pi/\omega$-periodic solution of NLW on $\mathcal{D}_\omega$. Then $e^{\alpha|x|}\varphi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all α satisfying*

$$\alpha^2 < f'(0) - \Big\lfloor\sqrt{\frac{f'(0)}{\omega^2}}\Big\rfloor^2\omega^2$$

where $\lfloor a \rfloor$ denotes the integer part of a.

By $L^2(S^1_\omega, H^1(\mathrm{R}^N))$ we mean functions $\psi : S^1_\omega \to H^1(\mathrm{R}^N)$, such that $\|\psi\|^2_{L^2(\mathrm{R}^N\times S^1_\omega)} + \|\nabla\psi\|^2_{L^2(\mathrm{R}^N\times S^1_\omega)} < \infty$. In Theorem 1.1, by nontrivial periodic solution we exclude time-independent solutions. Theorem 1.2, however, applies to time-independent solutions as well.

In this case φ solves the nonlinear elliptic equation

$$-\Delta\varphi + f(\varphi) = 0.$$

Conversely, any solution of this equation is a time independent solution of NLW. For time independent solutions we can take for the frequency ω any positive number. Taking ω sufficiently large results in $\lfloor\sqrt{\frac{f'(0)}{\omega^2}}\rfloor = 0$. On the other hand we have that if $f'(0) < 0$ then $\varphi = 0$. This result follows from Theorems 2.2 and 5.1 (below). The former theorem states, in this context, that if $f'(0) < 0$ then φ is exponentially bounded with arbitrarily large exponent. By the latter theorem φ is then in fact zero. Thus, we have the following corollary.

Corollary 1.3 *Let $f \in C^3(\mathrm{R},\mathrm{R})$ with $f(0) = 0$. Suppose $\varphi \in H^1(\mathrm{R}^N)$ is a solution of the equation*

$$\Delta\varphi = f(\varphi)$$

such that φ and $x \cdot \nabla\varphi \in L^\infty(\mathrm{R}^N)$ and vanish as $|x| \to \infty$. If $f'(0) < 0$, then $\varphi = 0$. If $f'(0) > 0$, then $e^{\alpha|x|}\varphi \in H^1(\mathrm{R}^N)$ for all $\alpha^2 < f'(0)$.

We comment on the condition $f(0) = 0$ in the above theorems. From a physical point of view one usually considers solutions with finite energy $E(\varphi)$, where

$$E(\varphi) = \int_{\mathrm{R}^N} \frac{1}{2}(\partial_t\varphi)^2 + \frac{1}{2}|\nabla\varphi|^2 + F(\varphi), \quad F' = f.$$

Note that the potential F is defined only up to a constant. We see that for solutions of finite energy,

$$\partial_t\varphi \text{ and } \nabla\varphi \to 0 \text{ as } |x| \to \infty$$

(at least along a subsequence). Therefore, $\varphi \to$ *const.* as $|x| \to \infty$ and so $(\partial_t^2 - \Delta)\varphi \to 0$ as $|x| \to \infty$. Since φ solves NLW, $\varphi \to c$ as $|x| \to \infty$ where $f(c) = 0$. If $c \neq 0$, then by defining $\varphi_c \equiv \varphi - c$ and $f_c(z) \equiv f(z + c)$ we have that $\partial_t^2\varphi_c - \Delta\varphi_c + f_c(\varphi_c) = 0$ with $f_c(0) = 0$. Thus, for finite energy solutions the assumption $f(0) = 0$ is not a restriction.

We also make several comments on extensions of these results to other equations. Complex valued solutions of NLW for nonlinearities of the form $f(\varphi) = g(|\varphi|)\varphi$ can be analysed using our method, the results being the same as above. In particular, standing wave solutions $\varphi(x,t) = e^{i\omega t}\phi(x)$ must satisfy $\omega^2 < g(0)$ while $\phi(x)$ is exponentially bounded with any exponent α such that $\alpha^2 < g(0) - \omega^2$. These results are consistent with previous studies of standing wave solutions (e.g. [B], [Str]).

The results of this paper can be extended to a rather wide class of hyperbolic equations and some of the results to the nonlinear Schrödinger equation. This will be done in a separate publication. Note however, that frequency bounds of the type of Theorem 1.1 are not valid for the nonlinear Schrödinger equation which conforms with the main premises of our approach (see the remarks below following the statement of Theorem 2.2).

To prove Theorems 1.1 and 1.2 we introduce a method into the area of nonlinear equations which is related to one that has been developed, to great success, in the scattering theory of Schrödinger operators: that of positive commutators, and by this we mean the following. Let K be a self-adjoint operator on a Hilbert space of functions. We say that K satisfies a positive commutator estimate on a set Ω if there is a self-adjoint operator A such that $\langle i[K,\, A]\rangle_\psi \geq \theta\|\psi\|^2$ for some $\theta > 0$ and for all $\psi \in \Omega$ where $\langle B\rangle_\psi \equiv \langle B\psi,\, \psi\rangle$. This is related to the Mourre estimate [M] of quantum mechanics (for a discussion of the Mourre estimate and its applications see [CFKS] or [HS]).

Our approach is motivated by pioneering studies of n-body Schrödinger operators of the form $H = -\Delta + V$ on $L^2(\mathrm{R}^N)$, where V is multiplication by a real-valued function, ([FH], [FHH-OH-O]). In these references positive commutator estimates based on the Mourre estimate are used to establish (among other results) the exponential localization of eigenfunctions of H. The situation we are considering here has some fundamental differences, though. One of them is related to the fact that we are dealing with hyperbolic, not elliptic, operators. Another, to the fact that the equations in question are nonlinear.

We review briefly previous results on the subject. Exponentially localized periodic solutions on the half-line R_+ were constructed in ([V],[W]) using techniques from invariant manifold theory, while radially symmetric exponentially decaying periodic solutions outside of a ball in R^N (the spatial domain) were found in ([S],[Sc]) using techniques similar to

([V],[W]). Unfortunately, even in theses cases no exponential bounds for a reasonable class of periodic solutions was obtained. The constraint $\omega^2 \leq f'(0)$ was obtained for classical periodic solutions in R^{1+1} in [C] and extended to radially symmetric solutions in arbitrary spatial dimensions in [L]. In the works above it was essential that the spatial variable be effectively one dimensional: the NLW was formulated as a dynamical system in a phase space of periodic functions with x playing the role of the dynamical variable. Our results on the frequency restrictions extend those of [C] and [L] in that we study NLW in arbitrary spatial dimensions and we assume considerably less regularity from the solutions (H^1 in space and time). Our results on exponential bounds seem to be new even in the one dimensional case.

The paper is organized as follows. In Section 2 we formulate NLW as a nonlinear eigenvalue problem in such a way that if φ is a $2\pi/\omega$ -periodic solution of NLW, then $K_\varphi \varphi = \lambda \varphi$ where, for a given $2\pi/\omega$ -periodic function $\varphi \in \mathcal{D}_\omega$, K_φ is a self-adjoint operator on a Hilbert space of $2\pi/\omega$ -periodic functions. Section 3 collects preliminary results that we use to prove exponential bounds for a class of hyperbolic operators, which is presented in Section 4. From this we obtain exponential bounds for periodic solutions (Theorem 2) by exploiting the relation mentioned above. Theorem 1.1 is proven in Section 5. An appendix describes an operator calculus that we employ.

<u>Notation:</u> For $x \in \mathrm{R}^N$, let $r = |x|$ and $\hat{x} = xr^{-1}$. ∇ denotes the gradient operator on R^N: $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$. For an operator B on $L^2(\mathrm{R}^N \times S^1_\omega)$, $\langle B \rangle_\psi$ will stand for the expectation value $\langle B\psi, \psi \rangle$. $\|\psi\|_{L^2(S^1_\omega, H^1(\mathrm{R}^N))}$ denotes the space-time norm of functions $\psi: \; S^1_\omega \to H^1(\mathrm{R}^N)$;

$$\|\psi\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} = \|\psi\|^2_{L^2(\mathrm{R}^N \times S^1_\omega)} + \|\nabla\psi\|^2_{L^2(\mathrm{R}^N \times S^1_\omega)}$$

where $H^r(\mathrm{R}^N)$ is the $L^2(\mathrm{R}^N)$ Sobolev space of order r. From now on the norm on $L^2(\mathrm{R}^N \times S^1_\omega)$ will be denoted simply by $\|\psi\|$. We decompose $L^2(\mathrm{R}^N \times S^1_\omega)$ into a direct sum using the eigenspaces of $i\partial_t$;

$$L^2(\mathrm{R}^N \times S^1_\omega) \simeq \oplus_{k \in \mathrm{Z}} \mathrm{E}_k$$

where $\mathrm{E}_k = e^{ik\omega t}\mathrm{R} \otimes L^2(\mathrm{R}^N)$. For $\psi \in L^2(\mathrm{R}^N \times S^1_\omega)$ we can write $\psi = \sum_{k\in\mathrm{Z}} e^{ik\omega t}\psi_k$ where $\psi_k(x) = (2\pi/\omega)^{-1} \int_{S^1_\omega} \psi(x,t)e^{-ik\omega t}\, dt \in L^2(\mathrm{R}^N)$ are the Fourier coefficients ("modes") of ψ. P_k will denote the projection onto E_k : $P_k\psi = e^{ik\omega t}\psi_k$. $\bar{P}_k = \mathbf{1} - P_k$, and $\Pi_m = \sum_{|k|\leq m} P_k$ with $\bar{\Pi}_m = \mathbf{1} - \Pi_m$.

For an interval $I \subset \mathrm{R}$ and self-adjoint operator H we define a smoothed-out spectral projection $E_I(H)$ as follows. Let $\sigma > 0$ be sufficiently small so that $2\sigma < |I|$ and let $g \in C^\infty_o(\mathrm{R},\mathrm{R})$ be such that $supp\,(g) \subset I$ and $g \equiv 1$ on I_σ where $I_\sigma = \{\lambda \in I\ ;\ dist\,(\lambda, \partial I) > \sigma\}$. We then set $E_I(H) \equiv g(H)$. The parameter σ is not made explicit since it is understood that it can be taken as small as needed.

2 NLW as an eigenvalue problem

To apply the technique of positive commutator estimates we first formulate NLW as a (non-linear) eigenvalue problem for a self-adjoint operator. Let

$$W(u) = \frac{f(u)}{u} - \kappa, \quad \kappa = f'(0). \tag{2.1}$$

We assume that $f \in C^3(\mathrm{R},\mathrm{R})$. Then, $W \in C^1(\mathrm{R},\mathrm{R})$. For a given function $\varphi \in L^2(\mathrm{R}^N \times S^1_\omega)$ define the "potential" $W_\varphi(x,t) \equiv W(\varphi(x,t))$ acting as an operator of multiplication, and the linear operators

$$K_\varphi = K_o + W_\varphi, \quad K_o \equiv \partial_t^2 - \Delta \tag{2.2}$$

on $L^2(\mathrm{R}^N \times S^1_\omega)$. Thus,

$$\varphi \text{ is a } 2\pi/\omega\text{-periodic solution to NLW} \iff K_\varphi\varphi = -\kappa\varphi.$$

Our analysis requires self-adjointness of the operators K_o and K_φ. It is clear that K_o is self-adjoint. To show that K_φ is also self-adjoint we require the following lemma.

Lemma 2.1 *If $\varphi \in \mathcal{D}_\omega$, then W_φ, $\partial_t W_\varphi$ and $x \cdot \nabla W_\varphi$ are of class $L^q(\mathrm{R}^N \times S^1_\omega)$ for all $q \in [2, \infty]$ and vanish as $|x| \to \infty$, uniformly in t.*

Proof:

We denote W_φ simply by W. By the chain rule, $\partial_t W = W'\partial_t \varphi$ and $x\cdot\nabla W = W' x\cdot\nabla\varphi$. Since $W(u)\in C^1(\mathrm{R},\mathrm{R})$ and φ is bounded, W and W' are members of $L^\infty(\mathrm{R}^N\times S^1_\omega)$. Therefore $\partial_t W$ and $x\cdot\nabla W$ are bounded. There exist constants c, c' and d, all strictly positive, such that $|W(u)| < c\,|u|$ and $|W'(u)| < c'\,|u|$ for $|u| < d$. For $q\in[2,\infty)$ we compute;

$$
\begin{aligned}
\|W\|^q_{L^q(\mathrm{R}^N,\times S^1_\omega)} &= \int_{|\varphi|\le d} |\,W\,|^q \;+\; \int_{|\varphi|>d} |\,W\,|^q \\
&\le c^q\|\varphi\|^q_{L^q(\mathrm{R}^N\times S^1_\omega)} \;+\; b^q\, meas\,\{|\,\varphi\,| > d\} \;=\; M,
\end{aligned}
\tag{2.3}
$$

where $b = sup\,\{|\,W(u)\,|\ ;\ |\,u\,| \le \|\varphi\|_{L^\infty(\mathrm{R}^N\times S^1_\omega)}\}$. Since $\|\varphi\|_{L^\infty(\mathrm{R}^N\times S^1_\omega)}$ is finite, the continuity of $W(u)$ implies that b is finite. Hence M is finite. If $\varphi\in L^\infty(\mathrm{R}^N\times S^1_\omega)$, then so is W. Therefore, $W\in L^q(\mathrm{R}^N\times S^1_\omega)$ for all $q\in[2,\infty]$. A similar argument applies to W', and therefore $\partial_t W$ and $x\cdot\nabla W\in L^q(\mathrm{R}^N\times S^1_\omega)$ for all $q\in[2,\infty]$. Since φ, $\partial_t\varphi$ and $x\cdot\nabla\varphi$ vanish as $|\,x\,|\to\infty$, uniformly in t, W, $\partial_t W$ and $x\cdot\nabla W$ vanish in the same manner □

By the results of Lemma 2.1, for $\varphi\in\mathcal{D}_\omega$, W_φ is a small perturbation of K_o and therefore by the Kato-Rellich theorem [RS-II] K_φ as defined by (2.2) is self-adjoint on $L^2(\mathrm{R}^N\times S^1_\omega)$ with domain $D(K_\varphi) = D(K_o) =$ closure of $H^2(\mathrm{R}^N\times S^1_\omega)$ in the graph norm $\|\psi\|^2_G = \|\psi\|^2_{L^2(\mathrm{R}^N\times S^1_\omega)} + \|K_o\psi\|^2_{L^2(\mathrm{R}^N\times S^1_\omega)}$.

We saw that if $\varphi\in L^2(\mathrm{R}^N\times S^1_\omega)$ is a $2\pi/\omega$-periodic solution to NLW, then $K_\varphi\varphi = -\kappa\varphi$. That is, φ is an eigenfunction of K_φ corresponding to the eigenvalue $-\kappa = -f'(0)$. For the remainder of the paper we will study properties of the eigenfunctions of K_φ which we will translate into properties of periodic solutions of NLW by considering, in particular, the eigenfunctions of the operator K_φ corresponding to the eigenvalue $-\kappa$.

We now formulate the theorem concerning exponential bounds for eigenfunctions of the operator K_φ, from which we will derive Theorem 1.2.

Theorem 2.2 *Let $K_\varphi = K_o + W_\varphi$, $\varphi\in\mathcal{D}_\omega$. Suppose $K_\varphi\psi = \lambda\psi$ for some $\psi\in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ and for some $\lambda\ne -m^2\omega^2$, $m\in\mathrm{Z}$. If $\lambda<0$, then $e^{\alpha r}\psi\in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all α satisfying*

$$\alpha^2 < -\lambda - \lfloor\sqrt{\frac{-\lambda}{\omega^2}}\rfloor^2\omega^2$$

where $\lfloor a \rfloor$ *denotes the integer part of* a*. If* $\lambda > 0$ *then* $e^{\alpha r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ *for all* α*.*

Remarks:

If φ is a $2\pi/\omega$ -periodic solution of NLW such that $\varphi \in \mathcal{D}_\omega$, then φ is an eigenfunction of $K_\varphi = K_o + W_\varphi$, corresponding to the eigenvalue $-\kappa = -f'(0)$. Theorem 1.2 thus follows from Theorem 2.2. We also have a kind of unique continuation at infinity theorem (Theorem 5.1 below) which states that if an eigenfunction of K_φ is in the space $e^{-\alpha r}L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all $\alpha > 0$ then this function is in fact zero. This, coupled with the last statement of Theorem 2.2, implies that K_φ has no positive eigenvalues. The nonexistence of positive eigenvalues is related to the frequency constraint $\omega^2 \leq f'(0)$ of Theorem 1.1.

By separation of variables we see that the spectrum of the operator K_o is composed of semi-infinite branches of essential (continuous) spectrum originating at the points $\{-m^2\omega^2\ ;\ m \in \mathrm{Z}\}$. We expect that the essential spectrum of K_o will be stable under the perturbation W_φ (this would certainly be true if W_φ was compact relative to K_o, by Weyl's theorem [RS-IV]). Thus, due to the nonresonance condition $\lambda \neq -m^2\omega^2$ and the assumption $\lambda < 0$, there is an integer $m_o \geq 1$ such that

$$-m_o^2\omega^2 \ < \ \lambda \ < \ -(m_o - 1)^2\omega^2.$$

Then, $\lfloor\sqrt{\frac{-\lambda}{\omega^2}}\rfloor^2 = (m_o - 1)^2$. In keeping with terminology used for Schrödinger operators, we call the set $\mathcal{E}(K_\varphi) = \{-m^2\omega^2\ ;\ m \in \mathrm{Z}\}$ the *thresholds* of K_φ. Therefore, Theorem 2.2 states that eigenfunctions of K_φ are exponentially bounded with any exponent α such that α^2 is less than the distance from λ to the nearest threshold above (i.e., greater than) λ (see Figure 1).

