Positive Commutator Methods for Nonlinear Wave Equations *

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Abstract

We discuss an application of positive commutators to the problem of periodic solutions of nonlinear wave equations.

1 Introduction

The use of positive commutators has played a key role in solving many problems in quantum mechanics, e.g., absence of bound states with positive energy, localization of bound states, local (in space) time decay of scattering states, and asymptotic completeness (see for example [CFKS] or [HS]).

In this article we announce recent results of ours concerning an application of positive commutator methods to the problem of periodic solutions of the nonlinear wave equation (NLW)

$$\partial_t^2\varphi - \Delta\varphi + f(\varphi) = 0. \tag{1.1}$$

Here $\varphi:\ \mathbb{R}^N_x \times \mathbb{R}_t \to \mathbb{R}$, $\ f:\ \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$, and $\partial_t^2 = \partial^2/\partial t^2$, $\Delta = \sum_{i=1}^N \partial^2/\partial {x_i}^2$. By a periodic solution we understand solutions that are periodic in time t, and L^2 in x. Full details of our results will be published elsewhere.

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We address two problems concerning periodic solutions: constraints on their frequencies and exponential bounds on their spatial localization. To state our results we introduce some notation. Let S^1_ω denote the circle of radius ω^{-1}. The class of solutions we consider is the following set:

$$\mathcal{D}_\omega \equiv \Big\{\varphi \in H^1(\mathbb{R}^N \times S^1_\omega) \quad ; \quad \text{if } \psi \text{ is any of } \varphi, \partial_t\varphi \text{ or } x\cdot\nabla\varphi, \text{ then } \|\psi\|_{L^\infty(\mathbb{R}^N\times S^1_\omega)} < \infty \\ \text{and } \lim_{|x|\to\infty} \mid \psi(x,t) \mid = 0 \text{ uniformly in t} \Big\}. \tag{1.2}$$

(This class of solutions can, probably, be enlarged). Here $H^1(\Omega)$ stands for the Sobolev space of order 1 for functions on Ω. Let $L^2(S^1_\omega, H^1(\mathbb{R}^N))$ denote functions $\psi: \ S^1_\omega \to H^1(\mathbb{R}^N)$, such that $\|\psi\|^2_{L^2(\mathbb{R}^N\times S^1_\omega)} + \|\nabla\psi\|^2_{L^2(\mathbb{R}^N\times S^1_\omega)} \equiv \|\psi\|^2_{L^2(S^1_\omega, H^1(\mathbb{R}^N))} < \infty$.

We are able to prove the following theorems.

Theorem 1.1 *Suppose $f \in C^3(\mathbb{R},\mathbb{R})$. Let φ be a nontrivial $2\pi/\omega$-periodic solution of NLW on $\mathcal{D}_\omega$. Then $\omega^2 \le f'(0)$.*

Theorem 1.2 *Suppose $f \in C^3(\mathbb{R},\mathbb{R})$ such that $f'(0) \neq m^2\omega^2$, $m \in \mathbb{Z}$. Let φ be a $2\pi/\omega$-periodic solution of NLW on $\mathcal{D}_\omega$. Then $e^{\alpha|x|}\varphi \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$ for all α satisfying*

$$\alpha^2 < f'(0) - \lfloor\sqrt{\frac{f'(0)}{\omega^2}}\rfloor^2\omega^2$$

where $\lfloor a \rfloor$ denotes the integer part of a.

To apply the technique of positive commutators we first formulate NLW as an eigenvalue problem. Let

$$W(u) = \frac{f(u)}{u} - \kappa, \qquad \kappa = f'(0). \tag{1.3}$$

For a given function $\varphi \in L^2(\mathbb{R}^N \times S^1_\omega)$ define the "potential" $W_\varphi(x,t) \equiv W(\varphi(x,t))$ acting as an operator of multiplication, and the linear operators

$$K_\varphi = K_o + W_\varphi, \qquad K_o \equiv \partial_t^2 - \Delta \tag{1.4}$$

on $L^2(\mathbb{R}^N \times S^1_\omega)$. By construction we have that

$$K_\varphi \varphi = -\kappa\varphi. \tag{1.5}$$

From (1.3) and the chain rule, it is clear that W_φ, $\partial_t W_\varphi$ and $x \cdot \nabla W_\varphi$ are bounded and vanish as $|x| \to \infty$, uniformly in t. In particular, this implies that for $\varphi \in \mathcal{D}$, K_φ as defined by (1.4) is self-adjoint on $L^2(\mathbb{R}^N \times S^1_\omega)$.

By separation of variables, $\sigma(K_o) = \sigma(\partial_t^2) + \sigma(-\Delta) = \bigcup_{m\in\mathbb{Z}}[-m^2\omega^2, \infty)$. That is, the essential spectrum of K_o consists of semi-infinite branches of continuous spectrum originating from the points $\{-m^2\omega^2;\ m \in \mathbb{Z}\}$. We expect the essential spectrum of K_o to be stable under the perturbation W_φ. Thus, we denote by $\mathcal{E}(K_\varphi) = \{-m^2\omega^2;\ m \in \mathbb{Z}\}$ the *thresholds* of K_φ. Due to the nonresonance condition $f'(0) \neq m^2\omega^2$, $m \in \mathbb{Z}$, and the hypothesis $f'(0) > 0$, there exists an $m_o \in \mathbb{N}$ such that

$$-m_o^2\omega^2 \ < \ -f'(0) \ < \ -(m_o - 1)^2\omega^2.$$

Then, $(m_o - 1)^2 = \lfloor\sqrt{\frac{f'(0)}{\omega^2}}\rfloor^2$ and Theorem 1.1 states that $e^{\alpha r}\varphi \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$ for all α such that α^2 is less than the distance from $-f'(0)$ to the nearest threshold above (i.e., greater than) $-f'(0)$.

2 Outline of proof

We will omit the subscript φ when discussing the operators K_φ and W_φ so that from now on $K = K_\varphi$ and $W = W_\varphi$. The main ingredients of the analysis are microlocalization in the operators K and $i\partial_t$ and the use of compactness. By the former we mean the decomposition of the space $L^2(\mathbb{R}^N \times S^1_\omega)$ according to the spectral projections associated to K and $i\partial_t$. This allows us to isolate certain properties of functions from this space. For example, an operator B may be bounded below globally on $L^2(\mathbb{R}^N \times S^1_\omega)$, i.e., $\langle B\rangle_\varphi \geq -c\|\varphi\|^2$ for all $\varphi \in L^2(\mathbb{R}^N \times S^1_\omega)$, where $\|\cdot\|$ denotes the $L^2(\mathbb{R}^N \times S^1_\omega)$ norm and $\langle B\rangle_\varphi$ denotes the expectation value $\langle B\varphi,\, \varphi\rangle$. However, B may be *essentially* positive in the sense that it may be positive when restricted to a certain subspace $\mathbb{E}$ and small otherwise. Here $\mathbb{E}$ will be the range of a spectral projection $E_I(H)$ associated to a self-adjoint operator H on $L^2(\mathbb{R}^N \times S^1_\omega)$ corresponding to some interval $I \subset \mathbb{R}$. Writing

$$\begin{aligned} B &= E_I(H)BE_I(H) + \bar{E}_I(H)B\bar{E}_I(H) \\ &\quad + \bar{E}_I(H)BE_I(H) + E_I(H)B\bar{E}_I(H), \end{aligned} \tag{2.1}$$

we require that the expectation values of the last three terms on the right of (2.1) are negligable (small).

To show how compactness is used, consider the term $\langle \bar{E}_I(H)B\bar{E}_I(H)\rangle_\varphi$. It may be possible to find a self-adjoint operator A such that for some interval $J \subset \mathbb{R}$, $\bar{E}_I(H)B\bar{E}_I(H)$ is a compact operator when restricted to the subspace $Ran\, E_J(A)$. If φ can be embedded into a family φ_ε that converges weakly to zero as $\varepsilon \to 0$, then $\langle E_J(A)\bar{E}_I(H)B\bar{E}_I(H)E_J(A)\rangle_{\varphi_\varepsilon}$ converges to zero. Another instance of compactness follows from the property that the potential W is bounded and vanishes as $|x| \to \infty$. This implies that W is compact relative to $-\Delta$ for each t (see for example [RS-IV]). This relative compactness of W will be important in the analysis.

We remark that compactness as just described arises naturally in many applications, that is, it is an inherent feature of the problem, and an extremely useful one.

A priori, the function φ under consideration may be microlocalized to some extent as a consequence of being an eigenfunction, or approximate eigenfunction, of a self-adjoint operator. In the case we are considering here, periodic solutions of NLW, φ is an eigenfunction of K. We use this strong localization of φ in the analysis.