The thresholds of K_φ are precisely the points in the spectrum of the operator $-\partial_t^2$ on $L^2(S^1_\omega)$, the operator associated to the time variable, while the branches of continuous spectrum of K_φ are due to the continuous spectrum of $-\Delta$ on $L^2(\mathrm{R}^N)$, the operator associated to the spatial variables. For other partial differential equations in space-time variables that are amenable to our method, the nature of the spectrum of the associated operator K_φ will depend upon the nature of the spectrum of the operators associated to the spatial and

Figure 1: (Rotated) Spectrum of K_φ

time variables. Consequently, the statements and conclusions of the above theorems will change accordingly. For example, in the case of the nonlinear Schrödinger equation (NLS) $i\partial_t\varphi - \Delta\varphi + g(|\varphi|)\varphi = 0$, the spectrum of $i\partial_t$ on $L^2(S^1_\omega)$ is $\{m\omega\ ;\ m \in \mathrm{Z}\}$. Therefore, the spectrum of $i\partial_t - \Delta$ on $L^2(\mathrm{R}^N \times S^1_\omega)$ has branches of continuous spectrum originating at each integral multiple of ω. This conforms with the fact that periodic solutions of NLS with arbitrarily large (negative) frequencies are possible, while we expect exponential bounds of the type discussed above for such solutions.

Outline of the proof of Theorems 1.1 and 2.2:

We will omit the subscript φ when discussing the operators K_φ and W_φ so that from now on $K \equiv K_\varphi$ and $W \equiv W_\varphi$. We begin by presenting an heuristic argument that delineates the structure of the proof of Theorem 2.2. Let $\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ and suppose that $K\psi = \lambda\psi$ for some $\lambda < 0$, $\lambda \neq -m^2\omega^2$, $m \in \mathrm{Z}$. For $R > 0$ and $\delta \geq 0$ set

$$\begin{aligned} \psi_R &= \chi_R e^{\delta h(r)}\psi, \quad \text{and} \\ K^h &= e^{\delta h(r)} K e^{-\delta h(r)} = K - \delta^2 |\nabla h|^2 + i\delta\gamma_h. \end{aligned}$$

Here $h(r) = 0$, $r < 2R$, $h(r) = r +$ *const*, $r > 3R$, and $\gamma_h = \frac{1}{i}(\nabla h \cdot \nabla + \nabla \cdot \nabla h)$. χ_R is a smooth cut-off function: $\chi_R(r) = 0$, $r < R$ and $\chi_R(r) = 1$, $r > 2R$. The important features of the function h are that $h = 0$ on $supp\,(\chi'_R)$, $h(r) = r$ near infinity, and $|h^{(m)}(r)| \leq c_m R^{1-m}$, c_m independent of R.

Our goal is to show that $\psi_R \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for some $\delta > 0$. In the rigorous analysis we will regularize ψ_R by cutting-off the function h near infinity. We will regain $e^{\delta h(r)}\psi$ in a limiting procedure at the end. For the present discussion, however, we will not regularize h and consequently we will be ignoring the possibiliy that ψ_R, as defined above, may not lie in $L^2(S^1_\omega, H^1(\mathrm{R}^N))$. This is just so we can elucidate the ideas behind the proof, the heuristics presented here will illustrate all the essential ideas of the full proof.

The significance of the operator K^h is that $e^{\delta h(r)}\psi$ is an eigenfunction of K^h corresponding to the eigenvalue λ. The factor χ_R causes ψ_R not to be a bona fide eigenfunction of K^h, but it is an approximate eigenfunction in the sense that

$$\|(K^h - \lambda)\psi_R\| \;\leq\; o_R(1)\|\psi\|_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} \tag{2.4}$$

where $o_R(1)$ denotes a quantity that vanishes as $R \to \infty$. This follows from the formula

$$\begin{aligned}(K^h - \lambda)\psi_R \;&=\; e^{\delta h(r)}\chi_R(K - \lambda)\psi \;+\; e^{\delta h(r)}[-\Delta,\, \chi_R]\psi \\ &=\; (-\Delta\chi_R)\psi \;-\; 2\nabla\chi_R \cdot \nabla\psi.\end{aligned} \tag{2.5}$$

To arrive at (2.4) from this we have used that $h = 0$ on $supp\,(\chi_R')$, and that $|\chi_R^{(m)}| \leq cR^{-m}$.

Let $A = \frac{1}{2i}(x \cdot \nabla + \nabla \cdot x)$. Since $e^{\delta h(r)}\psi$ is an eigenfunction of K^h, we have that

$$0 \;=\; Im\,\langle (K^h - \lambda)e^{\delta h(r)}\psi,\; Ae^{\delta h(r)}\psi\rangle.$$

This equation is related to the virial theorem of quantum mechanics [CFKS]. Expanding the inner product,

$$\begin{aligned}0 \;&=\; Im\,\langle (K^h - \lambda)e^{\delta h(r)}\psi,\; Ae^{\delta h(r)}\psi\rangle \\ &=\; \frac{1}{2}\langle i[K,\, A]\rangle_{e^{\delta h(r)}\psi} \;+\; \delta Re\,\langle \gamma_h A\rangle_{e^{\delta h(r)}\psi} - \frac{\delta^2}{2}\langle i[|\nabla h|^2,\, A]\rangle_{e^{\delta h(r)}\psi}.\end{aligned}$$

If we substitute ψ_R for $e^{\delta h(r)}\psi$ in this equation the left hand side is no longer zero, but since $(K^h - \lambda)\psi_R$ is localized to the support of χ_R' (cf. (2.5)) where $h = 0$, we have instead that

$$|Im\,\langle (K^h - \lambda)\psi_R,\, A\psi_R\rangle| \;\leq\; c\|\psi\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))},$$

where c is independent of R. Furthermore, since $\gamma_h = \sqrt{h'r^{-1}}A\sqrt{h'r^{-1}}$, a simple calculation gives

$$\begin{aligned} Re\,\langle \gamma_h A\rangle_{\psi_R} &= Re\,\langle \sqrt{h'r^{-1}}A\sqrt{h'r^{-1}}A\rangle_{\psi_R} \\ &= \langle \sqrt{h'r^{-1}}A^2\sqrt{h'r^{-1}}\rangle_{\psi_R} + Re\,\langle \sqrt{h'r^{-1}}A\,[\sqrt{h'r^{-1}},\, A]\rangle_{\psi_R} \\ &= \langle \sqrt{h'r^{-1}}A^2\sqrt{h'r^{-1}}\rangle_{\psi_R} + \frac{1}{2}\langle \sqrt{h'r^{-1}}[A,\,[\sqrt{h'r^{-1}},\, A]\,]\rangle_{\psi_R} \\ &= \text{ positive term } + o_R(1)\|\psi_R\|^2, \end{aligned}$$

where at the bottom we have used that $\sqrt{h'r^{-1}}[A,\,[\sqrt{h'r^{-1}},\, A]\,] \leq cr^{-1}$. We calculate

$$\begin{aligned} \Big|[|\nabla h|^2,\, A]\Big| &= \Big|2h'h''r\Big| \\ &\leq c, \end{aligned}$$

independently of R (here we are using the property $h''(r) = 0$ for $r > c'R$ for some fixed c', which will hold in the rigorous analysis), so that

$$\Big|\frac{\delta^2}{2}\langle i[|\nabla h|^2,\, A]\rangle_{\psi_R}\Big| \;\leq\; c\delta^2\|\psi_R\|^2.$$

These relations yield

$$\langle i[K, A]\rangle_{\psi_R} - o_R(1)\|\psi_R\|^2 - c\delta^2\|\psi_R\|^2 \;\leq\; c\|\psi\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))}. \tag{2.6}$$

If we can now show positivity of the commutator $i[K, A]$ evaluated on ψ_R,

$$\langle i[K,\, A]\rangle_{\psi_R} \;\geq\; \theta\|\psi_R\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))}, \quad \text{for some } \theta > 0, \tag{2.7}$$

then from (2.6) and (2.7) it follows that for R sufficiently large and δ sufficiently small,

$$\|\psi_R\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} \;\leq\; c\|\psi\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} \;<\; \infty.$$

Hence, $e^{\delta r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$. Although δ must be taken sufficiently small, we will be sure to show that it is still strictly positive.

Some effort is required to establish the estimate (2.7). Evaluating the commutator

$$i[K,\, A] \;=\; -2\Delta - x\cdot\nabla W$$

and writing $-\Delta = K - \partial_t^2 - W$, we have

$$\langle i[K,\, A]\rangle_{\psi_R} \;=\; \langle -\Delta\rangle_{\psi_R} + \langle K-\lambda\rangle_{\psi_R} \;+\; \langle \lambda\rangle_{\psi_R} \;+\; \langle -\partial_t^2\rangle_{\psi_R} \;-\; \langle W + x\cdot\nabla W\rangle_{\psi_R}.$$

Now,

$$K - \lambda \;=\; K^h - \lambda + \delta^2|\nabla h|^2 - i\delta\gamma_h, \tag{2.8}$$

and since $K-\lambda$ is self-adjoint and $i\gamma_h$ is skew-adjoint,

$$\langle K-\lambda\rangle_{\psi_R} \;=\; Re\,\langle K^h-\lambda\rangle_{\psi_R} + \delta^2\langle|\nabla h|^2\rangle_{\psi_R}$$

from which we derive (recalling (2.4))

$$\begin{aligned}
|\langle K-\lambda\rangle_{\psi_R}| \;&\leq\; |\langle K^h-\lambda\rangle_{\psi_R}| + c\delta^2\|\psi_R\|^2\\
&\leq\; \|(K^h-\lambda)\psi_R\|\,\|\psi_R\| + c\delta^2\|\psi_R\|^2\\
&\leq\; o_R(1)\|\psi\|_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}\,\|\psi_R\| + c\delta^2\|\psi_R\|^2\\
&\leq\; o_R(1)\Big[\|\psi\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} + \|\psi_R\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}\Big] + c\delta^2\|\psi_R\|^2.
\end{aligned}$$

By Lemma 2.1, W and $x \cdot \nabla W$ both vanish as $|x| \to \infty$ uniformly in t. We use this property and the fact that ψ_R is supported outside of a ball of radius $2R$ to obtain the estimate

$$|\langle W + x \cdot \nabla W \rangle_{\psi_R}| \;\; \leq \;\; o_R(1)\|\psi_R\|^2.$$

Therefore, we have the inequality

$$\begin{aligned}\langle i[K,\, A]\rangle_{\psi_R} \;\; \geq \;\; & \|\nabla \psi_R\|^2 + \lambda \|\psi_R\|^2 \; + \; \langle -\partial_t^2 \rangle_{\psi_R} \\ & -o_R(1)\Big[\|\psi\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} + \|\psi_R\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))}\Big] - c\delta^2\|\psi_R\|^2. \end{aligned} \tag{2.9}$$

To achieve (2.7) we require that

$$\langle -\partial_t^2 \rangle_{\psi_R} \;\; \geq \; (-\lambda + \nu)\|\psi_R\|^2, \quad \nu > 0$$

up to a remainder term that we can control. In fact, due to the discrete nature of the spectrum of $-\partial_t^2$, we can show that

$$\langle -\partial_t^2 \rangle_{\psi_R} \; \geq \; m_o^2 \omega^2 \|\psi_R\|^2, \tag{2.10}$$

up to a remainder term, where $m_o \geq 1$ is the integer characterized by the relation $-m_o\omega^2 < \lambda < -(m_o - 1)^2\omega^2$. To prove (2.10) we write

$$\psi_R \;\; = \;\; \bar{\Pi}_{m_o-1}\psi_R \; + \; \Pi_{m_o-1}\psi_R$$

where Π_m denotes projection onto the first m modes and $\bar{\Pi}_m = \mathbf{1} - \Pi_m$ (as defined at the end of Section 1). Note that $\bar{\Pi}_{m_o-1}\psi_R$ satisfies the estimate (2.10). Using that $-\partial_t^2\bar{\Pi}_{m_o-1} \geq m_o^2\omega^2\bar{\Pi}_{m_o-1}$ and that $-\partial_t^2\Pi_{m_o-1} \geq 0$ (these inequalities are in the sense of quadratic forms), we obtain

$$\begin{aligned}\langle -\partial_t^2 \rangle_{\psi_R} \;\; & = \;\; \langle -\partial_t^2 \rangle_{\bar{\Pi}_{m_o-1}\psi_R} \;\; + \;\; \langle -\partial_t^2 \rangle_{\Pi_{m_o-1}\psi_R} \\ & \geq \;\; m_o^2\omega^2 \langle \bar{\Pi}_{m_o-1} \rangle_{\psi_R}\end{aligned}$$

$$= \; m_o^2\omega^2\|\psi_R\|^2 \; - \; m_o^2\omega^2\langle\Pi_{m_o-1}\rangle_{\psi_R}. \tag{2.11}$$

To estimate the second term on the right hand side we proceed as follows.

If $P_k\psi = 0$, $|k| \le m_o - 1$, then (2.10) is achieved because $\Pi_{m_o-1}\psi_R = 0$. This will be the situation in Theorem 1.1 (as we will see). Otherwise, pick an interval $I \subset \mathrm{R}$ containing λ and such that $\sup(I) < -(m_o-1)^2\omega^2$. Let $E_I(K)$ be a smoothed-out spectral projection of K corresponding to the interval I (as described at the end of Section 1) and decompose Π_{m_o-1} with respect to $E_I(K)$ and $\bar{E}_I(K)$,

$$\begin{aligned}\Pi_{m_o-1} &= \Big(E_I(K)+\bar{E}_I(K)\Big)\Pi_{m_o-1}\Big(E_I(K)+\bar{E}_I(K)\Big) \\ &= E_I(K)\Pi_{m_o-1}E_I(K) \; + \; \bar{E}_I(K)\Pi_{m_o-1}E_I(K) \\ &\quad + E_I(K)\Pi_{m_o-1}\bar{E}_I(K) \; + \; \bar{E}_I(K)\Pi_{m_o-1}\bar{E}_I(K).\end{aligned} \tag{2.12}$$

From this and the Schwarz inequality we then have the bound

$$\Big|\langle\Pi_{m_o-1}\rangle_{\psi_R}\Big| \;\le\; 3\|\bar{E}_I(K)\psi_R\|\|\psi_R\| \; + \; \langle C\rangle_{\psi_R}, \tag{2.13}$$

where $C = E_I(K)\Pi_{m_o-1}E_I(K)$ is a compact operator, as we will see shortly. Because ψ_R has support that goes off to infinity as $R \to \infty$, this term is of order $o_R(1)\|\psi_R\|^2$ (Lemma 3.3 below).

To prove that the operator $E_I(K)\Pi_{m_o-1}E_I(K)$ is compact, we need some kind of relative compactness of W. To this end we take advantage of the natural decomposition of K_o along the eigenspaces E_k:

$$K_o \;=\; \oplus_{k\in\mathrm{Z}}(-k^2\omega^2 - \Delta),$$

from which it follows that

$$(K_o - z)^{-1} \;=\; \oplus_{k\in\mathrm{Z}}(-k^2\omega^2 - \Delta - z)^{-1}, \quad z \in \mathrm{C}\setminus\mathrm{R}.$$

Our assumptions on the solution φ guarantees that each mode $W_k(x)$ of W is compact relative to $-\Delta$ as an operator of multiplication on $L^2(\mathrm{R}^N)$. As a result, W is compact relative to K_o

when restricted to finitely many of the subspaces E_k. Introducing the spectral projections $E_I(K_o)$ associated to the operator K_o, we write

$$\begin{aligned} E_I(K)\Pi_{m_o-1}E_I(K) &= E_I(K_o)\Pi_{m_o-1}E_I(K_o) \\ &\quad + \big(E_I(K)-E_I(K_o)\big)\Pi_{m_o-1}E_I(K) \\ &\quad + E_I(K_o)\Pi_{m_o-1}\big(E_I(K)-E_I(K_o)\big). \end{aligned} \tag{2.14}$$

The first term on the right hand side is zero by conservation of energy. That is,

$$P_kE_I(K_o)=0 \quad \text{for } k<m_o. \tag{2.15}$$

This relation can be seen as follows. On $Ran\,P_k=\mathrm{E}_k,\;\; K_o=-k^2\omega^2-\Delta$ so that

$$P_kE_I(K_o) \;=\; E_I(-k^2\omega^2-\Delta).$$

Since $\sup(I)<-k^2\omega^2$, $\;spec\,(-k^2\omega^2-\Delta)=[-k^2\omega^2,\infty)$ is disjoint from I. Hence, $E_I(-k^2\omega^2-\Delta)=0$.

To treat the other two terms on the right hand side of (2.14) it is enough to consider the resolvents $R(z)=(K-z)^{-1}$ and $R_o(z)=(K_o-z)^{-1}$ in place of the projections $E_I(K)$ and $E_I(K_o)$. For the second term on the right, say, and using the second resolvent equation, we have, for any $m_1\in\mathrm{N}$,

$$R(z)WR_o(z)\Pi_{m_o-1}E_I(K) \;=\; R(z)\Pi_{m_1}WR_o(z)\Pi_{m_o-1}E_I(K)+R(z)\bar{\Pi}_{m_1}WR_o(z)\Pi_{m_o-1}E_I(K). \tag{2.16}$$

By the relative compactness of W, $\;WR_o(z)$ is a compact operator on each E_k, and so $\Pi_{m_1}WR_o(z)\Pi_{m_o-1}$ is a compact operator for each $m_1\in\mathrm{N}$ since it acts on finitely many E_k. Thus the first term on the right hand side of equation (2.16) is compact. By taking m_1 sufficiently large we can make the second term arbitrarily small in norm. This can be seen heuristically by noting that if W is time independent and if $m_1>m_o-1$ then, because W will commute with the projections P_k while $R_o(z)$ commutes with the P_k anyways, $\bar{\Pi}_{m_1}WR_o(z)\Pi_{m_o-1}=0$. The time dependence of W couples the space and time variables and can bridge the gap between $\bar{\Pi}_{m_1}$ and Π_{m_o-1}, but we can estimate this by writing

$$\begin{aligned} \bar{\Pi}_{m_1} W R_o(z) \Pi_{m_o-1} &= \partial_t^{-1} \partial_t \bar{\Pi}_{m_1} W R_o(z) \Pi_{m_o-1} \\ &= \partial_t^{-1} \bar{\Pi}_{m_1} (\partial_t W) R_o(z) \Pi_{m_o-1} \;+\; \partial_t^{-1} \bar{\Pi}_{m_1} W R_o(z) \partial_t \Pi_{m_o-1}. \end{aligned}$$

If $\partial_t W$ is bounded relative to K_o, as will be the case when, in particular, $\partial_t W$ is a bounded function, then $(\partial_t W) R_o(z)$ is a bounded operator. Combining this with the estimates

$$\|\partial_t^{-1} \bar{\Pi}_{m_1}\| \;\leq\; 1/m_1, \quad \text{and} \quad \|\partial_t \Pi_{m_o-1}\| \;\leq\; m_o - 1,$$

we see that $\|\bar{\Pi}_{m_1} W R_o(z) \Pi_{m_o-1}\|$ can be made arbitrarily small by taking m_1 sufficiently large. Therefore, referring to (2.16), $R(z) W R_o(z) \Pi_{m_o-1} E_I(K)$, and hence $\big(E_I(K) - E_I(K_o)\big) \Pi_{m_o-1} E_I(K)$ is compact.