We begin now a more detailed discussion of these ideas in the context of proving exponential bounds for periodic solutions of NLW. Recall that $\varphi \in \mathcal{D}_\omega$ is a $2\pi/\omega$ -periodic solution of NLW and that $K\varphi = -\kappa\varphi$, $\kappa = f'(0)$. Our objective is to show that for some $\delta > 0$, $e^{\delta r}\varphi \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$. To this end we first cut-off the function r at infinity, we denote this function by $h(r)$, with the cut-off depending on a parameter ε and such that $\lim_{\varepsilon\to 0} h(r) = r + const$. Then, $e^{\delta h(r)}\varphi \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$ for all $\varepsilon > 0$. We will show that

$\|e^{\delta h(r)}\varphi\|_{L^2(S^1_\omega,H^1(\mathbb{R}^N))} < \infty$ uniformly in ε. This implies then that $e^{\delta r}\varphi \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$. To utilize the compact operators that will arise, we multiply $e^{\delta h(r)}\varphi$ by a cut-off function χ_R whose support lies outside of a ball of radius R in $\mathbb{R}^N$. Then, for any compact operator C, $\langle C\rangle_{\chi e^{\delta h(r)}\varphi} \leq o_R(1)\|\chi_R e^{\delta h(r)}\varphi\|^2$ where $o_R(1)$ denotes a quantity that vanishes as $R \to \infty$. This is a stronger statement than the mere vanishing of $\langle C\rangle_{\chi e^{\delta h(r)}\varphi}$.

We make these definitions more precise. For $R > 0$ and $\delta \geq 0$ set

$$\varphi_R \;=\; \chi_R e^{\delta h(r)}\varphi, \quad \text{and} \tag{2.2}$$

$$K^h \;=\; e^{\delta h(r)} K e^{-\delta h(r)} \;=\; K - \delta^2|\nabla h|^2 + i\delta\gamma_h. \tag{2.3}$$

Here h is a smooth function such that $h(r) = 0,\ r < 2R,\ h(r) = r + \mathit{const},\quad r > 3R$ and $\gamma_h = \frac{1}{i}(\nabla h \cdot \nabla + \nabla \cdot \nabla h)$. χ_R is a smooth cut-off function: $\chi_R(r) = 0,\ r < R$ and $\chi_R(r) = 1,\ r > 2R$. The important features of the function h are that $h = 0$ on $\mathit{supp}\,(\chi_R^{'})$, $h(r) = r + \mathit{const}$ near infinity, with $|h^{(m)}(r)| \leq c_m R^{1-m}$, c_m independent of R. In the rigorous analysis the function h is cut-off in a neighborhood of infinity as mentioned above. However, to make the present discussion simpler we will not perform this regularization. This does not affect the presentation of the essential ideas.

Our goal is to show that for R sufficiently large and for some $\delta > 0$, $\varphi_R \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$ uniformly in ε. The function φ_R is an approximate eigenfunction of K^h in the sense that

$$\|(K^h + \kappa)\varphi_R\| \;\leq\; o_R(1)\|\varphi\|_{L^2(S^1_\omega,H^1(\mathbb{R}^N))}. \tag{2.4}$$

This follows from the formula

$$\begin{aligned}(K^h + \kappa)\varphi_R \;&=\; e^{\delta h(r)}\chi_R(K+\kappa)\varphi \;+\; e^{\delta h(r)}[-\Delta,\, \chi_R]\varphi \\ &=\; (-\Delta\chi_R)\varphi \;-\; 2\nabla\chi_R \cdot \nabla\varphi,\end{aligned} \tag{2.5}$$

where we have used that $h = 0$ on $\mathit{supp}\,(\chi_R^{'})$, and then the property $|\chi_R^{(m)}| \leq cR^{-m}$ to arrive at (2.5). Let $A = \frac{1}{2i}(x \cdot \nabla + \nabla \cdot x)$. Since $e^{\delta h(r)}\varphi$ is an eigenfunction of K^h, we have that

$$0 \;=\; Im\,\langle (K^h + \kappa)e^{\delta h(r)}\varphi,\; Ae^{\delta h(r)}\varphi\rangle.$$

This equation is related to the virial theorem of quantum mechanics [CFKS]. Expanding the inner product, we find

$$\begin{aligned} 0 &= Im\,\langle (K^h+\kappa)e^{\delta h(r)}\varphi,\ Ae^{\delta h(r)}\varphi\rangle \\ &= \frac{1}{2}\langle i[K,\,A]\rangle_{e^{\delta h(r)}\varphi} \ + \ \delta Re\,\langle \gamma_h A\rangle_{e^{\delta h(r)}\varphi} - \frac{\delta^2}{2}\langle i[|\nabla h|^2,\,A]\rangle_{e^{\delta h(r)}\varphi}. \end{aligned} \tag{2.6}$$

If we substitute φ_R for $e^{\delta h(r)}\varphi$ in this equation the left hand side is no longer zero, but since $(K^h+\kappa)\varphi_R$ is localized to the support of χ_R' (cf. (2.5)) where $h=0$, we find that

$$|Im\,\langle (K^h+\kappa)\varphi_R,\,A\varphi_R\rangle| \ \le\ c\|\varphi\|^2_{L^2(S^1_\omega,H^1(\mathbb{R}^N))}, \tag{2.7}$$

where c is independent of R. Furthermore, using that A and γ_h are parallel, a simple calculation gives

$$Re\,\langle\gamma_h A\rangle_{\varphi_R} \ = \ \text{positive term } + o_R(1)\|\varphi_R\|^2. \tag{2.8}$$

By design of h we have the estimate

$$\Big|\frac{\delta^2}{2}\langle i[|\nabla h|^2,\,A]\rangle_{\varphi_R}\Big| \ \le\ c\delta^2\|\varphi_R\|^2. \tag{2.9}$$

These relations yield

$$\langle i[K,A]\rangle_{\varphi_R} - o_R(1)\|\varphi_R\|^2 - c\delta^2\|\varphi_R\|^2 \ \le\ c\|\varphi\|^2_{L^2(S^1_\omega,H^1(\mathbb{R}^N))}. \tag{2.10}$$

We see how positivity of $i[K,\,A]$ enters: if we can show that

$$\langle i[K,\,A]\rangle_{\varphi_R} \ \ge\ \theta\|\varphi_R\|^2_{L^2(S^1_\omega,H^1(\mathbb{R}^N))}, \qquad \text{for some } \theta>0, \tag{2.11}$$

then from (2.10) and (2.11) it follows that for R sufficiently large and δ sufficiently small,

$$\|\varphi_R\|^2_{L^2(S^1_\omega,H^1(\mathbb{R}^N))} \ \le\ c\|\varphi\|^2_{L^2(S^1_\omega,H^1(\mathbb{R}^N))} \ <\ \infty. \tag{2.12}$$

Hence, $e^{\delta r}\varphi \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$.

To prove (2.11) we begin by evaluating the commutator

$$i[K,\, A] \;=\; -2\Delta - x\cdot\nabla W$$

and writing $-\Delta = K - \partial_t^2 - W$. We have then that

$$\langle i[K,\, A]\rangle_{\varphi_R} \;=\; \langle -\Delta\rangle_{\varphi_R} + \langle K+\kappa\rangle_{\varphi_R} \;-\; \langle \kappa\rangle_{\varphi_R} \;+\; \langle -\partial_t^2\rangle_{\varphi_R} \;-\; \langle W + x\cdot\nabla W\rangle_{\varphi_R}.$$

This allows us to take advantage of the localization of φ with respect to K. Now,

$$K+\kappa \;=\; K^h + \kappa + \delta^2|\nabla h|^2 - i\delta\gamma_h \tag{2.13}$$

from which it follows that, since $K+\kappa$ is self-adjoint and $i\gamma_h$ is skew-adjoint,

$$\langle K+\kappa\rangle_{\varphi_R} \;=\; Re\,\langle K^h+\kappa\rangle_{\varphi_R} + \delta^2\langle|\nabla h|^2\rangle_{\varphi_R}.$$

Using this and (2.4),

$$\begin{aligned} |\langle K+\kappa\rangle_{\varphi_R}| \;&\le\; |\langle K^h+\kappa\rangle_{\varphi_R}| + c\delta^2\|\varphi_R\|^2 \\ &\le\; \|(K^h+\kappa)\varphi_R\|\,\|\varphi_R\| + c\delta^2\|\varphi_R\|^2 \\ &\le\; o_R(1)\Big[\|\varphi\|^2_{L^2(S^1_\omega,H^1(\mathbb{R}^N))} + \|\varphi_R\|^2_{L^2(S^1_\omega,H^1(\mathbb{R}^N))}\Big] + c\delta^2\|\varphi_R\|^2. \end{aligned} \tag{2.14}$$

Because W and $x\cdot\nabla W$ both vanish as $|x|\to\infty$ uniformly in t and φ_R is supported outside of a ball of radius $2R$, we obtain the estimate

$$|\langle 2W + x\cdot\nabla W\rangle_{\varphi_R}| \;\le\; o_R(1)\|\varphi_R\|^2. \tag{2.15}$$