Going back to (2.13), we use the fact that ψ is an eigenfunction of K corresponding to the eigenvalue λ to show that ψ_R is essentially localized in I with respect to the spectral decomposition of K, i.e., that $\|\bar{E}_I(K)\psi_R\|$ is small. More precisely, we will estimate

$$\begin{aligned} \|\bar{E}_I(K)\psi_R\| \, \|\psi_R\| \;&\leq\; d^{-1} o_R(1) \Big[\|\psi\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} + \|\psi_R\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} \Big] \\ &\quad + d^{-1} \delta(\delta+1) \|\psi_R\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} \end{aligned} \tag{2.17}$$

where $d = \mathit{dist}\,(\partial I,\, \lambda)$. This follows from the formula, derived using the functional calculus,

$$\|\bar{E}_I(K)\psi_R\| \;\leq\; d^{-1} \|(K - \lambda)\psi_R\|,$$

and the estimate

$$\|(K - \lambda)\psi_R\| \;\leq\; o_R(1) \|\psi\|_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} \;+\; \delta(\delta+1) \|\psi_R\|_{L^2(S^1_\omega, H^1(\mathrm{R}^N))}$$

which follows from (2.4), (2.8) and the triangle inequality.

We want to emphasize that here we are localizing simultaneously in two non-commuting operators; in K and in $i\partial_t$. That is, we are using the spectral projections associated with

Figure 2: Phase space decomposition of $\langle -\partial_t^2 \rangle_{\psi_R}$.

K and $i\partial_t$, $E_I(K)$ and P_k respectively, to decompose phase space $(= L^2(S^1_\omega, H^1(\mathrm{R}^N)))$ into regions where $-\partial_t^2$ has certain properties. In particular, on $Ran\,\bar{\Pi}_{m_o-1}$, $-\partial_t^2 \geq m_o^2\omega^2$, i.e., is strictly positive, while on $Ran\,\Pi_{m_o-1}$, $-\partial_t^2 \leq (m_o-1)^2\omega^2$, i.e., is bounded. We further decompose $Ran\,\Pi_{m_o-1}$ according to the subspaces $Ran\,E_I(K)$ and $Ran\,\bar{E}_I(K)$ where Π_{m_o-1} acts either as a compact operator $(E_I(K)\Pi_{m_o-1}E_I(K))$, or else is proportional to $\bar{E}_I(K)$ $(\bar{E}_I(K)\Pi_{m_o-1}E_I(K)$, etc.); see Figure 2. Either of these latter cases lead to expectation values, evaluated at ψ_R, that can be made arbitrarily small by varying the parameters R and δ associated to ψ_R. This illustrates another important aspect of our analysis: using compactness in this way allows us to do away with any smallness assumption on the solution φ.

Combining (2.11), (2.13) and (2.17), we have that

$$\begin{aligned}\langle -\partial_t^2 \rangle_{\psi_R} \;\geq\; & m_o^2\omega^2\|\psi_{\cdot}\|^2 \;-\; d^{-1}o_R(1)\Big[\|\psi\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} + \|\psi_R\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}\Big] \\ & - o_R(1)\|\psi_R\|^2 \;-\; d^{-1}\delta(\delta+1)\|\psi_R\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}, \end{aligned} \tag{2.18}$$

and so

$$\langle i[K,\,A]\rangle_{\psi_R} \;\geq\; b\|\psi_R\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} \;-\; (d^{-1}+1)o_R(1)\Big[\|\psi\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} + \|\psi_R\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}\Big] - d^{-1}\delta(\delta+1)\|\psi_R\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} \tag{2.19}$$

where $b = \min{(\lambda + m_o^2\omega^2,\,1)} > 0$. Thus we have achieved (2.7) for R sufficiently large and δ sufficiently small.

Since δ must be taken sufficiently small we cannot prove arbitrarily large exponential bounds at once. Therefore we will iterate this procedure to obtain greater bounds.

Regularization and iterative scheme

To make these heuristics rigorous we cut-off the function $h(r) = h_R(r)$ at infinity so that $e^{\delta h(r)}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$. The cut-off depends on a parameter ε: $h(r) = h_{R,\varepsilon}(r)$, and is such that $\lim_{\varepsilon\to 0} h_{R,\varepsilon}(r) = r + const.$ for $r > 3R$. Our regularized function is then $\chi_R e^{\delta h_{R,\varepsilon}(r)}\psi$ where χ_R is a cut-off function with support in a neighborhood of infinity. We then show, following the ideas outlined in the heuristics above, that for R sufficiently large and δ sufficiently small but nonzero, $\|\chi_R e^{\delta h_{R,\varepsilon}(r)}\psi\|_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} < c < \infty$ uniformly in ε. By taking the limit $\varepsilon \to 0$ we conclude that $e^{\delta r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$.

We discuss the regularization procedure. For $R > 1$ let $\chi_R(r)$ be a smooth function such that $\chi_R = 0$ for $r \leq R$, $\chi_R = 1$ for $r \geq 2R$ and $|\,\chi_R^{(m)}\,| \leq c_m R^{-m}$. For $\varepsilon > 0$ and the same R let $h(r) = h_{R,\varepsilon}(r)$ be a smooth function such that

$$\begin{aligned} h(r) &= 0, & r &\leq 2R \\ &= r - 3R + 1, \quad 3R < & r &< 3R + \tfrac{1}{\varepsilon} \\ &= 2 + \tfrac{1}{\varepsilon}, & r &\geq 2(3R + \tfrac{1}{\varepsilon}), \end{aligned}$$

with h defined on the intervals $[2R,\,3R]$ and $[3R + \frac{1}{\varepsilon},\; 2(3R + \frac{1}{\varepsilon})]$ in such a way that

$$\begin{aligned} |\,h^{(m)}(r)\,| &\leq c'_m R^{1-m}, & 2R < \; r &< 3R \\ &\leq c''_m (3R + \tfrac{1}{\varepsilon})^{1-m}, \quad & (3R + \tfrac{1}{\varepsilon}) < \; r &< 2(3R + \tfrac{1}{\varepsilon}) \end{aligned}$$

Figure 3: The functions $\chi_R(r)$ and $h_{R,\varepsilon}(r)$

(see Figure 3). We remark that the constants $c_m, c_m^{'}$ and $c_m^{''}$ are independent of R and ε. The function h_R we define as

$$h_R(r) \;=\; \lim_{\varepsilon\to 0} h_{R,\varepsilon}(r)$$

and is equal to $r + const.$ in a neighborhood of infinity.

The method outlined above secures *some* exponential bound δ for ψ. To achieve a better bound we iterate this method, incrementally approaching the optimal bound. We begin the iteration by assuming that $\psi_\alpha \equiv e^{\alpha r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for some $\alpha \geq 0$. To prove that there exists a $\delta > 0$ such that $e^{\delta r}\psi_\alpha \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$, we regularize the function $e^{\delta r}\psi_\alpha$ as $\chi_R e^{\delta h(r)}\psi_\alpha \equiv \xi_\alpha$ which was described above. We then show, in the same way as for $\chi_R e^{\delta h(r)}\psi$, that for R sufficiently large and δ sufficiently small but nonzero, call this $\delta(\alpha)$, that $\|\chi_R e^{\delta(\alpha)h(r)}\psi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} \;<\; \infty$ uniformly in ε (the ε appearing in $h = h_{R,\varepsilon}$). Therefore, after taking the limit $\varepsilon \to 0$, we conclude that $e^{\delta(\alpha)r+\alpha r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ and hence our new exponential bound for ψ has exponent $\alpha + \delta(\alpha)$. Now we set $\psi_{\alpha_1} = e^{\alpha_1 r}\psi$, where $\alpha_1 = \alpha + \delta(\alpha)$, regularize this function as $\chi_R e^{\delta h(r)}\psi_{\alpha_1}$, and repeat the above analysis to determine $\alpha_2 = \alpha_1 + \delta(\alpha_1)$. Finally, we show that as $n \to \infty$, $\lambda + \alpha_n^2 \to -(m_o - 1)^2\omega^2$ if $\lambda < 0$, or else becomes arbitrarily large if $\lambda > 0$ (recall that $m_o \geq 1$ is the largest integer m such that $\lambda < -(m-1)^2\omega^2$).

Figure 4: Spectrum of K_φ when $\omega^2 > f'(0)$.

The next main result after Theorem 2.2 is a sort of unique continuation theorem at infinity (Theorem 5.1). It states that if eigenfunctions of K are exponentially bounded with arbitrarily large exponent, then the function is zero. This is used in the proof of Theorem 1.1. However, with Theorem 1.1 we work with the time dependent part of φ: $\varphi - P_0\varphi \equiv \bar{\varphi}$ (recall that $\big(P_o\varphi\big)(x) = (2\pi/\omega)^{-1}\int_{S^1_\omega}\varphi(x,t)\,dt$). We first show that $\bar{\varphi}$ is an eigenfunction of an operator $\bar{K}$ which is constructed in a similar way as was K after first projecting NLW onto $\mathrm{E}_0^\perp$: $\bar{K}\bar{\varphi} = -\kappa\bar{\varphi}$, where $\kappa = f'(0)$. From this we prove, along analogous lines as outlined in the heuristics above, that $e^{\alpha r}\bar{\varphi} \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all α. We are able to show this because by using $\bar{K}$ instead of K we have removed the only threshold above $-\kappa$; the threshold 0 (that this is the only threshold above $-\kappa$ follows from the assumption that $\omega^2 > f'(0)$), see Figure 4. The unique continuation theorem holds also for $\bar{K}$ from which it follows that $\bar{\varphi} = 0$. Therefore $\varphi = P_0\varphi$. That is, φ is independent of time.

The proof of Theorem 1.1 is less involved than the proof of Theorem 2.2 because we do not require a microlocalization of $\langle -\partial_t^2\rangle_\psi$ (equation (2.11) and the subsequent analysis). In Theorem 1.1, we will be dealing with the function $\bar{\varphi} = \varphi - P_0\varphi$ so that $\langle -\partial_t^2\rangle_{\bar{\varphi}} \geq \omega^2\|\bar{\varphi}\|^2$. Therefore, equation (2.10) is satisfied with $m_o = 1$, which is sufficient for the purposes considered in the situation of Theorem 1.1. For the proof of Theorem 1.1, then, the reader may proceed directly to Section 5.2 for a (mostly) self-contained proof.

3 Preliminary Results

In this section we present a series of preliminary results that will be used in the proof of Theorem 2.2. We remind the reader that we will be denoting the operators K_φ and W_φ simply as K and W. The symbol $o_R(1)$ will denote a positive function that depends only on x and vanishes as $R \to \infty$. Positive constants will be denoted generically by c and will always be independent of the parameters R, δ, ε and α. In this way the explicit dependence on these parameters of the estimates will be apparent.

The aim of this section is to prove the estimates (2.7) and (2.10) - this is carried out in Propositions 3.8 and 3.7 respectively. Recall that we are considering an eigenfunction ψ of K corresponding to the eigenvalue λ: $K\psi = \lambda\psi$. We assume that for some $\alpha \geq 0$, $\psi_\alpha \equiv e^{\alpha r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ and for $\delta > 0$ we define $\xi_\alpha \equiv \chi_R e^{\delta h(r)}\psi_\alpha$ where the functions $\chi_R(r)$ and $h(r)$ have been described in the previous section.

3.1 Spectral Localization

Here we determine the localization of the function ξ_α with respect to the spectral decomposition of K. Since $K\psi = \lambda\psi$, if we set $E_I = E_{\{\lambda\}}$, then $\bar{E}_{\{\lambda\}}\psi = 0$. There is no reason to expect that ξ_α should also be so well localized. However, it will be enough if we can obtain an upper bound for its localization. This will follow from Lemma 3.2 below via the calculation described presently.

From spectral theory and noting that for $\beta \in I$, $(K - \beta)$ is invertible on $\bar{E}_I$, we compute $\bar{E}_I\psi$ with the formula

$$\bar{E}_I = \bar{E}_I(K-\beta)^{-1}(K-\beta) = \int_{\mu \notin I} (\mu - \beta)^{-1}(\mu - \beta)\, dE_\mu, \tag{3.1}$$

where dE_μ is the spectral measure on R associated to K. This leads to the estimate

$$\|\bar{E}_I\psi\| \leq (dist\,(\partial I,\ \beta))^{-1}\, \|(K-\beta)\psi\|. \tag{3.2}$$

The exponential factor $e^{\alpha r}$ effectively boosts the energy (eigenvalue) of ψ from λ to $\lambda + \alpha^2$ (this can be seen from equation (3.10) below; see also Section 5.1) so that we expect ξ_α to be localized near $\lambda+\alpha^2$. Lemma 3.2 provides an estimate for $\|(K-\lambda-\alpha^2)\xi_\alpha\|$, but before that we require the following estimate.

Lemma 3.1 *Let $K^H = e^H K e^{-H}$ where, for $\delta > 0$ and $\alpha \geq 0$, $H(r) = \delta h(r) + \alpha r$ with $h(r)$ as defined in Section 2, and let ψ_α and ξ_α be as defined above. Then*

$$\|(K^H - \lambda)\xi_\alpha\| \leq (1+\alpha) o_R(1) \|\psi_\alpha\|_{L^2(S^1_\omega, H^1(\mathrm{R}^N))}. \tag{3.3}$$

Proof:

$$\begin{aligned} (K^H-\lambda)\xi_\alpha &= e^{H(r)}\chi_R(K-\lambda)\psi \;+\; e^{H(r)}[K,\ \chi_R]\psi \\ &= e^{\alpha r}[-\Delta, \chi_R]\psi \\ &= e^{\alpha r}\left((-\Delta\chi_R)\psi - 2\nabla\chi_R\cdot\nabla\psi\right) \\ &= (-\Delta\chi_R)\psi_\alpha - 2\nabla\chi_R\cdot\nabla\psi_\alpha + 2\alpha\nabla\chi_R\cdot\hat{x}\psi_\alpha. \end{aligned} \tag{3.4}$$

Here we have used that $(K-\lambda)\psi = 0$, that $h=0$ on *supp* $(\chi_R^{'})$, and that

$$e^{\alpha r}\nabla\psi = \nabla\psi_\alpha - \alpha\hat{x}\psi_\alpha. \tag{3.5}$$

Let c_1 and c_2 be constants (independent of R) that satisfy the inequalities

$$|\ \chi_R^{'}\ | \leq c_1 R^{-1}, \quad |\ \chi_R^{''}\ | \leq c_2 R^{-2}. \tag{3.6}$$

Combining this with (3.4) and noting that $\nabla\chi_R = \chi_R^{'}\hat{x}$, and $\Delta\chi_R = \chi_R^{''} + \frac{N-1}{r}\chi_R^{'}$, we have,

$$\|(K^H-\lambda)\xi_\alpha\| \leq (c_2 + (N-1)c_1/2)R^{-2}\|\psi_\alpha\| + 2c_1R^{-1}\|\nabla\psi_\alpha\| + 2c_1\alpha R^{-1}\|\psi_\alpha\| \tag{3.7}$$

from which (3.3) follows $\square$

Lemma 3.2

$$\|(K-\lambda-\alpha^2)\xi_\alpha\| \;\leq\; \mu(\delta,\alpha)o_R(1)\Big[\|\psi_\alpha\|_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} \;+\; \|\xi_\alpha\|_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}\Big]$$

$$+\ 2\delta(\delta+2\alpha)\|\xi_\alpha\| \tag{3.8}$$

where μ is an increasing function of both variables.

Proof:

Recall from the previous lemma that $K^H = e^H K e^{-H}$ where $H(r) = \delta h(r) + \alpha r$. Pulling e^{-H} through K we find

$$K^H \ = \ K - |\nabla H|^2 \ + i\gamma_H \tag{3.9}$$

where $\gamma_H = \frac{1}{i}(\nabla H \cdot \nabla + \nabla \cdot \nabla H)$. From the relations

$$\begin{aligned} K^H - \lambda \ &= \ K - \lambda - |\nabla H|^2 + i\gamma_H \\ &= \ K - \lambda - \alpha^2 - \delta^2 \mid \nabla h \mid^2 - 2\delta\alpha\nabla h \cdot \hat{x} + i\gamma_H \end{aligned} \tag{3.10}$$

and

$$\|(K^H - \lambda)\xi_\alpha\|^2 = \langle (K^H - \lambda)\xi_\alpha, \ \ (K^H - \lambda)\xi_\alpha \rangle, \tag{3.11}$$

we obtain

$$\begin{aligned} \|(K^H - \lambda)\xi_\alpha\|^2 \ &= \ \|(K - \lambda - \alpha^2)\xi_\alpha\|^2 + \|(\delta^2(h')^2 + 2\delta\alpha h' - i\gamma_H)\xi_\alpha\|^2 \\ &\quad -2Re\,\langle (K - \lambda - \alpha^2)\xi_\alpha, \ (\delta^2(h')^2 + 2\delta\alpha h')\xi_\alpha \rangle \\ &\quad +\langle i[K, \ \gamma_H] \rangle_{\xi_\alpha}. \end{aligned} \tag{3.12}$$

Here we have written $|\nabla h|^2 = (h')^2$ and $\nabla h \cdot \hat{x} = h'$. Since we are interested in an upper bound to $\|(K - \lambda - \alpha^2)\xi_\alpha\|$ we need only estimate the nonpositive terms on the right.