Combining these results we have the inequality

$$\begin{aligned} \langle i[K,\, A]\rangle_{\varphi_R} \;\ge\; &\|\nabla\varphi_R\|^2 - \kappa\|\varphi_R\|^2 \;+\; \langle -\partial_t^2\rangle_{\varphi_R} \\ &- o_R(1)\Big[\|\varphi\|^2_{L^2(S^1_\omega,H^1(\mathbb{R}^N))} + \|\varphi_R\|^2_{L^2(S^1_\omega,H^1(\mathbb{R}^N))}\Big] - c\delta^2\|\varphi_R\|^2. \end{aligned} \tag{2.16}$$

It remains to show that

$$\langle -\partial_t^2 \rangle_{\varphi_R} \;\geq\; m_o^2 \omega^2 \|\varphi_R\|^2, \tag{2.17}$$

up to a remainder term that we can control through the parameters R and δ, where $m_o \geq 1$ is the integer characterized by the relation $-m_o\omega^2 < -\kappa < -(m_o-1)^2\omega^2$. To prove (2.17) we first write $L^2(\mathbb{R}^N \times S^1_\omega) \simeq \oplus_{k\in\mathbb{Z}} \mathbb{E}_k$ where $\mathbb{E}_k = e^{ik\omega t} \otimes L^2(\mathbb{R}^N)$ is an eigenspace of $i\partial_t$. For $\psi \in L^2(\mathbb{R}^N \times S^1_\omega)$ we write $\psi = \sum_k e^{ik\omega t}\psi_k$ where $\psi_k(x) = (2\pi/\omega)^{-1} \int_{S^1_\omega} \psi(x,t) e^{-ik\omega t}\, dt$. Let P_k denote the projection onto the k^{th} mode: $P_k\psi = e^{ik\omega t}\psi_k$, and $\Pi_m = \sum_{|k|\leq m} P_k$, $\bar{\Pi}_m = \mathbf{1} - \Pi_m$. Using that $-\partial_t^2 \bar{\Pi}_{m_o-1} \geq m_o^2\omega^2 \bar{\Pi}_{m_o-1}$ and that $-\partial_t^2 \Pi_{m_o-1} \geq 0$ (these inequalities are in the sense of quadratic forms), we obtain

$$\begin{aligned}
\langle -\partial_t^2 \rangle_{\varphi_R} &= \langle -\partial_t^2 \rangle_{\bar{\Pi}_{m_o-1}\varphi_R} \;+\; \langle -\partial_t^2 \rangle_{\Pi_{m_o-1}\varphi_R} \\
&\geq m_o^2\omega^2 \langle \bar{\Pi}_{m_o-1} \rangle_{\varphi_R} \\
&= m_o^2\omega^2 \|\varphi_R\|^2 \;-\; m_o^2\omega^2 \langle \Pi_{m_o-1} \rangle_{\varphi_R}
\end{aligned} \tag{2.18}$$

To estimate the second term on the right hand side we microlocalize with respect to K and $i\partial_t$ and use the $-\Delta$ compactness of W.

Pick an interval $I \subset \mathbb{R}$ containing $-\kappa$ and such that $\sup(I) < -(m_o-1)^2\omega^2$. Let $E_I(K)$ be a smoothed-out spectral projection of K corresponding to the interval I and decompose Π_{m_o-1} with respect to $E_I(K)$ and $\bar{E}_I(K)$ to obtain

$$\begin{aligned}
\Pi_{m_o-1} &= \Big(E_I(K) + \bar{E}_I(K)\Big)\Pi_{m_o-1}\Big(E_I(K) + \bar{E}_I(K)\Big) \\
&= E_I(K)\Pi_{m_o-1}E_I(K) \;+\; \bar{E}_I(K)\Pi_{m_o-1}E_I(K) \\
&\qquad + E_I(K)\Pi_{m_o-1}\bar{E}_I(K) \;+\; \bar{E}_I(K)\Pi_{m_o-1}\bar{E}_I(K).
\end{aligned} \tag{2.19}$$

From this and the Schwarz inequality we then have the bound

$$\Big|\langle \Pi_{m_o-1} \rangle_{\varphi_R}\Big| \;\leq\; 3\|\bar{E}_I(K)\varphi_R\|\|\varphi_R\| \;+\; \langle C \rangle_{\varphi_R}, \tag{2.20}$$

where $C = E_I(K)\Pi_{m_o-1}E_I(K)$ is a compact operator, as we will see shortly. Because φ_R has support that goes off to infinity as $R \to \infty$, this term is of order $o_R(1)\|\varphi_R\|^2$.