Because $h' \leq 1$, independently of ε and R, from the Schwarz inequality we have that

$$\left| 2Re\,\langle (K - \lambda - \alpha^2)\xi_\alpha, \ (\delta^2(h')^2 + 2\delta\alpha h')\xi_\alpha \rangle \right| \ \leq \ 2\delta(\delta + 2\alpha)\|(K - \lambda - \alpha^2)\xi_\alpha\| \ \|\xi_\alpha\|. \tag{3.13}$$

Expanding the commutator in the last line of (3.12) we obtain

$$i[K, \gamma_H] \ = \ i\delta[K, \gamma_h] \ + i\alpha\,[K, \gamma] \tag{3.14}$$

where $\gamma_h = \frac{1}{i}(\nabla h \cdot \nabla + \nabla \cdot \nabla h)$ and $\gamma = \frac{1}{i}(\nabla r \cdot \nabla + \nabla \cdot \nabla r) \;=\; \frac{1}{i}(\hat{x} \cdot \nabla + \nabla \cdot \hat{x})$. For the first commutator we calculate,

$$\begin{aligned} i[K,\ \gamma_h] &= i[-\Delta + W,\ \gamma_h] \\ &= [-\Delta,\ 2h'r^{-1}x \cdot \nabla] \;+\; [-\Delta,\ h'' + (N-1)h'r^{-1}] \\ &\qquad + [W,\ 2h'r^{-1}x \cdot \nabla]. \end{aligned} \tag{3.15}$$

Because

$$\begin{aligned} [-\Delta,\ 2h'r^{-1}x \cdot \nabla] &= 2h'r^{-1}[-\Delta,\ x \cdot \nabla] + 2[-\Delta,\ h'r^{-1}]x \cdot \nabla \\ &= -4h'r^{-1}\Delta + 4(h'r^{-3} - h''r^{-2})(x \cdot \nabla)^2 \\ &\qquad - \Big((N+3)h'r^{-3} + (3-N)h''r^{-2} + h'''r^{-1}\Big)\, x \cdot \nabla \;, \end{aligned} \tag{3.16}$$

with $|h^{(m)}(r)| \leq c_m R^{1-m}$, we have that

$$\begin{aligned} \langle [-\Delta,\, 2h'r^{-1}x \cdot \nabla] \rangle_{\xi_\alpha} &= \langle 2h'r^{-1}[-\Delta,\, x \cdot \nabla] \rangle_{\xi_\alpha} + 4\langle \nabla \cdot x\, (h'r^{-3} - h''r^{-2}) x \cdot \nabla \rangle_{\xi_\alpha} \\ &\qquad + cR^{-1}\|\xi_\alpha\|^2. \end{aligned} \tag{3.17}$$

The first term on the right hand side is positive while the second term is bounded above in absolute value by $cR^{-1}\|\nabla \xi_\alpha\|^2$. The absolute value of the real parts of the expectation values of the other terms in (3.15) can be bounded above by $o_R(1)\|\xi_\alpha\|^2$. Thus,

$$\langle i[K,\ \gamma_h] \rangle_{\xi_\alpha} \;\geq\; -o_R(1)\|\xi_\alpha\|^2_{L^2(S^1_\omega, H^1(\mathbb{R}^N))}. \tag{3.18}$$

The second commutator, $i[K,\ \gamma]$, satisfies the same estimate since $i[K,\ \gamma]$ is as (3.15) with $h' = 1$ and $h'' = 0$.

Combining the above estimates and dropping the positive terms, we have that

$$\begin{aligned} \|(K^H - \lambda)\xi_\alpha\|^2 \;&\geq\; \|(K - \lambda - \alpha^2)\xi_\alpha\|^2 - 2\delta(\delta + 2\alpha)\|(K - \lambda - \alpha^2)\xi_\alpha\|\ \|\xi_\alpha\| \\ &\qquad -(\delta + \alpha)o_R(1)\|\xi_\alpha\|^2_{L^2(S^1_\omega, H^1(\mathbb{R}^N))}. \end{aligned} \tag{3.19}$$

Rearranging, we write this as

$$\Big[\, \|(K-\lambda-\alpha^2)\xi_\alpha\| - \delta(\delta+2\alpha)\|\xi_\alpha\|\,\Big]^2 \;\le\; \|(K^H-\lambda)\xi_\alpha\|^2 + (\delta+\alpha)o_R(1)\|\xi_\alpha\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} + \Big(\delta(\delta+2\alpha)\|\xi_\alpha\|\Big)^2. \tag{3.20}$$

Taking the square root of both sides of (3.20), using Lemma 3.1 to estimate $\|(K^H-\lambda)\xi_\alpha\|$, and writing $\mu(\delta,\alpha) = \max\,\Big(\,(\alpha+\delta)^{1/2},\,(1+\alpha)\,\Big)$, we arrive at (3.8) □

3.2 Commutator Estimates

Our goal in this section is to establish an analogous estimate to (2.7) for the commutator $i[K,\,A]$ (Proposition 3.8). For this we will require the estimates on the spectral localization of ξ_α obtained in the previous subsection as well as an estimate analogous to (2.10) for the expectation value $\langle -\partial_t^2\rangle_{\xi_\alpha}$ (Proposition 3.7). In order to establish the latter we first show that certain operators constructed from the projections $E_I(K)$ and P_k are compact. Several of the following lemmata are for this purpose.

The compact operators that will arise in our analysis are not necessarily small in norm, but we take advantage of the structure of the functions ξ_α to control these terms as the following lemma demonstrates.

Lemma 3.3 *If C is a compact operator, then*

$$|\,\langle C\rangle_{\xi_\alpha}\,| \le\; o_R(1)\|\xi_\alpha\|^2. \tag{3.21}$$

Here, as elsewhere in the paper, $o_R(1)$ vanishes as $R\to\infty$ uniformly in ε and δ.

Proof:

Recall that $\xi_\alpha = \chi_R e^{\delta h(r)}\psi_\alpha$. Let F be a finite rank operator;

$$F\psi = \sum_{i=1}^{M}\langle\psi,\;\varphi_i\rangle\zeta_i, \tag{3.22}$$

for some $\varphi_i, \zeta_i \in L^2(\mathrm{R}^N \times S^1_\omega)$. By the Schwarz inequality,

$$\begin{aligned} |\langle \xi_\alpha,\ \varphi_i\rangle| &= |\langle \xi_\alpha,\ \chi_{R/2}\varphi_i\rangle| \\ &\leq \|\xi_\alpha\|\ \|\chi_{R/2}\varphi_i\| \end{aligned} \tag{3.23}$$

where the R appearing in $\chi_{R/2}$ is the R associated to ξ_α. For $\eta > 0$ choose R_i, $i = 1, \ldots, M$, sufficiently large such that for $R' \geq R_i$, $\|\chi_{R'/2}\varphi_i\| \leq \eta\|(M\|\zeta_i\|)^{-1}$. Then for $R \geq \ max\,(R_1, \ldots, R_M)$,

$$\begin{aligned} \|F\,\xi_\alpha\| &\leq \sum_{i=1}^{M} |\langle \xi_\alpha, \varphi_i\rangle|\ \|\zeta_i\| \\ &\leq \eta\,\|\xi_\alpha\| \end{aligned} \tag{3.24}$$

independently of ε and δ (the ε and δ appearing in ξ_α). Now, for an arbitrary compact operator C and any $\eta > 0$ there is a finite rank operator F_η and operator B_η with $\|B_\eta\| < \eta$, such that $C = F_\eta + B_\eta$. By the preceding analysis,

$$\begin{aligned} \|C\,\xi_\alpha\| &\leq \|F_\eta\xi_\alpha\| + \|B_\eta\,\xi_\alpha\| \\ &\leq 2\eta\|\xi_\alpha\| \end{aligned} \tag{3.25}$$

for R sufficiently large. Applying the Schwarz inequality to $\langle C\rangle_{\xi_\alpha}$, we obtain the desired result □

Lemma 3.4 *For any integers $m_1 \geq 0$, $m_2 > 0$ and any interval $I \subset \mathrm{R}$,*

$$\|\Pi_{m_1}\,E_I\,\bar{\Pi}_{m_2}\| \leq const.\left(\frac{m_1+1}{m_2}\right) \quad \text{and} \quad \|\bar{\Pi}_{m_2}E_I\Pi_{m_1}\| \leq const.\left(\frac{m_1+1}{m_2}\right). \tag{3.26}$$

Proof:

Let $g \in C_o^\infty(\mathrm{R},\mathrm{R})$ be a smoothed-out characteristic function of the interval I such that $E_I = g(K)$ as described at the end of section 1. We are denoting $E_I(K)$ simply by E_I. Noting that $\partial_t\partial_t^{-1} =$ Id on range $\bar{\Pi}_{m_2}$, we begin by writing

$$\Pi_{m_1}E_I\bar{\Pi}_{m_2} = \Pi_{m_1}E_I\partial_t\partial_t^{-1}\bar{\Pi}_{m_2}$$

$$= \quad \partial_t \Pi_{m_1} E_I \partial_t^{-1} \bar{\Pi}_{m_2} + \Pi_{m_1} [E_I, \partial_t] \partial_t^{-1} \bar{\Pi}_{m_2}. \qquad (3.27)$$

We use the operator calculus described in the appendix to express the commutator;

$$\begin{aligned} [E_I, \partial_t] &= \int_{\mathrm{R}^2} d\tilde{g}(z)\,(K-z)^{-1}\,[K, \partial_t]\,(K-z)^{-1} \\ &= -\int_{\mathrm{R}^2} d\tilde{g}(z)\,(K-z)^{-1}\ \partial_t W\ (K-z)^{-1}. \end{aligned} \qquad (3.28)$$

Here $z = u + iv$ and $\tilde{g}(z)$ is an almost analytic extension of g. As described in the appendix, $d\tilde{g}(z) = a(z)\,dudv + b(z)\,dudv$, with $a(z)$ and $b(z)$ compactly supported, $a(z)$ supported away from $v = 0$, and $\mid b(z) \mid < const\ v^2$. This, combined with the estimate $\|(K-z)^{-1}\| < v^{-1}$ and the boundedness of $\partial_t W$ imply that $\|[E_I, \partial_t]\| < \infty$. Going back to (3.27), $\|\partial_t \Pi_{m_1}\| \leq m_1 \omega$ and $\|\partial_t^{-1} \bar{\Pi}_{m_2}\| \leq 1/m_2\omega$ imply that $\|\Pi_{m_1} E_I \bar{\Pi}_{m_2}\| \leq m_1/m_2 + \|[E_I, \partial_t]\|/m_2\omega \leq const\ (\frac{m_1+1}{m_2})$. The same result holds for $\bar{\Pi}_{m_2} E_I \Pi_{m_1}$ since $(\bar{\Pi}_{m_2} E_I \Pi_m) = (\Pi_{m_1} E_I \bar{\Pi}_{m_2})^*$. □

Lemma 3.5 *For any $m_1, m_2 \in \mathrm{N}$ and $z \in \mathrm{C} \setminus \mathrm{R}$, $\Pi_{m_1} W\,(K_o - z)^{-1}\,\Pi_{m_2}$ is a compact operator on $L^2(\mathrm{R}^N \times S^1_\omega)$.*

Proof:

First note that $(K_o - z)^{-1}$ decomposes along the eigenspaces of $i\partial_t$ as

$$(K_o - z)^{-1} P_k = (-k^2\omega^2 - \Delta - z)^{-1}, \qquad (3.29)$$

and that for $\psi \in L^2(\mathrm{R}^N \times S^1_\omega)$,

$$(W\psi)_l(x) = \sum_k W_{lk}(x) \psi_k(x) \qquad (3.30)$$

where

$$W_{lk}(x) = (2\pi/\omega)^{-1} \int_{S^1_\omega} W(x,t)\, e^{-i(l-k)\omega t}\, dt.$$

Thus,

$$\Pi_{m_1} W\,(K_o - z)^{-1}\,\Pi_{m_2} = \sum_{\substack{|k| \leq m_2 \\ |l| \leq m_1}} e^{il\omega t} W_{lk}(x) \left(-k^2\omega^2 - \Delta - z\right)^{-1} P_k. \qquad (3.31)$$

The sum in (3.31) is finite so it is enough to show that $e^{il\omega t} W_{lk} \left(-k^2\omega^2 - \Delta - z\right)^{-1} P_k$ is compact for each l, k. Since $W_{lk} \in L^\infty(\mathrm{R}^N)$ and vanishes as $|x| \to \infty$ (this is a consequence of Lemma 2.1), $W_{lk} \left(-k^2\omega^2 - \Delta - z\right)^{-1}$ is a compact operator on $L^2(\mathrm{R}^N)$ (see for example [RS-IV]). Due to the relation

$$e^{il\omega t} W_{lk} \left(-k^2\omega^2 - \Delta - z\right)^{-1} = e^{i(l-k)\omega t} \otimes W_{lk}(-k^2\omega^2 - \Delta - z)^{-1} \tag{3.32}$$

on E_k, each term on the right hand side of (3.31) is compact. Thus, $\Pi_{m_1} W (K_o - z)^{-1} \Pi_{m_2}$ is compact □

Lemma 3.6 *Let $E_I = E_I(K)$ and $E_I^o = E_I(K_o)$. For any interval $I \subset \mathrm{R}$ and any integer $m \in \mathrm{N}$, $(E_I - E_I^o)\,\Pi_m$ and $\Pi_m(E_I - E_I^o)$ are compact operators on $L^2(\mathrm{R}^N \times S^1_\omega)$.*

Proof:

We prove that $(E_I - E_I^o)\Pi_m$ is compact. Since $\Pi_m(E_I - E_I^o) = ((E_I - E_I^o)\Pi_m)^*$, this will imply that $\Pi_m(E_I - E_I^o)$ is also compact.

Let $m_1 > m$. Then

$$\begin{aligned} (E_I - E_I^o)\Pi_m &= \Pi_{m_1}(E_I - E_I^o)\Pi_m + \bar{\Pi}_{m_1}(E_I - E_I^o)\Pi_m \\ &= \Pi_{m_1}(E_I - E_I^o)\Pi_m + \bar{\Pi}_{m_1}\, E_I\, \Pi_m. \end{aligned} \tag{3.33}$$

We have used the commutativity between Π_m and E_I^o in the last line. To compute the first term on the right hand side we use the operator calculus described in the appendix to represent E_I as

$$E_I = \int_{\mathrm{R}^2} d\tilde{g}(z)\, (K - z)^{-1} \tag{3.34}$$

and similarly for E_I^o (recall that $E_I = g(K)$ and $E_I^o = g(K_o)$). Next, for $m_2 > m_1$ and using the second resolvent equation, we have that

$$\begin{aligned} \Pi_{m_1}(E_I - E_I^o)\Pi_m &= \Pi_{m_1} \left\{ \int_{\mathrm{R}^2} d\tilde{g}(z) \left((K - z)^{-1} - (K_o - z)^{-1} \right) \right\} \Pi_m \\ &= \Pi_{m_1} \left\{ \int_{\mathrm{R}^2} d\tilde{g}(z)\, (K - z)^{-1}\, W\, (K_o - z)^{-1} \right\} \Pi_m \\ &= \int_{\mathrm{R}^2} d\tilde{g}(z)\, \Pi_{m_1} (K - z)^{-1}\, W\, (K_o - z)^{-1}\, \Pi_m \end{aligned}$$

$$= \int_{\mathbb{R}^2} d\tilde{g}(z)\, \Pi_{m_1}\, (K-z)^{-1}\, \Pi_{m_2}\, W\, (K_o - z)^{-1}\, \Pi_m$$
$$+ \int_{\mathbb{R}^2} d\tilde{g}(z)\, \Pi_{m_1}\, (K-z)^{-1}\, \bar{\Pi}_{m_2}\, W\, (K_o - z)^{-1}\, \Pi_m. \quad (3.35)$$

By Lemma 3.5, the first integrand on the right hand side in the last line is compact for each $z \in \mathbb{C} \setminus \mathbb{R}$. Since this integral is the norm limit of Riemann sums, each of which is a compact operator, the integral itself is a compact operator (which depends on m_1 and m_2). To estimate the second integral we write

$$\begin{aligned} &\Pi_{m_1}(K-z)^{-1}\bar{\Pi}_{m_2}\, W\, (K_o - z)^{-1}\Pi_m \\ &= \Pi_{m_1}(K-z)^{-1}\partial_t^{-1}\partial_t\bar{\Pi}_{m_2}\, W\, (K_o - z)^{-1}\Pi_m \\ &= \Pi_{m_1}(K-z)^{-1}\partial_t^{-1}\bar{\Pi}_{m_2}(\partial_t W)(K_o - z)^{-1}\Pi_m \\ &\quad + \Pi_{m_1}(K-z)^{-1}\partial_t^{-1}\bar{\Pi}_{m_2}\, W\, (K_o - z)^{-1}\partial_t\Pi_m \end{aligned} \quad (3.36)$$

where we have commuted ∂_t through $(K_o - z)^{-1}$ in the last line. Recall from Lemma 3.4 that $d\tilde{g}(z) = a(z)\, dudv + b(z)\, dudv$ with $a(z)$ and $b(z)$ as described there. This, combined with the estimates

$$\|(K-z)^{-1}\| \le v^{-1}, \quad \|(K_o - z)^{-1}\| \le v^{-1}, \quad \|\partial_t^{-1}\bar{\Pi}_{m_2}\| \le 1/m_2\omega, \quad \text{and} \quad \|\partial_t\Pi_m\| \le m\omega, \quad (3.37)$$

plus the boundedness of W and $\partial_t W$, implies that

$$\| \int_{\mathbb{R}^2} d\tilde{g}(z)\, \Pi_{m_1}(K-z)^{-1}\, \bar{\Pi}_{m_2}\, W\, (K_o - z)^{-1}\, \Pi_m\| \le c/m_2 \quad (3.38)$$

(here c depends on m but m is fixed).

Now consider the second term on the right hand side in (3.33). Applying Lemma 3.4 we find

$$\|\bar{\Pi}_{m_1}\, E_I\, \Pi_m\| \le const\, (1/m_1). \quad (3.39)$$

Therefore,

$$(E_I - E_I^o)\Pi_m = C_{m_1,m_2} + B_{m_1} + B_{m_2} \quad (3.40)$$

where C_{m_1,m_2} is compact for each m_1, m_2 and $\|B_{m_1}\| \leq const\,(1/m_1), \quad \|B_{m_2}\| \leq const\,(1/m_2)$. Since m_1 and m_2 are arbitrary, we see that $(E_I - E_I^o)\Pi_m$ is the norm limit of compact operators, and as such is itself compact $\square$

Proposition 3.7 *Let $I \subset \mathrm{R}$ and*

$$\bar{m} = \min\{m \in \mathrm{N} \mid \; -m^2\omega^2 < \sup(I)\}. \tag{3.41}$$

Then for any $\psi \in L^2(\mathrm{R}^N \times S^1_\omega)$,

$$\langle -\partial_t^2 \rangle_\psi \;\; \geq \;\; \bar{m}^2\omega^2\|\psi\|^2 - 3\bar{m}^2\omega^2\|\bar{E}_I\psi\| \, \|\psi\| + \langle C\rangle_\psi \tag{3.42}$$

where C is a compact operator.

Remark:

Of the terms on the right hand side in (3.42) it is the positive term $\bar{m}^2\omega^2\|\psi\|^2$ that plays a key role in the proof of Theorem 2.2. Notice that (3.42) does not necessarily mean that the expectation value $\langle -\partial_t^2\rangle_\psi$ is positive. The usefulness of (3.42) for us is that we will be able to control the last two terms on the right.

Proof of Proposition 3.7:

Since $\Pi_{\bar{m}-1} + \bar{\Pi}_{\bar{m}-1} = \mathbf{1}$ we have

$$-\partial_t^2 = -\partial_t^2\Pi_{\bar{m}-1} - \partial_t^2\bar{\Pi}_{\bar{m}-1}. \tag{3.43}$$

Using that the first term on the right hand side is positive and that

$$\begin{aligned} -\partial_t^2\bar{\Pi}_{\bar{m}-1} \;\; &\geq \;\; \bar{m}\omega^2\bar{\Pi}_{\bar{m}-1} \\ &= \;\; \bar{m}\omega^2 \; - \; \bar{m}\omega^2\Pi_{\bar{m}-1}, \end{aligned} \tag{3.44}$$

we obtain

$$-\partial_t^2 \; \geq \; \bar{m}^2\omega^2 \; - \; \bar{m}^2\omega^2\Pi_{\bar{m}-1}. \tag{3.45}$$

For the second term on the right we write

$$\begin{aligned}\Pi_{\bar{m}-1} &= E_I \Pi_{\bar{m}-1} E_I \\ &\quad + \bar{E}_I \Pi_{\bar{m}-1} E_I + E_I \Pi_{\bar{m}-1} \bar{E}_I \\ &\quad + \bar{E}_I \Pi_{\bar{m}-1} \bar{E}_I. \end{aligned} \tag{3.46}$$

The expectation values of the last three terms on the right are estimated as follows. Let $\psi \in L^2(\mathrm{R}^N \times S^1_\omega)$. Then

$$\begin{aligned} | \langle \bar{E}_I \Pi_{\bar{m}-1} E_I \rangle_\psi | &= | \langle \Pi_{\bar{m}-1} E_I \psi, \ \bar{E}_I \psi \rangle | \\ &\leq \|E_I \psi\| \, \|\bar{E}_I \psi\| \\ &\leq \|\bar{E}_I \psi\| \, \|\psi\|, \end{aligned} \tag{3.47}$$

using the Schwarz inequality. The last two terms on the right in (3.46) satisfy this same estimate.

The first term on the right in (3.46) we write as

$$\begin{aligned} E_I \Pi_{\bar{m}-1} E_I &= E_I^o \Pi_{\bar{m}-1} E_I^o \\ &\quad + (E_I - E_I^o) \Pi_{\bar{m}-1} E_I \\ &\quad + E_I^o \Pi_{\bar{m}-1} (E_I - E_I^o). \end{aligned} \tag{3.48}$$

The first term on the right in this equation is zero because

$$P_k E_I^o = 0 \tag{3.49}$$

for $|k| < \bar{m}$. This was explained in Section 2, but for convenience we repeat the argument here. On $Ran\, P_k = \mathrm{E}_k$, $K_o = -k^2\omega^2 - \Delta$ so that

$$P_k E_I(K_o) = E_I(-k^2\omega^2 - \Delta).$$

Since $\sup(I) < -k^2\omega^2$, $spec\,(-k^2\omega^2 - \Delta) = [-k^2\omega^2, \infty)$ is disjoint from I. Hence,

$E_I(-k^2\omega^2 - \Delta) = 0$.

Going back to (3.48), $(E_I - E_I^o)\Pi_{\bar{m}-1}$ and $\Pi_{\bar{m}-1}(E_I - E_I^o)$ are compact by Lemma 3.6. Thus, $E_I\Pi_{\bar{m}-1}E_I$ is compact. Equations (3.46), (3.47), and (3.48) together show that

$$\langle \bar{m}^2\omega^2\Pi_{\bar{m}-1}\rangle_\psi \;\leq\; 3\bar{m}^2\omega^2\|\bar{E}_I\psi\|\,\|\psi\| + \langle C\rangle_\psi \tag{3.50}$$

where $C = E_I\Pi_{\bar{m}-1}E_I$ is compact. Combining this with (3.45) yields (3.42) □

Proposition 3.8 *Let $I \subset \mathrm{R}$ be any interval containing $\lambda + \alpha^2$ and let $A = \frac{1}{2i}(x \cdot \nabla + \nabla \cdot x)$. Then*

$$\begin{aligned}\langle i[K,\ A]\rangle_{\xi_\alpha} \geq{} & (\lambda + \alpha^2 + \bar{m}^2\omega^2)\|\xi_\alpha\|^2 + \|\nabla\xi_\alpha\|^2 \\ & -\Big(\frac{3\bar{m}^2\omega^2}{d_\alpha} + 1\Big)\Big\{(\mu(\delta,\alpha)o_R(1)\Big[\|\psi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} + \|\xi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}\Big] \\ & \qquad +2\,\delta(\delta + 2\alpha)\|\xi_\alpha\|^2\Big\}\end{aligned} \tag{3.51}$$

where $d_\alpha = dist\,(\partial I, \lambda + \alpha^2)$, μ is as in Lemma 3.2, and $\bar{m}$ is as in Proposition 3.7.

Remark:

The choice of the interval I is somewhat flexible at this point. The only requirement now is that it contain $\lambda + \alpha^2$. However, by making α smaller if necessary we can assume that I is any interval containing the eigenvalue λ. During the proof of exponential bounds, below, we will require that I satisfy additional conditions. There it will be important that $\sup(I)$ be less than the next threshold above λ : $\sup(I) < -(m_o - 1)^2\omega^2$ where m_o is the integer characterized by the relation

$$-m_o^2\omega^2 \;<\; \lambda \;<\; -(m_o - 1)^2\omega^2.$$

At the conclusion of the proof of Theorem 2.2 it will also become apparent that $\sup(I)$ limits how large of an exponential bound we can prove for ψ. That is, we will be able to show that $e^{\alpha r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all α such that $\lambda + \alpha^2 \;<\; \sup(I)$. But because $\sup(I)$ can be arbitrarily close to $-(m_o-1)^2\omega^2$, the condition on α is really that $\lambda+\alpha^2 < -(m_o-1)^2\omega^2$, as stated in Theorem 2.2

Proof of Proposition 3.8:

We begin by writing

$$-\Delta = K - \partial_t^2 - W. \tag{3.52}$$

Then,

$$\begin{aligned}\langle i[K,\ A]\rangle_{\xi_\alpha} &= \langle -2\Delta\ - x\cdot\nabla W\rangle_{\xi_\alpha} \\ &= \langle -\Delta\rangle_{\xi_\alpha} + \langle K-\lambda-\alpha^2\rangle_{\xi_\alpha} + \langle\lambda+\alpha^2\rangle_{\xi_\alpha} + \langle -\partial_t^2\rangle_{\xi_\alpha} \\ &\qquad -\langle W + x\cdot\nabla W\rangle_{\xi_\alpha}. \end{aligned}\tag{3.53}$$

Here we have separated out a $\langle -\Delta\rangle_{\xi_\alpha} = \|\nabla\xi_\alpha\|^2$ term because we will need this to compensate for the negative term $-o_R(1)\|\nabla\xi_\alpha\|^2$ that will appear below in the estimate for $\langle K-\lambda-\alpha^2\rangle_{\xi_\alpha}$ via the use of Lemma 3.2.

The potential term in (3.53) we estimate as

$$|\ \langle W + x\cdot\nabla W\rangle_{\xi_\alpha}\ | = \ o_R(1)\|\xi_\alpha\|^2, \tag{3.54}$$

since $|\ W + x\cdot\nabla W\ |\to 0$ as $|\ x\ |\to\infty$ uniformly in t (cf. Lemma 2.1) and $supp\,(\xi_\alpha)$ is contained outside of a ball of radius R.

Letting $I\subset \mathrm{R}$ be as in the statement of the Proposition, from Proposition 3.7 and Lemma 3.1 we have

$$\langle -\partial_t^2\rangle_{\xi_\alpha} \ \geq\ \bar m^2\omega^2\|\xi_\alpha\|^2 - 3\bar m^2\omega^2\|\bar E_I\xi_\alpha\|\,\|\xi_\alpha\| - o_R(1)\|\xi_\alpha\|^2, \tag{3.55}$$

where, recall, $\bar m = \min\{m\in \mathrm{N}\ |\ -m^2\omega^2 < \sup(I)\}$.

Applying the usual functional calculus we derive the estimate (cf. equations (3.1) and (3.2))

$$\|\bar E_I\xi_\alpha\| \ = \ \|\bar E_I(K-\lambda-\alpha^2)^{-1}\,(K-\lambda-\alpha^2)\,\xi_\alpha\|$$

$$\leq \frac{1}{d_\alpha}\|(K-\lambda-\alpha^2)\xi_\alpha\| \tag{3.56}$$

where $d_\alpha = dist\,(\partial I, \lambda+\alpha^2)$. From the Schwarz inequality we have that

$$\left|\langle K-\lambda-\alpha^2\rangle_{\xi_\alpha}\right| \leq \|(K-\lambda-\alpha^2)\xi_\alpha\|\,\|\xi_\alpha\|. \tag{3.57}$$

Thus,

$$\begin{aligned}\langle i[K,\,A]\rangle_{\xi_\alpha} \geq\ & (\lambda+\alpha^2+\bar{m}^2\omega^2)\|\xi_\alpha\|^2 \ +\ \|\nabla\xi_\alpha\|^2 \\ & -\Big(\frac{3\bar{m}^2\omega^2}{d_\alpha}+1\Big)\|(K-\lambda-\alpha^2)\xi_\alpha\|\,\|\xi_\alpha\| \\ & -o_R(1)\|\xi_\alpha\|^2. \end{aligned} \tag{3.58}$$

From Lemma 3.2 we obtain

$$\begin{aligned}\|(K-\lambda-\alpha^2)\xi_\alpha\|\,\|\xi_\alpha\| \leq\ & \mu(\delta,\,\alpha)o_R(1)\Big[\|\psi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}+\|\xi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}\Big] \\ & +2\delta(\delta+2\alpha)\|\xi_\alpha\|^2. \end{aligned} \tag{3.59}$$

This, combined with (3.58) gives (3.51) □

4 Proof of Theorem 2.2

Our first task here is to use the results of the previous section to show that if $\psi_\alpha = e^{\alpha r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$, for some $\alpha \geq 0$ where $K\psi = \lambda\psi$, then there exists a $\delta > 0$ such that $e^{\delta r}\psi_\alpha \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$.

4.1 Exponential Bounds

Using the relation

$$K^H = K - |\nabla H|^2 + i\gamma_H \tag{4.1}$$

we compute

$$Im\,\langle (K^H - \lambda)\xi_\alpha,\ A\xi_\alpha\rangle \;=\; \frac{1}{2}\langle i[K,\ A]\rangle_{\xi_\alpha} - \frac{1}{2}\langle i[\mid \nabla H \mid^2,\ A]\rangle_{\xi_\alpha} \;+\; Re\,\langle \gamma_{\scriptscriptstyle H} A\rangle_{\xi_\alpha}. \quad (4.2)$$

The middle commutator on the right hand side is equal to

$$\begin{aligned} i[|\nabla H|^2,\ A] &= i[|\nabla(\delta h(r) + \alpha r)|^2,\ A] \\ &= i\delta^2[|\nabla h|^2,\ A] \;+\; 2i\delta\alpha[\nabla h\cdot\hat{x},\ A] \\ &= -2\delta^2 h' h'' r \;-\; 2\delta\alpha h'' r. \end{aligned} \quad (4.3)$$

Let c_h' and c_h'' be constants (independent of R and ε) such that

$$\begin{aligned} \mid h''(r)\mid &\le c_h' R^{-1}, & 2R <\ & r < 3R \\ \mid h''(r)\mid &\le c_h''(3R + \tfrac{1}{\varepsilon})^{-1}, & 3R + \tfrac{1}{\varepsilon} <\ & r < 2(3R + \tfrac{1}{\varepsilon}) \end{aligned} \quad (4.4)$$

(recall the definition of h) and set $c_h = \max\,(c_h', c_h'')$. Noting that $h'(r) \le 1$ for all R, ε and that $h''(r) = 0$ for $3R < r < 3R + \frac{1}{\varepsilon}$ and for $r \ge 2(3R + \frac{1}{\varepsilon})$, we have

$$\begin{aligned} \mid h' h'' r\mid &\le 3c_h, \\ \mid h'' r\mid &\le 3c_h. \end{aligned} \quad (4.5)$$

Therefore,

$$\left|\langle i[|\nabla H|^2,\ A]\rangle_{\xi_\alpha}\right| \;\le\; 6c_h\delta(\delta + \alpha)\|\xi_\alpha\|^2. \quad (4.6)$$

Recall that $\langle\gamma_{\scriptscriptstyle H} A\rangle_{\xi_\alpha} = \delta\langle\gamma_h A\rangle_{\xi_\alpha} + \alpha\langle\gamma A\rangle_{\xi_\alpha}$ with $\gamma_h = \frac{1}{i}(\nabla h\cdot\nabla + \nabla\cdot\nabla h)$ and $\gamma = \frac{1}{i}(\hat{x}\cdot\nabla + \nabla\cdot\hat{x})$. Since $\gamma = r^{-1/2} A r^{-1/2}$,

$$\begin{aligned} Re\,\langle\gamma A\rangle_{\xi_\alpha} &= Re\,\langle r^{-1/2} A r^{-1/2} A\rangle_{\xi_\alpha} \\ &= \langle r^{-1/2} A^2 r^{-1/2}\rangle_{\xi_\alpha} + Re\,\langle r^{-1/2} A\,[r^{-1/2},\ A]\rangle_{\xi_\alpha} \\ &= \langle r^{-1/2} A^2 r^{-1/2}\rangle_{\xi_\alpha} + \frac{1}{2}\langle r^{-1/2}\,[A,\ [r^{-1/2},\ A]\,]\rangle_{\xi_\alpha} \end{aligned}$$

$$= \quad \text{positive term } + o_R(1)\|\xi_\alpha\|^2, \tag{4.7}$$

so that

$$Re\,\langle\gamma A\rangle_{\xi_\alpha} \;\geq\; -cR^{-1}\|\xi_\alpha\|^2 \;\geq\; -cR^{-1}\|\xi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}.$$

Writing $\gamma_h = \sqrt{h'r^{-1}}A\sqrt{h'r^{-1}}$ one sees that the same estimate holds for $Re\,\langle\gamma_h A\rangle_{\xi_\alpha}$.

Combining these results with Proposition 3.8 we obtain

$$\begin{aligned}
Im\,\langle (K^H-\lambda)\xi_\alpha,\ A\xi_\alpha\rangle \;\geq\; & \frac{1}{2}(\lambda+\alpha^2+\bar{m}^2\omega^2)\|\xi_\alpha\|^2+\frac{1}{2}\|\nabla\xi_\alpha\|^2 \\
& -\Big(\frac{3\bar{m}^2\omega^2}{d_\alpha}+1\Big)\Big\{\mu(\delta,\,\alpha)o_R(1)(\|\xi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}+\|\psi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))})+2\delta(\delta+2\alpha)\|\xi_\alpha\|^2\Big\} \\
& -(\delta+\alpha)o_R(1)\|\xi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} \\
& -6c_h\delta(\delta+\alpha)\|\xi_\alpha\|^2
\end{aligned} \tag{4.8}$$

where $d_\alpha = dist\,(\partial I,\ \lambda+\alpha^2)$. Now write this as

$$\begin{aligned}
& \frac{1}{2}(\lambda+\alpha^2+\bar{m}^2\omega^2)\|\xi_\alpha\|^2+\frac{1}{2}\|\nabla\xi_\alpha\|^2 \\
& \qquad -\Big(\frac{3\bar{m}^2\omega^2}{d_\alpha}+1\Big)\Big\{\mu_1(\delta,\,\alpha)o_R(1)\|\xi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}+10c_h\delta(\delta+\alpha)\|\xi_\alpha\|^2\Big\} \\
& \quad \leq Im\,\langle (K^H-\lambda)\xi_\alpha,\ A\xi_\alpha\rangle+\left(\frac{3\bar{m}^2\omega^2}{d_\alpha}+1\right)\mu_1(\delta,\,\alpha)o_R(1)\|\psi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} \\
& \quad \leq \;|\,\langle (K^H-\lambda)\xi_\alpha,\ A\xi_\alpha\rangle\,|+\left(\frac{3\bar{m}^2\omega^2}{d_\alpha}+1\right)\mu_1(\delta,\,\alpha)o_R(1)\|\psi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}
\end{aligned} \tag{4.9}$$

where $\mu_1(\delta,\alpha)=\mu(\delta,\alpha)+(\delta+\alpha)$. The term $10c_h\delta(\delta+\alpha)\|\xi_\alpha\|^2$ on the left hand side of the inequality comes from combining the $6c_h\delta(\delta+\alpha)\|\xi_\alpha\|^2$ and $2\delta(\delta+2\alpha)\|\xi_\alpha\|^2$ terms on the right hand side of (4.8) and estimating their sum from above, which preserves the inequality. To proceed further we require the following estimate.