To prove that the operator $E_I(K)\Pi_{m_o-1}E_I(K)$ is compact, we use the relative compactness of W. The operator K_o has a natural decomposition along the eigenspaces $\mathbb{E}_k$:

$$K_o \;=\; \oplus_{k\in\mathbb{Z}}(-k^2\omega^2 - \Delta),$$

from which it follows that

$$(K_o - z)^{-1} \;=\; \oplus_{k\in\mathbb{Z}}(-k^2\omega^2 - \Delta - z)^{-1}, \quad z \in \mathbb{C}\setminus\mathbb{R}.$$

Because W is compact relative to $-\Delta$ for each t, each mode $W_k(x)$ of W is compact relative to $-\Delta$. As a result, W is compact relative to K_o when restricted to finitely many of the subspaces $\mathbb{E}_k$. Introducing the spectral projections $E_I(K_o)$ associated to the operator K_o, we write

$$\begin{aligned} E_I(K)\Pi_{m_o-1}E_I(K) \;&=\; E_I(K_o)\Pi_{m_o-1}E_I(K_o) \\ &\quad + \Big(E_I(K) - E_I(K_o)\Big)\Pi_{m_o-1}E_I(K) \\ &\quad + E_I(K_o)\Pi_{m_o-1}\Big(E_I(K) - E_I(K_o)\Big). \end{aligned} \tag{2.21}$$

The first term on the right hand side is zero by conservation of energy. That is,

$$P_k E_I(K_o) = 0 \quad \text{for } k < m_o. \tag{2.22}$$

This relation can be seen as follows. On $Ran\, P_k = \mathbb{E}_k,\;\; K_o = -k^2\omega^2 - \Delta$ so that

$$P_k E_I(K_o) \;=\; P_k E_I(-k^2\omega^2 - \Delta).$$

Since $\sup(I) < -k^2\omega^2$, $\;spec\,(-k^2\omega^2 - \Delta) = [-k^2\omega^2, \infty)$ is disjoint from I. Hence, $E_I(-k^2\omega^2 - \Delta) = 0$.

To treat the other two terms on the right hand side of (2.21) it is enough to consider the resolvents $R(z) = (K-z)^{-1}$ and $R_o(z) = (K_o - z)^{-1}$ in place of the projections $E_I(K)$ and $E_I(K_o)$. For the second term on the right, say, and using the second resolvent equation, we have, for any $m_1 \in \mathbb{N}$,

$$R(z)WR_o(z)\Pi_{m_o-1}E_I(K) \;=\; R(z)\Pi_{m_1}WR_o(z)\Pi_{m_o-1}E_I(K) + R(z)\bar{\Pi}_{m_1}WR_o(z)\Pi_{m_o-1}E_I(K). \tag{2.23}$$

By the relative compactness of W, $WR_o(z)$ is a compact operator on each $\mathbb{E}_k$, and so $\Pi_{m_1}WR_o(z)\Pi_{m_o-1}$ is a compact operator for each $m_1 \in \mathbb{N}$ since it acts on finitely many $\mathbb{E}_k$. Thus the first term on the right hand side of equation (2.23) is compact. By taking m_1 sufficiently large we can make the second term arbitrarily small in norm. This can be seen by noting that if W is time independent and if $m_1 > m_o-1$ then, because W will commute with the projections P_k, $\bar{\Pi}_{m_1}WR_o(z)\Pi_{m_o-1} = 0$. The time dependence of W couples the space and time variables and can bridge the gap between $\bar{\Pi}_{m_1}$ and Π_{m_o-1}, but we can estimate this by writing

$$\begin{aligned}\bar{\Pi}_{m_1}WR_o(z)\Pi_{m_o-1} \;&=\; \partial_t^{-1}\partial_t\bar{\Pi}_{m_1}WR_o(z)\Pi_{m_o-1} \\ &=\; \partial_t^{-1}\bar{\Pi}_{m_1}(\partial_t W)R_o(z)\Pi_{m_o-1} \;+\; \partial_t^{-1}\bar{\Pi}_{m_1}WR_o(z)\partial_t\Pi_{m_o-1}.\end{aligned} \tag{2.24}$$

Combining this with the estimates

$$\|\partial_t^{-1}\bar{\Pi}_{m_1}\| \;\leq\; 1/m_1, \quad \text{and} \quad \|\partial_t\Pi_{m_o-1}\| \;\leq\; m_o - 1, \tag{2.25}$$

we see that $\|\bar{\Pi}_{m_1}WR_o(z)\Pi_{m_o-1}\|$ can be made arbitrarily small by taking m_1 sufficiently large. Therefore, referring to (2.23), $R(z)WR_o(z)\Pi_{m_o-1}E_I(K)$, and hence $\big(E_I(K)-E_I(K_o)\big)\Pi_{m_o-1}E_I(K)$ is compact.