Lemma 4.1

$$|\,\langle (K^H-\lambda)\xi_\alpha,\ A\xi_\alpha\rangle\,| \leq\; c(1+R^{-1})(1+\alpha)\|\psi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}. \tag{4.10}$$

Proof:

Recall from Lemma 3.1 that

$$(K^H - \lambda)\xi_\alpha = (-\Delta\chi_R)\psi_\alpha - 2\nabla\chi_R \cdot \nabla\psi_\alpha + 2\alpha\nabla\chi_R \cdot \hat{x}\psi_\alpha \tag{4.11}$$

and that from the definition of h that $h = 0$ on $supp\,(\chi_R^{(m)})$, $m \geq 1$. Hence, when restricted to $supp\,(\chi_R^{(m)})$, $\xi_\alpha = \chi_R\psi_\alpha$ and

$$\begin{aligned} A\xi_\alpha &= -i(x \cdot \nabla + \frac{N}{2})\chi_R\psi_\alpha \\ &= -i(x \cdot \nabla\chi_R)\psi_\alpha - i\chi_R x \cdot \nabla\psi_\alpha - i\frac{N}{2}\chi_R\psi_\alpha. \end{aligned} \tag{4.12}$$

Using (4.11) and (4.12) and noting that $\mid \Delta\chi_R \mid \leq cR^{-2}$, $\mid \nabla\chi_R \mid \leq cR^{-1}$, $\mid x \mid < 2R$ on $supp\,(\chi_R')$, and so $\mid x \cdot \nabla\chi_R \mid \leq c$, an application of the triangle and Schwarz inequalities then leads to (4.10) □

Rewriting (4.9) and using (4.10) on the right hand side and using the inequality $(1+\alpha) \leq \mu_1(\delta,\, \alpha)$, we obtain

$$\begin{aligned} &\frac{1}{2}(\lambda + \alpha^2 + \bar{m}^2\omega^2)\|\xi_\alpha\|^2 + \frac{1}{2}\|\nabla\xi_\alpha\|^2 \\ &\qquad -\Big(\frac{3\bar{m}^2\omega^2}{d_\alpha} + 1\Big)\Big\{\mu_1(\delta,\, \alpha)o_R(1)\|\xi_\alpha\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} + 10c_h\delta(\delta+\alpha)\|\xi_\alpha\|^2\Big\} \\ &\leq \mu_1(\delta,\, \alpha)\Big[1 + \left(\frac{3\bar{m}^2\omega^2}{d_\alpha} + 1\right) o_R(1)\Big]\|\psi_\alpha\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))}. \end{aligned} \tag{4.13}$$

For sake of clarity we write this here as

$$\frac{1}{2}(\lambda + \alpha^2 + \bar{m}^2\omega^2)\|\xi_\alpha\|^2 - \left(\frac{3\bar{m}^2\omega^2}{d_\alpha} + 1\right) 10c_h\delta(\delta+\alpha)\|\xi_\alpha\|^2 \;\; + \;\; e \;\; \leq \;\; g, \tag{4.14}$$

to distinguish the negative terms containing $\|\xi_\alpha\|^2$ that are controlled by δ alone (rather than by R). We will come back to equation (4.14) in a moment (at equation (4.17)).

We would like to point out that equation (4.13) is rather general in that it is valid for any function ξ_α of the form $\xi_\alpha = e^{\delta h}\psi_\alpha$, $\psi_\alpha = e^{\alpha r}\psi$, $\delta > 0, \alpha \geq 0$, where $\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$

is an eigenfunction of K corresponding to the eigenvalue λ. The numbers $\bar{m}$ and d_α are determined by the interval I which, as was pointed out above in the remark following the statement of Proposition 3.8, can be any interval containing $\lambda + \alpha^2$ (or even just containing λ; cf. the same remark). This inequality (equation (4.13)) is fundamental in proving exponential bounds. It is with this purpose in mind that we now proceed to determine more precisely what the interval I should be. This is where the generality of (4.13) is used: we first determine the integer $\bar{m}$ and then design I accordingly. Once this is done d_α will be determined and then it remains only to determine δ and R (which up until now have been free parameters).

Our objective now is to show that for a suitable choice of the parameters $\bar{m}, \delta$ and R (which we will be denoting by $m_o, \delta(\alpha)$ and R_o) and interval I, there exists an $a > 0$ independent of ε such that

$$\text{left hand side of (4.13)} \;\geq\; a\|\xi_\alpha\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))}. \tag{4.15}$$

This will then imply that

$$\begin{aligned}\|\xi_{\alpha,R_o,\delta(\alpha),\varepsilon}\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} &\leq a^{-1}\mu_1(\delta(\alpha),\,\alpha)\Big[1+\left(\frac{3\bar{m}^2\omega^2}{d_\alpha}+1\right)o_R(1)\Big]\|\psi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} \\ &< \infty \end{aligned} \tag{4.16}$$

for all $\varepsilon > 0$. Note that the right hand side of this inequality is independent of ε. Subsequently (and as will be explained in detail below), from (4.16) and the monotone convergence theorem it follows that the function $\chi_{R_o} e^{\delta(\alpha)h_{R_o}(r)}\psi_\alpha = \lim_{\varepsilon\to 0}\xi_{\alpha,R_o,\delta(\alpha),\varepsilon}$ is in $L^2(S^1_\omega, H^1(\mathrm{R}^N))$ and therefore that $e^{(\delta+\alpha)r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$.

We proceed to demonstrate (4.15). First we determine what the integer $\bar{m}$ should be. This in turn will determine the interval I via Proposition 3.7 (that is, choose I in Proposition 3.7 so that $\bar{m}$ is the desired integer). Then $\delta(\alpha)$ and finally R_o will be chosen. Recall from Proposition 3.8 that the only requirement on I up to this point has been that $\lambda + \alpha^2 \in I$ which implies that $\sup(I) > \lambda$. Now we show that in order to obtain the positivity estimate

Figure 5: Location of the interval I.

(4.15), $\sup(I)$ cannot be too large. More precisely, it must be below the next threshold above λ. This follows from Proposition 3.7, as we will presently describe.

To obtain (4.15) we require that the quantity $\lambda + \bar{m}^2\omega^2$ appearing on the left hand side of (4.13) be positive. Let m_o to be the smallest integer $m \geq 0$ such that $\lambda + m^2\omega^2 > 0$. This characterizes the location of λ with respect to the thresholds $\mathcal{E}(K)$: In the case $\lambda < 0$ we have

$$-m_o^2\omega^2 \ < \ \lambda \ < \ -(m_o - 1)^2\omega^2$$

for some $m_o \geq 1$; in the case $\lambda > 0$ we have $m_o = 0$. If $m_o \geq 1$ then, referring to Proposition 3.7 with m_o in place of $\bar{m}$ there (see equations (3.41) and (3.42)), we see that the interval I must satisfy $-m_o^2\omega^2 < \sup(I) < -(m_o - 1)^2\omega^2$. This is why m_o must be the *smallest* integer such that $\lambda + m^2\omega^2 > 0$: otherwise $\sup(I) < \lambda$ and I will not contain λ. Unlike $\sup(I)$, $\inf(I)$ can be freely specified. We take advantage of this fact to simplify the quantity $d_\alpha = \mathit{dist}\,(\partial I,\, \lambda + \alpha^2)$. We will assume that $\inf(I) < \lambda - (\sup(I) - \lambda)$ so that $d_\alpha = \sup(I) - \lambda - \alpha^2$. An interval I satisfying these conditions is illustrated in Figure 5.

In the case $\lambda > 0$ we do not need to estimate the spectral localization of ξ_α with respect to K (and hence there is no interval I to discuss) because then $\lambda + \bar{m}^2\omega^2$ is positive with $\bar{m} = 0$ and the a priori estimate $\langle -\partial_t^2 \rangle_{\xi_\alpha} \geq 0$ is enough for our purposes.

For $\lambda < 0$ we have denoted the distance from $\lambda + \alpha^2$ to $\sup(I)$ by d_α . We will denote the distance from λ to $\sup(I)$ by d_0: $d_o = \sup(I) - \lambda$. Then, $d_\alpha = d_o - \alpha^2$. Now we solve the equation

$$\left(\frac{3m_o^2\omega^2}{d_\alpha} + 1\right) 10c_h\delta(\delta + \alpha) \ = \ \frac{1}{8}(\lambda + m_o^2\omega^2 + \alpha^2) \tag{4.17}$$

for δ : see equation (4.14). Recall that c_h was given in (4.5). In the case $\lambda > 0$ we have $m_o = 0$. In solving (4.17) we have determined how small δ must be in order that the positive coefficient dominates the negative coefficient of $\|\xi_\alpha\|^2$ in (4.14). Denote by $\delta(\alpha)$ the solution to (4.17). That is,

$$\delta(\alpha) \;=\; \frac{-\alpha + \sqrt{\alpha^2 + \frac{1}{20c_h a(\alpha)}(\lambda + m_o^2\omega^2 + \alpha^2)}}{2} \tag{4.18}$$

where $a(\alpha) = (\frac{3m_o^2\omega^2}{d_\alpha} + 1)$. Note that $\delta(\alpha) > 0$. From this it follows that for $\delta \le \delta(\alpha)$,

$$\left(\frac{3m_o^2\omega^2}{d_\alpha} + 1\right) 10c_h\delta(\delta + \alpha)\|\xi_\alpha\|^2 \le \frac{1}{8}(\lambda + m_o^2\omega^2 + \alpha^2)\|\xi_\alpha\|^2. \tag{4.19}$$

Having found $\delta(\alpha)$ we next determine how large R should be. Let

$$b = \min\left(\lambda + m_o^2\omega^2 + \alpha^2,\ 1\right). \tag{4.20}$$

Referring to the left hand side of (4.13) and recalling that the functions $o_R(1)$ vanish as $R \to \infty$, let R_o be such that

$$\Big(\frac{3m_o^2\omega^2}{d_\alpha} + 1\Big)\mu_1(\delta(\alpha), \alpha)o_R(1) \;<\; \frac{1}{8}b \tag{4.21}$$

for $R \ge R_o$. R_o is uniform in $\varepsilon > 0$ and in δ for $\delta \le \delta(\alpha)$ because the functions $o_R(1)$ are independent of ε and δ and the function μ_1 is an increasing function of δ which is also independent of ε.

We have then from (4.19) and (4.21) that if $R \ge R_o$, $\delta \le \delta(\alpha)$ and $\bar{m} = m_o$, the left side of (4.13) is bounded below by $\frac{1}{4}b\|\xi_\alpha\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))}$ where $b > 0$ is given by (4.20), and therefore that

$$\|\xi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} \;\le\; 4b^{-1}\mu_1(\delta, \alpha)\Big[1 + \left(\frac{3m_o^2\omega^2}{d_\alpha} + 1\right)o_R(1)\Big]\|\psi_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} \;<\; \infty \tag{4.22}$$

uniformly in ε (cf. (4.16)). In particular,

$$\|\xi_{\alpha,R_o,\delta(\alpha),\varepsilon}\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} < c < \infty \tag{4.23}$$

for all $\varepsilon > 0$ where c is the right hand side of (4.22) with $\delta(\alpha)$ and R_o in place of δ and R. We point out again that the right hand side of (4.22) is independent of ε.

Now, because $h_{R_o,\varepsilon}$ and $h'_{R_o,\varepsilon}$ increase pointwise to h_{R_o} and h'_{R_o} respectively as $\varepsilon \to 0$, $|\ \xi_{\alpha,R_o,\delta(\alpha),\varepsilon}\ |^2 = |\ \chi_{R_o} e^{\delta(\alpha)h_{R_o,\varepsilon}}\psi_\alpha\ |^2$ increases pointwise to $|\ \chi_{R_o} e^{\delta(\alpha)h_{R_o}}\psi_\alpha\ |^2 = |\ \xi_{\alpha,R_o,\delta(\alpha),\varepsilon=0}\ |^2$, and $|\ \nabla\xi_{\alpha,R_o,\delta(\alpha),\varepsilon}\ |^2 = e^{2\delta(\alpha)h_{R_o,\varepsilon}}\ |\ \delta(\alpha)h'_{R_o,\varepsilon}\hat{x}\chi_{R_o}\psi_\alpha + \nabla(\chi_{R_o}\psi_\alpha)\ |^2$ increases pointwise to $e^{2\delta(\alpha)h_{R_o}}\ |\ \delta(\alpha)h'_{R_o}\hat{x}\chi_{R_o}\psi_\alpha + \nabla(\chi_{R_o}\psi_\alpha)\ |^2 = |\ \nabla\xi_{\alpha,R_o,\delta(\alpha),\varepsilon=0}\ |^2$ as $\varepsilon \to 0$, with both $\|\xi_{\alpha,R_o,\delta(\alpha),\varepsilon}\|^2$ and $\|\nabla\xi_{\alpha,R_o,\delta(\alpha),\varepsilon}\|^2$ uniformly bounded in ε by the right hand side of (4.22). The monotone convergence theorem implies then that $\|\xi_{\alpha,R_o,\delta(\alpha),\varepsilon=0}\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}$ is bounded above by the right hand side of (4.22). From this it follows that $e^{\alpha_1 r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ where $\alpha_1 = \alpha + \delta(\alpha)$.

We pause here to comment on the formula (4.22). We can write it as

$$\|\chi_{R_o} e^{(\alpha+\delta(\alpha))r}\psi\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} \ \le\ 4b^{-1}\mu_1(\delta(\alpha),\alpha)\Big[1 + \Big(\frac{3m_o^2\omega^2}{d_\alpha} + 1\Big)o_R(1)\Big]\ \|e^{\alpha r}\psi\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} \tag{4.24}$$

where R_o and $\delta(\alpha)$ are as defined above. This inequality estimates the norm of the function $e^{(\alpha+\delta(\alpha))r}\psi$ restricted to the exterior of a ball of radius R_o in R^N by the norm of $e^{\alpha r}\psi$. α is an exponential bound for ψ and $\alpha + \delta(\alpha)$ is the new exponential bound for ψ. Recall that $d_\alpha = dist\,(\partial I,\ \lambda + \alpha^2) = \sup(I) - (\lambda + \alpha^2)$ where $\sup(I)$ can be chosen arbitrarily close to, but below, $-(m_o - 1)^2\omega^2$. Since $d_\alpha \to 0$ as $\lambda + \alpha^2 \to \sup(I)$, we see that the right hand side of (4.24) $\nearrow \infty$ as $\lambda + \alpha^2 \to \sup(I)$. Therefore, it is consistent with our analysis that $e^{\alpha r}\psi$ leaves $L^2(S^1_\omega, H^1(\mathrm{R}^N))$ as $\lambda + \alpha^2 \to -(m_o - 1)^2\omega^2$, although we do not prove this.

4.2 Iterative Step

For a given $\alpha \ge 0$ the analysis of the preceding subsection defined a function $\delta(\alpha) > 0$ which is the incremental exponential bound for $\psi_\alpha = e^{\alpha r}\psi$. That is, we showed that $e^{\delta(\alpha)r}\psi_\alpha \in$

$L^2(S^1_\omega, H^1(\mathrm{R}^N))$. We use this abstract set up to iterate this procedure. For example, at the next iteration we start with the function $\psi_{\alpha_1 r} \equiv e^{\alpha_1 r}\psi$ where $\alpha_1 = \alpha + \delta(\alpha)$ and find $\delta(\alpha_1) > 0$ such that $e^{\delta(\alpha_1)r}\psi_{\alpha_1} \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ by considering the regularized function $\xi_{\alpha_1} \equiv e^{\delta h(r)}\psi_{\alpha_1}$. In the end we will arrive at the inequality (4.13) and subsequently will be lead to the equation (4.17) with α_1 in place of α in these two equations. Keep in mind that the interval I has been fixed at the start, that is, it is the same for each iteration so that $\sup(I)$ and m_o are fixed (see the remark following Proposition 3.8 and the discussion in the paragraphs preceding equation (4.17)). Denote by Φ the function

$$\Phi(\alpha) = \alpha + \delta(\alpha) \tag{4.25}$$

where $\delta(\alpha)$ is defined by (4.18). Then, $\Phi(0)$ is the exponential bound we find for ψ after the first iteration and $\Phi^n(0)$ is the exponential bound we find after n iterations of the procedure described in Section 4.1.

Lemma 4.2 *If $m_o \neq 0$ then $\lim_{n\to\infty}\left(\Phi^n(0)\right)^2 = d_o$, where $d_o = \sup(I) - \lambda$. If $m_o = 0$ then $\lim_{n\to\infty}\Phi^n(0) = \infty$.*

Proof:

Setting $\alpha_n = \Phi^n(0)$, suppose that $m_o \neq 0$ and that $\lim_{n\to\infty}\alpha_n^2 < d_o$. This implies that $a(\alpha_n)^{-1}$ is bounded away from 0 for all n (since d_{α_n} will be bounded away from zero for all n). It follows then that

$$\begin{aligned}\alpha_n &= \alpha_{n-1} + \delta(\alpha_{n-1}) \\ &= \frac{\alpha_{n-1} + \sqrt{\alpha_{n-1}^2 + \frac{1}{20c_h a(\alpha_{n-1})}(\lambda + m_o^2\omega^2 + \alpha_{n-1}^2)}}{2} \geq \alpha_{n-1} + c\end{aligned} \tag{4.26}$$

where $c > 0$ is independent of n. Hence, $\alpha_n \to \infty$ which contradicts the assumption that $\lim_{n\to\infty}\alpha_n^2 < d_o$. On the other hand, $\lim_{n\to\infty}\alpha_n^2$ cannot exceed d_o because $\delta(\alpha_n) \to 0$ as $\alpha_n^2 \to d_o$. Thus, $\lim_{n\to\infty}\alpha_n^2 = d_o$.

If $m_o = 0$ then

$$\delta(\alpha_n) = \frac{-\alpha_{n-1} + \sqrt{\alpha_{n-1}^2 + \frac{1}{20c_h}(\lambda + \alpha_{n-1}^2)}}{2}$$

$$\geq \frac{-\alpha_{n-1} + \sqrt{\alpha_{n-1}^2 + \frac{\lambda}{20c_h}}}{2}$$
$$> \; c \; > \; 0$$

for all n and hence $\Phi^n(0) \; > \; nc$ □

If $\lambda < 0$ the results of Lemma 4.2 imply that by iterating the procedure described in Section 4.1, $e^{\alpha r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all α such that $\alpha^2 < d_o = \sup(I) - \lambda$. Since $\sup(I)$ can be arbitrarily close to, but below, $-(m_o - 1)^2\omega^2$, we have then that $e^{\alpha r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all α such that $\lambda + \alpha^2 < -(m_o^2 - 1)^2\omega^2$. If $\lambda > 0$ then $m_o = 0$ so that $\Phi^n(0) \to \infty$. Thus in this latter case we can show that $e^{\alpha r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all α. This completes the proof of Theorem 2.2 □

5 Constraints on Periods: Proof of Theorem 1.1

5.1 Absence of positive eigenvalues

We first prove a kind of unique continuation theorem at infinity for eigenfunctions of the operator K which we will use in the proof of Theorem 1.1. We show that if an eigenfunction has arbitrarily fast exponential decay (in the L^2 sense) then in fact the function is zero. A consequence of this, when combined with Theorem 2.2, is that K has no positive eigenvalues.

Theorem 5.1 *Under the definitions and hypothesis of Theorem 2.2, if* $K\psi = \lambda\psi$ *where* $e^{\alpha r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ *for all* α*, then* $\psi = 0$.

Our proof follows that of [HS]. We will obtain a contradiction by assuming that $\psi \neq 0$. We begin by fixing a such that

$$\int_{S^1_\omega}\int_{r<a} |\,\psi\,|^2 \leq \int_{S^1}\int_{r>2a} |\,\psi\,|^2 \,. \tag{5.1}$$

Let $g(r) \le r, \quad g'(r) \ge 0$ with $g(r) = r$ for $r > a$ and set $\phi_\alpha = e^{\alpha g(r)}\psi \|e^{\alpha g}\psi\|^{-1}$. Then $\phi_\alpha \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all α and

$$
\begin{aligned}
\int_{S^1_\omega}\int_{r<a} |\phi_\alpha|^2 &= \|e^{\alpha g}\psi\|^{-2}\int_{S^1_\omega}\int_{r<a} e^{2\alpha g(r)} |\psi|^2 \\
&\le e^{2\alpha a}\|e^{\alpha g}\psi\|^{-2}\int_{S^1_\omega}\int_{r<a} |\psi|^2 \\
&\le e^{2\alpha a}\|e^{\alpha g}\psi\|^{-2}\int_{S^1_\omega}\int_{r>2a} |\psi|^2 \\
&= e^{-2\alpha a}\|e^{\alpha g}\psi\|^{-2}\int_{S^1_\omega}\int_{r>2a} e^{4\alpha a} |\psi|^2 \\
&\le e^{-2\alpha a}\|e^{\alpha g}\psi\|^{-2}\int_{S^1_\omega}\int_{r>2a} e^{2\alpha r} |\psi|^2 \\
&\le e^{-2\alpha a}.
\end{aligned} \tag{5.2}
$$

We will show that the function ϕ_α is an approximate eigenfunction of K with eigenvalue $\lambda + \alpha^2$. Therefore, $\langle K\rangle_{\phi_\alpha} \approx \lambda + \alpha^2$. On the other hand, as an approximate eigenfunction the virial theorem implies that the expectation value $\langle i[K,\, A]\rangle_{\phi_\alpha}$ will be small. The fact that $\langle K\rangle_{\phi_\alpha}$ eventually dominates $\langle i[K,\, A]\rangle_{\phi_\alpha}$ as α becomes large will lead to a contradiction.

Lemma 5.2

$$\langle i[K,\, A]\rangle_{\phi_\alpha} \le c_1\alpha^2 e^{-2\alpha a} + \alpha c_1. \tag{5.3}$$

for some constant c_1 (not necessarily positive) and all $\alpha > 0$.

<u>Proof</u>:

Set

$$K_\alpha = e^{\alpha g}Ke^{-\alpha g} = K - \alpha^2 |\nabla g|^2 + i\alpha\gamma_g, \tag{5.4}$$

$$\gamma_g = \frac{1}{i}(\nabla g\cdot\nabla + \nabla\cdot\nabla g). \tag{5.5}$$

Then

$$\langle i[K, A]\rangle_{\phi_\alpha} = 2Im\,\langle (K_\alpha - \lambda)\phi_\alpha,\; A\phi_\alpha\rangle + \alpha^2\left\langle i[|\nabla g|^2, A]\right\rangle_{\phi_\alpha} - 2\alpha Re\,\langle \gamma_g A\rangle_{\phi_\alpha}. \tag{5.6}$$

We have

$$
\begin{aligned}
(K_\alpha - \lambda)\phi_\alpha &= 0, && (5.7)\\
\left\langle i[\mid \nabla g \mid^2, A]\right\rangle_{\phi_\alpha} &= \int_{S^1_\omega}\int_{r<a} -(2rg'g'') \mid \phi_\alpha \mid^2 \leq ce^{-2\alpha a}, && (5.8)\\
Re\,\langle \gamma_g A\rangle_{\phi_\alpha} &= \int_{S^1_\omega}\int_{\mathrm{R}^N} 2r^{-1}g' \mid x\cdot\nabla\phi_\alpha \mid^2 \quad + \quad \text{bounded term}, && (5.9)
\end{aligned}
$$

where the integral on the right in the last line is positive □

Lemma 5.3

$$\langle K\rangle_{\phi_\alpha} \geq \lambda + \alpha^2(1-e^{-2\alpha a}) \tag{5.10}$$

for all $\alpha > 0$.

Proof:

$$\langle K\rangle_{\phi_\alpha} = \langle K_\alpha\rangle_{\phi_\alpha} + \alpha^2\left\langle \mid \nabla g\mid^2\right\rangle_{\phi_\alpha} - \alpha\langle i\gamma_g\rangle_{\phi_\alpha}. \tag{5.11}$$

We have

$$
\begin{aligned}
\langle K_\alpha\rangle_{\phi_\alpha} &= \lambda, && (5.12)\\
\langle\mid \nabla g\mid^2\rangle_{\phi_\alpha} &\geq \int_{S^1_\omega}\int_{r>a} \mid \phi_\alpha\mid^2 \;\geq\; (1-e^{-2\alpha a}), && (5.13)\\
\alpha\langle i\gamma_g\rangle_{\phi_\alpha} &= Im\langle K_\alpha\rangle_{\phi_\alpha} \;=\; 0 \qquad \Box && (5.14)
\end{aligned}
$$

Proof of Theorem 5.1:

Subtracting (5.10) from (5.3) we obtain

$$
\begin{aligned}
\langle i[K,A]\rangle_{\phi_\alpha} - \langle K\rangle_{\phi_\alpha} &= \langle -\partial_t^2 - \Delta - x\cdot\nabla W - W\rangle_{\phi_\alpha}\\
&\leq \alpha^2\left(c_1e^{-2\alpha a} - (1-e^{-2\alpha a})\right) - \lambda + \alpha c_1. && (5.15)
\end{aligned}
$$

As $\alpha\to\infty$ the right hand side of the inequality tends to $-\infty$ but the left side is bounded from below ($-\partial_t^2 - \Delta$ is a positive operator while $x\cdot\nabla W + W$ is bounded). This contradiction completes the proof of Theorem 5.1 $\square$

During the proof of Theorem 1.1 we will require a slightly modified version of Theorem 5.1, the difference being with the potential W. We will require a unique continuation theorem for operators $\bar{K}$ of the form $\bar{K} = K_o + \bar{\Pi}_m V \bar{\Pi}_m$ where V is a real-valued function such that V and $x \cdot \nabla V$ are bounded and vanish as $|x| \to \infty$ uniformly in t, just like the potentials W_φ, and for eigenfunctions of the form $\bar{\psi} = \bar{\Pi}_m \psi$. That is, if $\bar{K}\bar{\psi} = \lambda\bar{\psi}$ and $e^{\alpha r}\bar{\psi} \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all α, then $\bar{\psi} = 0$. That this is true follows from the fact that the projections $\bar{\Pi}_m$ commute with functions of x and with derivatives with respect to x. Therefore, the estimates of Lemmas 5.2 and 5.3 hold for $\bar{K}$ and $\bar{\phi}_\alpha$, where $\bar{\phi}_\alpha$ is defined in an analogous way as was ϕ_α.

5.2 Proof of Theorem 1.1

Preparatory discussion

The spectrum of K depends on the frequency ω. In particular, the thresholds $\mathcal{E}(K) = \{-m^2\omega^2;\ m \in \mathrm{Z}\}$ depend on ω. Therefore the positioning of the eigenvalue $-\kappa = -f'(0)$ with respect to the thresholds will change as ω changes. As ω increases, fewer and fewer thresholds will lie above the point $-\kappa$. For $\omega^2 > f'(0)$ the only threshold above $-\kappa$ is zero (see Figure 4). It is the presence of the zero threshold (the existence of a zero mode of φ) that prevents us from proving the exponential bound for φ is arbitrarily large and hence, by Theorem 5.1, that $\varphi = 0$. To prove Theorem 1.1, then, we first remove the this threshold in the following way.

Each threshold corresponds to a point of the spectrum of ∂_t^2. By projecting-out spectral subspaces E_k we can remove points from the spectrum of ∂_t^2. Consequently, we can remove thresholds from K. (However, then we will be considering K restricted to the compliment of these subspaces.) Thus, in the case when $\omega^2 > f'(0)$, by projecting-out the subspace E_0 we can remove the one and only threshold above $-\kappa$. In removing the zero threshold we also remove the zero modes from the set of periodic functions we are considering so that now any conclusions drawn from the analysis will only be about the time-dependent part of a function (the modes other than the zero mode).

Proof of Theorem 1.1:

Let $\varphi_0 \equiv P_0\varphi$ and $\bar{\varphi} \equiv \varphi - \varphi_0$. We will show that if $\omega^2 > f'(0)$ then $\bar{\varphi} = 0$. This implies then that $\varphi = \varphi_0$, i.e., φ is independent of time.

Project $\partial_t^2\varphi - \Delta\varphi + f(\varphi) = 0$ onto $\mathrm{E}_0^{\perp}$:

$$0 = \bar{P}_0\Big(\partial_t^2\varphi - \Delta\varphi + f(\varphi)\Big) = K_o\bar{\varphi} + \bar{P}_0 f(\varphi). \tag{5.16}$$

From the identity

$$f(u+v) = f(u) + f(v) + uv\int_0^1\int_0^1 f''(au+bv)\,dadb, \tag{5.17}$$

we see that we can write

$$f(\varphi) = f(\varphi_0 + \bar{\varphi}) = f(\varphi_0) + f(\bar{\varphi}) + \varphi_0\bar{\varphi}\,g(\varphi_0,\bar{\varphi}) \tag{5.18}$$

where $g(u,v) = \int_0^1\int_0^1 f''(au+bv)\,dadb$. Since $\bar{P}_0 f(\varphi_0) = 0$ we have that

$$\bar{P}_0 f(\varphi) = \bar{P}_0 f(\varphi_0+\bar{\varphi}) = \bar{P}_0\Big(f(\varphi_0+\bar{\varphi}) - f(\varphi_0)\Big). \tag{5.19}$$

We remark here that the property $\bar{P}_0 f(\varphi_0) = 0$ means that the branch $[0,\infty)$ of the spectrum of K is uncoupled (through f) to the other branches (we will discuss this again below).

Combining (5.18) and (5.19) we obtain

$$\begin{aligned}
\bar{P}_0 f(\varphi) &= \bar{P}_0\Big(f(\bar{\varphi}) + \varphi_0\bar{\varphi}\,g(\varphi_0,\bar{\varphi})\Big) \\
&= \bar{P}_0\Big(\frac{f(\bar{\varphi})}{\bar{\varphi}} + \varphi_0\,g(\varphi_0,\bar{\varphi})\Big)\bar{P}_0\bar{\varphi} \qquad \text{(note that } \bar{P}_0\bar{\varphi} = \bar{\varphi}\text{)} \\
&= (U+\kappa)\bar{\varphi},
\end{aligned} \tag{5.20}$$

where

$$U = \bar{P}_0 V_{\bar{\varphi}}\bar{P}_0, \quad V_{\bar{\varphi}} = \frac{f(\bar{\varphi})}{\bar{\varphi}} - \kappa + \varphi_0\,g(\varphi_0,\bar{\varphi}), \quad \kappa = f'(0). \tag{5.21}$$

Thus,

$$\bar{K}\bar{\varphi} \;=\; -\kappa\bar{\varphi} \;\text{ with }\; \bar{K} \;=\; K_o + U. \tag{5.22}$$

Because U is bounded and symmetric, $\bar{K}$ is self-adjoint.

Our task now is to show that $e^{\alpha r}\bar{\varphi} \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all α. As in Theorem 2.2, set $\bar{\xi}_\alpha = \chi_R e^{\delta h}\bar{\psi}_\alpha$, $\bar{\psi}_\alpha = e^{\alpha r}\bar{\varphi}$, where $\delta > 0$ and $\alpha \geq 0$. Let $h(r)$ be as described in Section 2, and $H(r) = \delta h(r) + \alpha r$. Then,

$$\begin{aligned} \bar{K}^H \;&=\; e^{H(r)}\bar{K}e^{-H(r)} \;=\; e^{H(r)}K_o e^{-H(r)} \;+\; U \\ &=\; \bar{K} - |\nabla H|^2 + i\gamma_H \end{aligned} \tag{5.23}$$

where $\gamma_H = \frac{1}{i}(\nabla H \cdot \nabla + \nabla \cdot \nabla H)$ as before. Since $P_0\bar{\varphi} = 0$ we have the a priori estimate

$$\langle -\partial_t^2 \rangle_{\bar{\xi}_\alpha} \;\geq\; \omega^2 \|\bar{\xi}_\alpha\|^2. \tag{5.24}$$

The estimate (5.24) is all we require to prove exponential bounds for $\bar{\varphi}$. Contrast this to the more general situation encountered in Theorem 2.2 where a microlocalization of $\langle -\partial_t^2 \rangle_{\xi_\alpha}$ was required.

Proceeding now as in the proof of Theorem 2.2, we have

$$Im\, \langle (\bar{K}^H + \kappa)\bar{\xi}_\alpha,\; A\bar{\xi}_\alpha \rangle \;=\; \frac{1}{2}\langle i[\bar{K},\; A]\rangle_{\bar{\xi}_\alpha} - \frac{1}{2}\langle i[|\; \nabla H\; |^2,\; A]\rangle_{\bar{\xi}_\alpha} + Re\, \langle \gamma_H A \rangle_{\bar{\xi}_\alpha} \tag{5.25}$$

(cf. equation (4.2)). We compute:

$$\begin{aligned} \langle i[\bar{K},\, A]\rangle_{\bar{\xi}_\alpha} \;&=\; \langle -\Delta \rangle_{\bar{\xi}_\alpha} + \langle \bar{K} + \kappa - \alpha^2 \rangle_{\bar{\xi}_\alpha} + \langle -\kappa + \alpha^2 \rangle_{\bar{\xi}_\alpha} + \langle -\partial_t^2 \rangle_{\bar{\xi}_\alpha} \\ &\quad -\langle V_{\bar{\varphi}} + x \cdot \nabla V_{\bar{\varphi}} \rangle_{\bar{\xi}_\alpha}, \end{aligned} \tag{5.26}$$

(cf. Proposition 3.8). To obtain the expectation value $\langle V_{\bar{\varphi}} + x \cdot \nabla V_{\bar{\varphi}} \rangle_{\bar{\xi}_\alpha}$ from $\langle U + x \cdot \nabla U \rangle_{\bar{\xi}_\alpha}$ we have used the fact that $\bar{P}_0$ commutes with functions of x and derivatives with respect to x, this implies that

$$i[U,\, A] \;=\; i\bar{P}_0[V_{\bar{\varphi}},\, A]\bar{P}_0 \;=\; \bar{P}_0(x \cdot \nabla V_{\bar{\varphi}})\bar{P}_0 \tag{5.27}$$

and that $\bar{P}_0\bar{\xi}_\alpha = \bar{\xi}_\alpha$. From this we see that

$$\langle i[U,\, A]\rangle_{\bar{\xi}_\alpha} \;=\; \langle \bar{P}_0(x\cdot\nabla V_{\bar{\varphi}})\bar{P}_0\rangle_{\bar{\xi}_n} \;=\; \langle x\cdot\nabla V_{\bar{\varphi}}\rangle_{\bar{\xi}_n} \tag{5.28}$$

and

$$\langle U\rangle_{\bar{\xi}_\alpha} \;=\; \langle V_{\bar{\varphi}}\rangle_{\bar{\xi}_\alpha}. \tag{5.29}$$

Using again the commutativity of $\bar{P}_0$ with functions of x we have

$$\begin{aligned}(\bar{K}^H+\kappa)\bar{\xi}_\alpha &= e^{H(r)}\chi_R(\bar{K}+\kappa)\bar{\varphi} \;+\; e^{H(r)}[\bar{K},\, \chi_R]\bar{\varphi}\\ &= e^{\alpha r}[-\Delta,\, \chi_R]\bar{\varphi}\\ &= (-\Delta\chi_R)\bar{\psi}_\alpha - 2\nabla\chi_R\cdot\nabla\bar{\psi}_\alpha + 2\alpha\nabla\chi_R\cdot\hat{x}\bar{\psi}_\alpha. \end{aligned}\tag{5.30}$$

Because of this the estimates of Lemmas 3.2 and 4.1 remain valid;

$$\begin{aligned}\|(\bar{K}+\kappa-\alpha^2)\bar{\xi}_\alpha\| &\le \mu(\delta,\, \alpha)o_R(1)\Big[\|\bar{\psi}_\alpha\|_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} + \|\bar{\xi}_\alpha\|_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}\Big]\\ &\qquad + 2\delta(\delta+2\alpha)\|\bar{\xi}_\alpha\|^2 \end{aligned}\tag{5.31}$$

$$|\langle(\bar{K}^H+\kappa)\bar{\xi}_\alpha,\, A\bar{\xi}_\alpha\rangle| \le c(1+R^{-1})(1+\alpha)\|\bar{\psi}_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}. \tag{5.32}$$