Going back to (2.20), we use the fact that φ is an eigenfunction of K corresponding to the eigenvalue λ to show that φ_R is essentially localized in I with respect to the spectral decomposition of K, i.e., that $\|\bar{E}_I(K)\varphi_R\|$ is small. More precisely, we can establish the estimate

$$\begin{aligned}\|\bar{E}_I(K)\varphi_R\|\,\|\varphi_R\| \;&\leq\; d^{-1}o_R(1)\Big[\|\varphi\|^2_{L^2(S^1_\omega,H^1(\mathbb{R}^N))} + \|\varphi_R\|^2_{L^2(S^1_\omega,H^1(\mathbb{R}^N))}\Big] \\ &\quad + d^{-1}\delta(\delta+1)\|\varphi_R\|^2_{L^2(S^1_\omega,H^1(\mathbb{R}^N))}\end{aligned} \tag{2.26}$$

where $d = dist\,(\partial I,\, \lambda)$. This follows from the formula, derived using the functional calculus,

$$\|\bar{E}_I(K)\varphi_R\| \;\; \leq \;\; d^{-1}\|(K-\lambda)\varphi_R\|, \tag{2.27}$$

and the estimate

$$\|(K+\kappa)\varphi_R\| \;\; \leq \;\; o_R(1)\|\varphi\|_{L^2(S^1_\omega, H^1(\mathbb{R}^N))} \; + \; \delta(\delta+1)\|\varphi_R\|_{L^2(S^1_\omega, H^1(\mathbb{R}^N))} \tag{2.28}$$

which follows from (2.4), (2.13) and the triangle inequality.

Combining (2.18), (2.20) and (2.26), we have that

$$\begin{aligned}\langle -\partial_t^2\rangle_{\varphi_R} \;\; \geq \;\; & m_o^2\omega^2\|\varphi_\cdot\|^2 \; - \; d^{-1}o_R(1)\Big[\|\varphi\|^2_{L^2(S^1_\omega, H^1(\mathbb{R}^N))} + \|\varphi_R\|^2_{L^2(S^1_\omega, H^1(\mathbb{R}^N))}\Big] \\ & - o_R(1)\|\varphi_R\|^2 \; - \; d^{-1}\delta(\delta+1)\|\varphi_R\|^2_{L^2(S^1_\omega, H^1(\mathbb{R}^N))},\end{aligned} \tag{2.29}$$

and so

$$\begin{aligned}\langle i[K,\, A]\rangle_{\varphi_R} \;\; \geq \;\; & b\|\varphi_R\|^2_{L^2(S^1_\omega, H^1(\mathbb{R}^N))} \; - \; (d^{-1}+1)o_R(1)\Big[\|\varphi\|^2_{L^2(S^1_\omega, H^1(\mathbb{R}^N))} + \|\varphi_R\|^2_{L^2(S^1_\omega, H^1(\mathbb{R}^N))}\Big] \\ & - d^{-1}\delta(\delta+1)\|\varphi_R\|^2_{L^2(S^1_\omega, H^1(\mathbb{R}^N))}\end{aligned} \tag{2.30}$$

where $b = \min\,(-\kappa + m_o^2\omega^2,\, 1) > 0$. Thus, we have achieved (2.11) for R sufficiently large and δ sufficiently small and therefore have an exponential bound for φ.

This method secures *some* exponential bound δ for φ. To achieve a better bound we iterate this method, incrementally approaching the optimal bound. We begin the iteration by assuming that $\varphi_\alpha \equiv e^{\alpha r}\varphi \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$ for some $\alpha \geq 0$. To prove that there exists a $\delta > 0$ such that $e^{\delta r}\varphi_\alpha \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$, we perform the above analysis on the function $\chi_R e^{\delta h(r)}\varphi_\alpha$. Denote by $\delta(\alpha)$ the exponential bound found for φ_α found in this way. Therefore our new exponential bound for φ has exponent $\alpha + \delta(\alpha) \equiv \alpha_1$. Repeating the analysis we determine $\alpha_2 = \alpha_1 + \delta(\alpha_1)$, etc. Finally, we show that as $n \to \infty$, $\lambda + \alpha_n^2 \to -(m_o-1)^2\omega^2$ if $\lambda < 0$ (recall that $m_o \geq 1$ is the largest integer m such that $-f'(0) = -\kappa < -(m-1)^2\omega^2$), or else becomes arbitrarily large if $f'(0) > 0$.