Here $\mu(\delta,\, \alpha)$ is as in Lemma 3.2. From the first inequality we estimate the expectation value $\langle\bar{K}+\kappa-\alpha^2\rangle_{\bar{\xi}_\alpha}$;

$$\begin{aligned}|\,\langle\bar{K}+\kappa-\alpha^2\rangle_{\bar{\xi}_\alpha}\,| &\le \|(\bar{K}+\kappa-\alpha^2)\bar{\xi}_\alpha\|\,\|\bar{\xi}_\alpha\|\\ &\le \mu(\delta,\alpha)o_R(1)\Big[\|\bar{\xi}_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} + \|\bar{\psi}_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}\Big]\\ &\qquad + 2\delta(\delta+2\alpha)\|\bar{\xi}_\alpha\|^2. \end{aligned}\tag{5.33}$$

The potential $V_{\bar{\varphi}}$ vanishes as $|\,x\,|\to\infty$ uniformly in t. We also require that $x\cdot\nabla V_{\bar{\varphi}}$ vanish in the same manner. That this is true for $\frac{f(\bar{\varphi})}{\bar{\varphi}}-\kappa$ follows from the same arguments presented in Lemma 2.1 for W_φ. For the other terms in $V_{\bar{\varphi}}$, $\varphi_o g(\varphi_o,\, \bar{\varphi})$, we note that since

$$\nabla g(\varphi_0,\, \bar{\varphi}) \;=\; \nabla\varphi_0\, g_u(\varphi_0,\, \bar{\varphi}) + \nabla\bar{\varphi}\, g_v(\varphi_0,\, \bar{\varphi}), \tag{5.34}$$

where g_u and g_v denote partial derivatives, and $x \cdot \nabla\varphi_0$ and $x \cdot \nabla\bar{\varphi}$ vanish uniformly in t as $|\, x \,| \to \infty$, it is enough that g_u and g_v be bounded in a neighborhood of $(0,0)$. But this is clear from the formulae

$$g_u(u,v) = \int_0^1 \int_0^1 a f^{(3)}(au+bv)\, dadb, \tag{5.35}$$

$$g_v(u,v) = \int_0^1 \int_0^1 b f^{(3)}(au+bv)\, dadb. \tag{5.36}$$

Therefore,

$$\left| \langle V_{\bar{\varphi}} + x \cdot \nabla V_{\bar{\varphi}} \rangle_{\bar{\xi}_\alpha} \right| \le o_R(1) \|\bar{\xi}_\alpha\|^2. \tag{5.37}$$

Combining this with (5.24), (5.26), and (5.33) we obtain

$$\begin{aligned} \langle i[\bar{K},\, A] \rangle_{\bar{\xi}_\alpha} \ \ge\ & (-\kappa + \alpha^2 + \omega^2)\|\bar{\xi}_\alpha\|^2 + \|\nabla\bar{\xi}_\alpha\|^2 \\ & -\mu(\delta,\alpha) o_R(1) \left[\|\bar{\xi}_\alpha\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} + \|\bar{\psi}_\alpha\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} \right] \\ & -2\delta(\delta + 2\alpha)\|\bar{\xi}_\alpha\|^2. \end{aligned} \tag{5.38}$$

Using this in (5.25) along with the estimates

$$\begin{aligned} \left| \frac{1}{2} \langle i[|\nabla H|^2,\, A] \rangle_{\bar{\xi}_\alpha} \right| \ =\ & \left| \frac{\delta^2}{2} \langle i[|\nabla h|^2,\, A] \rangle_{\bar{\xi}_n} \ +\ \delta\alpha \langle i[\nabla h \cdot \hat{x},\, A] \rangle_{\bar{\xi}_\alpha} \right| \\ \le\ & 3 c_h \delta(\delta + \alpha) \|\bar{\xi}_\alpha\|^2 \end{aligned} \tag{5.39}$$

(cf. (4.3) and following) and

$$Re\, \langle \gamma_H A \rangle_{\bar{\xi}_\alpha} = \delta\, Re\, \langle \gamma_h A \rangle_{\bar{\xi}_\alpha} + \alpha\, Re \langle \gamma A \rangle_{\bar{\xi}_\alpha} \ \ge\ -(\delta + \alpha) o_R(1) \|\bar{\xi}_\alpha\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} \tag{5.40}$$

(cf. (4.7) and following), we obtain

$$\begin{aligned} Im\, \langle (\bar{K} + \kappa)\bar{\xi}_\alpha,\ A\bar{\xi}_\alpha \rangle \ \ge\ & \frac{1}{2}(-\kappa + \alpha^2 + \omega^2)\|\bar{\xi}_\alpha\|^2 + \frac{1}{2}\|\nabla\bar{\xi}_\alpha\|^2 \\ & -\mu(\delta,\alpha) o_R(1) \left[\|\bar{\xi}_\alpha\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} + \|\bar{\psi}_\alpha\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} \right] \\ & -(\delta + \alpha) o_R(1) \|\bar{\xi}_\alpha\|^2_{L^2(S^1_\omega, H^1(\mathrm{R}^N))} \\ & -6 c_h \delta(\delta + \alpha)\|\bar{\xi}_\alpha\|^2. \end{aligned} \tag{5.41}$$

This is analogous to equation (4.8) obtained in the proof of Theorem 2.2. Proceeding as there and using (5.32) we have,

$$\begin{aligned}
&\frac{1}{2}(-\kappa+\alpha^2+\omega^2)\|\bar{\xi}_\alpha\|^2+\frac{1}{2}\|\nabla\bar{\xi}_\alpha\|^2\\
&\qquad -\mu_1(\delta,\alpha)o_R(1)\|\bar{\xi}_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))}\\
&\qquad -10c_h\delta(\delta+\alpha)\|\bar{\xi}_\alpha\|^2\\
&\leq \mu_1(\delta,\,\alpha)\Big(1+o_R(1)\Big)\|\bar{\psi}_\alpha\|^2_{L^2(S^1_\omega,H^1(\mathrm{R}^N))} \qquad (5.42)
\end{aligned}$$

(cf. (4.13)). Since $\omega^2 > f'(0) = \kappa$, $-\kappa+\omega^2 > 0$. Thus, we solve

$$10c_h\delta(\delta+\alpha) \;=\; \frac{1}{8}(-\kappa+\omega^2+\alpha^2) \qquad (5.43)$$

for δ (cf. (4.17) and the discussion there), the solution of which we denote by $\delta(\alpha)$. Arguing as in Lemma 4.2 with $m_o = 0$, we have that $\Phi^n(0)\to\infty$ where Φ is the map $\Phi:\;\alpha\mapsto\alpha+\delta(\alpha)$. By applying the same iteration argument described at the end of Section 4.2 we can show that $e^{\alpha r}\bar{\varphi}\in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all α. The unique continuation theorem for $\bar{K}$ (see the paragraph at the end of section 5.1) then implies that $\bar{\varphi}=0$. This completes the proof of Theorem 1.1 $\square$.

<u>Discussion:</u>

One may attempt a similar analysis on $\Pi_{m-1}\varphi$ for $m>1$. That is, perhaps one can prove that if $m^2\omega^2 > f'(0)$ then $\bar{\Pi}_{m-1}\varphi=0$. However, this would require a very special, and hence nongeneric, relationship between the solution and nonlinearity. For, applying the method used above in the proof of Theorem 1.1, we first project NLW onto *Ran* $\bar{\Pi}_{m-1}$ to derive an equation for $\bar{\Pi}_{m-1}\varphi$. Let us define

$$\varphi_{<m}\equiv\Pi_{m-1}\;,\qquad \varphi_{\geq m}\equiv\bar{\Pi}_{m-1}$$

and set $\Pi \equiv \Pi_{m-1}$. Then,

$$0 = \bar{\Pi}\Big(\partial_t^2 \varphi - \Delta\varphi + f(\varphi)\Big) = K_o \varphi_{\geq m} + \bar{\Pi} f(\varphi). \tag{5.44}$$

We would like to form a potential from the term $\bar{\Pi} f(\varphi)$. Writing

$$f(\varphi) = f(\varphi_{<m} + \varphi_{\geq m}) = f(\varphi_{<m}) + f(\varphi_{\geq m}) + \varphi_{<m}\varphi_{\geq m}\, g(\varphi_{<m}, \varphi_{\geq m}), \tag{5.45}$$

we have

$$\begin{aligned} \bar{\Pi} f(\varphi) &= \bar{\Pi}\Big(f(\varphi_{<m} + \varphi_{\geq m}) - f(\varphi_{<m})\Big) + \bar{\Pi} f(\varphi_{<m}) \\ &= \bar{\Pi} V_{\bar{\varphi}} \bar{\Pi} + \bar{\Pi} f(\varphi_{<m}) \\ &= U \varphi_{\geq m} + \bar{\Pi} f(\varphi_{<m}) \end{aligned} \tag{5.46}$$

where here

$$V_{\bar{\varphi}} = \frac{f(\varphi_{\geq m})}{\varphi_{\geq m}} - \kappa + \varphi_{<m}\, g(\varphi_{<m}, \varphi_{\geq m}), \quad \kappa = f'(0). \tag{5.47}$$

But now the term $\bar{\Pi} f(\varphi_{<m})$ may be nonzero because, although $\varphi_{<m}$ has no modes $\geq m$, $f(\varphi_{<m})$ *may* contain nonzero modes $\geq m$. That is, f may generate those modes from the lower modes of φ. For $\bar{\Pi} f(\varphi_{<m})$ to vanish requires a special relationship among the modes of φ, a relationship determined by the structure of f. In the case $m = 1$ no such special relationship is needed: $\bar{P}_0 f(P_0 \varphi)$ is always zero because f cannot generate time dependent modes from a time independent function. In terms of the spectrum of K, f couples the branches $[-m^2\omega^2, \infty)$, $m > 0$ to one another while the zero branch remains uncoupled.

Going back to (5.44) and using (5.46), we see that the equation satisfied by $\varphi_{\geq m}$ is in fact

$$K_o \varphi_{\geq m} + U \varphi_{\geq m} \;=\; -\bar{\Pi} f(\varphi_{<m}). \tag{5.48}$$

We can think of $-\bar{\Pi} f(\varphi_{<m})$ as a forcing term: the system is not closed. The case considered in Theorem 1.1 ($m = 1$) is then special in that projecting NLW onto $\mathrm{E}_0^{\perp}$ results in a *closed* dynamical system ($\bar{P}_0 f(\varphi_0) = 0$).

It is worth remarking that in the special case when $P_k \varphi = 0$ for all $|k| < m$ then, because $f(0) = 0$, $\bar{\Pi} f(\varphi_{<m}) = 0$ and we *do* have a closed system. In this case the branches $[-k^2\omega^2, \infty)$, $|k| < m$, of the spectrum of K are decoupled from the higher branches so that it is possible to project them out.

5.3 A nonexistence result

The proof of Theorem 5.1 relied only on the fact that $e^{\alpha r}\psi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all $\alpha \geq 0$. By considering the eigenvalue $\lambda = -f'(0)$ and corresponding eigenfunction φ where φ is a $2\pi/\omega$ -periodic solution of NLW, that theorem implies the following.

Theorem 5.5 *Let φ be a $2\pi/\omega$-periodic solution of NLW on $\mathcal{D}_\omega$. If $e^{\alpha r}\varphi \in L^2(S^1_\omega, H^1(\mathrm{R}^N))$ for all α, then $\varphi = 0$.*

For example, periodic solutions from $\mathcal{D}_\omega$ cannot decay like $e^{-\alpha r^2}$ for any $\alpha > 0$, nor can they have compact spatial support.

6 Appendix : Operator calculus

We outline here a convenient operator calculus first introduced in [Hel,Sjö] (see also [D]) for functions $g(H)$ of self-adjoint operators based on the representation

$$g(H) = \frac{1}{2\pi} \int_{\mathbb{R}^2} (H-z)^{-1} \, \partial_{\bar{z}} \tilde{g}(z) \, dudv \tag{A1}$$

where $z = u + iv$ and $\partial_{\bar{z}} = \partial_x + i\partial_y$. Here g is a complex-valued function in $C_o^\infty(\mathbb{R})$ and $\tilde{g}$ an *almost analytic extension* of g into the complex plane. By almost analytic we mean that $\tilde{g}$ satisfies the Cauchy-Riemann equations on $\mathbb{R}$:

$$\partial_{\bar{z}} \tilde{g}(z) = 0 \quad for \;\; z \in \mathbb{R}.$$

We abbreviate (A1) by writing

$$g(H) = \int d\tilde{g}(z) \; (H-z)^{-1} \;\; ; \quad d\tilde{g}(z) \equiv \frac{1}{2\pi} \partial_{\bar{z}} \tilde{g}(z) \, dudv. \tag{A2}$$

For example, we can construct an almost analytic extension of g by the formula

$$\tilde{g}(z) = \left(g(u) + ivg'(u) \right) \chi(z) \tag{A3}$$

where $\chi \in C_o^\infty(\mathbb{C})$ with $\chi \equiv 1$ on some complex neighborhood of $supp\,(g)$.

Lemma A1 *If* $g \in C_o^\infty(\mathbb{R})$ *and* $\tilde{g}$ *is an almost analytic extension of* g *as given above, then* $g(H)$ *is given by (A1).*

Proof:

$\partial_{\bar{z}}\tilde{g}$ has compact support and vanishes on the real axis, so that $| \partial_{\bar{z}} \tilde{g}(z) | < const \; | v |$. Also, $\| (H-z)^{-1} \| < | v |^{-1}$ so that the integral (A1) exists as the norm limit of Riemann sums. It is enough now to show that

$$g_\varepsilon(t) \equiv \int_{|v|>\varepsilon} d\tilde{g}(z) \, (t-z)^{-1}, \quad t \in \mathbb{R} \tag{A4}$$

converges pointwise to $g(t)$ as $\varepsilon \to 0$. First note that, since $\partial_{\bar z}(t-z)^{-1} = 0$ for $z \notin \mathrm{R}$, $\partial_{\bar z}\tilde g(z)(z-t)^{-1} = \partial_{\bar z}\Big(\tilde g(z)(z-t)^{-1}\Big)$. Integrating by parts in u and v,

$$g_\varepsilon(t) \;=\; \frac{i}{2\pi}\int_{\mathrm{R}} \tilde g(u+iv)(u+iv-t)^{-1}\,\Big|_{v=\varepsilon}^{v=-\varepsilon} \;\; du. \tag{A5}$$

Expanding $\tilde g(u+iv) = g(u) \pm i\varepsilon g'(u) + \mathcal{O}(\varepsilon^2)$ we find,

$$\tilde g(u+iv)(u+iv-t)^{-1}\,\Big|_{v=\varepsilon}^{v=-\varepsilon} \;=\; \frac{1}{\pi} g(u)\frac{\varepsilon}{(u-t)^2+\varepsilon^2} \;-\; \frac{1}{\pi} g'(u)\frac{\varepsilon(u-t)}{(u-t)^2+\varepsilon^2} \;+\; \mathcal{O}(\varepsilon). \tag{A6}$$

The last two terms are bounded and vanish pointwise for $u \neq t$ as $\varepsilon \to 0$. Therefore,

$$\lim_{\varepsilon\to 0}\; g_\varepsilon(t) \;=\; \lim_{\varepsilon\to 0}\; \frac{1}{\pi}\int_{\mathrm{R}} g(u)\frac{\varepsilon}{(u-t)^2+\varepsilon^2}\,du \;=\; g(t) \qquad \square \tag{A7}$$

Commutator Estimates

Suppose H and A are unbounded self-adjoint operators on a Hilbert space $\mathcal{H}$. If $[H,\ A]$ is H-bounded, then for $\psi \in \mathcal{H}$,

$$\begin{aligned} \|[H,\ A]\,(H-z)^{-1}\,\psi\| \;&\le\; a\|H\,(H-z)^{-1}\,\psi\| + b\|\,(H-z)^{-1}\,\psi\|, \qquad a,b>0,\\ &=\; a\|\psi + z\,(H-z)^{-1}\,\psi\| + b\|\,(H-z)^{-1}\,\psi\|\\ &\le\; a\|\psi\| + (\mid z\mid + b)\,\|\,(H-z)^{-1}\,\psi\|\\ &\le\; a\|\psi\| + (\mid z\mid + b)\,\mid v\mid^{-1}\,\|\psi\|\\ &\le\; c\,(1+\mid z\mid)\,\mid v\mid^{-1}\,\|\psi\|. \end{aligned}$$

That is,

$$\|[H,\ A]\,(H-z)^{-1}\,\| \le const\,(1+\mid z\mid)\,\mid v\mid^{-1}. \tag{A8}$$

In particular, this holds when $[H,\ A]$ is bounded. Thus, if $g \in C_o^\infty(\mathrm{R})$ then the representation

$$[g(H),\ A] = \int d\tilde g(z)\;(H-z)^{-1}\,[H,\ A]\,(H-z)^{-1} \tag{A9}$$

requires an almost analytic extension $\tilde g$ that satisfies $\mid\partial_{\bar z}\tilde g(z)\mid \le\; const\, v^2$. To this end let

$\chi \in C_o^\infty(\mathrm{R})$ with $\chi \equiv 1$ on some open interval containing 0. We define an almost analytic extension of g by the formula

$$\tilde{g}(z) = \chi(v/\langle u\rangle) \sum_{k=0}^{2} g^{(k)}(u) \frac{(iv)^k}{k!} \tag{A10}$$

where $\langle u\rangle = (1+u^2)^{1/2}$. Then,

$$\begin{aligned} \partial_{\bar{z}}\tilde{g}(z) &= \langle u\rangle^{-1}\chi'(v/\langle u\rangle)\left(i - \frac{uv}{\langle u\rangle^2}\right) \sum_{k=0}^{2} g^{(k)}(u)\frac{(iv)^k}{k!} \quad + \quad \chi(v/\langle u\rangle) g^{(3)}(u) \frac{(iv)^2}{2} \\ &= a(z) \quad + \quad b(z). \end{aligned}$$

We see that both $a(z)$ and $b(z)$ are compactly supported, $a(z)$ away from $v = 0$ (since $\chi'(t) \equiv 0$ in a neighborhood of 0). Furthermore, $a(z)$ is bounded and $| b(z) | \leq$ *const* v^2. Thus, $| \partial_{\bar{z}}\tilde{g}(z) | \leq$ *const* v^2.

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