This completes the discussion of the proof of exponential bounds. To prove Theorem 1.1 we first show that under the hypothesis $\omega^2 > f'(0)$, $e^{\alpha r}\bar{\varphi} \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$ for all α where $\bar{\varphi} \equiv \varphi - P_0\varphi$. Note that $\bar{\varphi}$ contains all the time dependence of φ. By unique continuation at infinity (see Theorem 3.1 below) it then follows that $\bar{\varphi} = 0$. Therefore, φ is independent of time.

It is the presence of the threshold $-(m_o - 1)^2\omega^2$ above $-f'(0)$ that prevents us from proving a larger exponential bound for φ. In the case $\omega^2 > f'(0)$, zero is the only threshold above $-f'(0)$. We remove this threshold by projecting NLW onto $\mathbb{E}_0^\perp$:

$$0 = \bar{P}_0\Big(\partial_t^2\varphi - \Delta\varphi + f(\varphi)\Big) = K_o\bar{\varphi} + \bar{P}_0 f(\varphi) \tag{2.31}$$

where $\bar{\varphi} \equiv \bar{P}_0\varphi$. Setting $\varphi_0 \equiv P_0\varphi$, we can write

$$f(\varphi) = f(\varphi_0 + \bar{\varphi}) = f(\varphi_0) + f(\bar{\varphi}) + \varphi_0\bar{\varphi}\, g(\varphi_0, \bar{\varphi}) \tag{2.32}$$

for some C^1 function g. Since $\bar{P}_0 f(\varphi_0) = 0$, we have that

$$\bar{P}_0 f(\varphi) = \bar{P}_0 f(\varphi_0 + \bar{\varphi}) = \bar{P}_0\Big(f(\varphi_0 + \bar{\varphi}) - f(\varphi_0)\Big). \tag{2.33}$$

Therefore

$$\begin{aligned}
\bar{P}_0 f(\varphi) &= \bar{P}_0\Big(f(\bar{\varphi}) + \varphi_0\bar{\varphi}\, g(\varphi_0, \bar{\varphi})\Big) \\
&= \bar{P}_0\Big(\frac{f(\bar{\varphi})}{\bar{\varphi}} + \varphi_0\, g(\varphi_0, \bar{\varphi})\Big)\bar{P}_0\bar{\varphi} \qquad (\text{note that } \bar{P}_0\bar{\varphi} = \bar{\varphi}) \\
&= (U + \kappa)\bar{\varphi},
\end{aligned} \tag{2.34}$$

where

$$U = \bar{P}_0 V_{\bar{\varphi}} \bar{P}_0, \quad V_{\bar{\varphi}} = \frac{f(\bar{\varphi})}{\bar{\varphi}} - \kappa + \varphi_0\, g(\varphi_0, \bar{\varphi}), \quad \kappa = f'(0). \tag{2.35}$$

Thus,

$$\bar{K}\bar{\varphi} \;=\; -\kappa\bar{\varphi} \;\text{ with }\; \bar{K} \;=\; K_o + U. \tag{2.36}$$

Observe that $\bar{K}_{\bar{\varphi}}$ has no zero threshold and therefore no zero threshold above $-\kappa$. This allows us to prove, by the same method used to prove exponential bounds, that $e^{\alpha r}\bar{\varphi} \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$ for all α. Theorem 1.1 now follows from the next theorem, a kind of unique continuation at infinity.

Theorem 2.1 *Suppose $\bar{K}_{\bar{\varphi}}\psi = \lambda\psi$ for some $\psi \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$. If $e^{\alpha r}\psi \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$ for all α, then $\psi = 0$.*

We point out the the zero threshold can always be removed without removing the eigenvalue $-f'(0)$ because f does not couple the branch $[0, \infty)$ to the other branches. This follows from the fact that $\bar{P}_0 f(\varphi_0) = 0$; f cannot generate time dependent modes from a time independent function.

The idea of the proof of Theorem 3.1 is based on the observation that if $K\psi = \lambda\psi$ and $e^{\alpha r}\psi \in L^2(S^1_\omega, H^1(\mathbb{R}^N))$, then ψ is an approximate eigenfunction of K with eigenvalue $\lambda + \alpha^2$. Therefore, $\langle K\rangle_\psi \approx \lambda + \alpha^2$. On the other hand, as an approximate eigenfunction the virial theorem implies that $\langle i[K,\, A]\rangle_\psi$ is small. Since $\langle i[K\, A]\rangle - \langle K\rangle = \langle -\partial_t^2 - \Delta - x\cdot\nabla W - W\rangle$ is bounded below, we obtain a contradiction by taking α sufficiently large.

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