Asymptotical Stability of Solitary Waves

An incomplete set of notes by R. Pyke[1], November, 2002

Introduction

We consider the nonlinear Schrödinger equation (NLS);

$$i\partial_t u + \Delta u + f(|u|^2)u = 0 \tag{1}$$

where $u\ :\ \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{C}$, $f : \mathbf{R} \to \mathbf{R}$ (for example, $f(z) = az^{p-1}$).

A soliton is a solution of the form

$$\psi_\omega(x,t) = e^{i\omega t}\varphi_\omega(x), \tag{2}$$

where φ_ω solves

$$\Delta\varphi + f(|\varphi|^2)\varphi = \omega\varphi. \tag{3}$$

A solution of (3) exists for all $\omega > 0$, decays exponentially as $\mid x \mid \to \infty$, and we can choose φ_ω to be real-valued. The *ground state* is the unique positive, radially symmetric solution. (See [S] pp59, [SS] pp77, [SW1] pp 123, [W1] pp472, [W2] pp54 and the references therein).

There are numerous equations that posses soliton solutions. In addition to the NLS, there is the Korteweg-deVries equation which describes (among other phenomena) water waves in a shallow channel; $\partial_t u + u\partial_x u + \partial_x^3 u = 0$, and the sine-Gordon equation; $\partial_t^2 u - \partial_x^2 u + \sin(u) = 0$ which arises in solid-state physics, for example. There are special nonlinear equations which are called *integrable* and for which are completely solvable by the Inverse Scattering Method (IST). For a general discussion about solitons for nonlinear wave equations and the IST, see for example the books [DEGM, DJ, N, Rem, Wh]. For specific discussion about the IST see for example, [FT, NMPZ].

<u>Symmetries of NLS</u>

Let $\Phi(\psi) \equiv i\partial_t\psi + \Delta\psi + f(|\psi|^2)\psi$, and for $\nu = (\gamma, v, D_o) \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n$ and any function $\psi(x,t) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{C}$, let

$$(T_\nu\psi)(x,t) = e^{i(\frac{1}{2}v\cdot x - \frac{1}{4}v^2 t + \gamma)}\psi(x - vt - D_o,\ t). \tag{4}$$

Then T_ν is a $2n+1$ parameter group of symmetries of NLS:

$$\Phi \circ T_\nu \ = \ T_\nu \circ \Phi. \tag{5}$$

In particular, if $\psi(x,t)$ is a solution of NLS, then so is $(T_\nu\psi)(x,t)$.

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Thus, the parameters $\sigma = (\omega, \gamma, v, D_o) \in \{\mathbf{R} \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n\}$ describe a a family of soliton states (travelling waves) ψ_σ of NLS;

$$\psi_\sigma \;=\; T_\nu \psi_\omega \;=\; \psi_\sigma(x,t) \equiv e^{i(\omega t + \frac{1}{2} v \cdot x - \frac{1}{4} v^2 t + \gamma)} \varphi_\omega(x - vt - D_o). \qquad (6)$$

Bound states and Scattering States:

The most natural, and simplest, space-time classification for solutions of dispersive wave equations are bound states (particles) and scattering states (radiation);

- **Bound states** in space-time: essentially localized in space, uniformly in time; $\|\bar{\chi}\varphi(t)\| < \epsilon \;\forall\; t$. (Here, $\bar{\chi}$ is the characteristic function of the complement of some ball in $\mathbf{R}^n$ and $\|\;\cdot\;\|$ denotes some spatial norm.)
- **Scattering states** in space-time: locally decaying in time (i.e., dispersive); $\|\chi\varphi(t)\| \to 0$ as $t \to \infty$. (χ is the characteristic function of some ball in $\mathbf{R}^n$.)

For linear wave equations there is a complete classification of behaviour of general solutions in terms of bound states and scattering states. Consider the linear Schrödinger equation;

$$i\partial_t \varphi = H\varphi.$$

The following well-known theorem in scattering theory states that all solutions can be described in terms of bound states and scattering states.

RAGE Theorem (**R**uelle, **A**mrein, **G**eorgescu, **E**nss)
[AG, CFKS, P, R, RSIII].

Suppose $H = -\Delta + V$ *acting on* $\mathcal{H} = L^2(\mathbf{R}^n)$ *is a locally compact operator. Let* $\mathcal{H}_d$ *and* $\mathcal{H}_c$ *be the discrete and continuous spectral subspaces of* H *respectively*, and $\varphi(t) = e^{-iHt}\varphi$. Then,

(a) $\varphi \in \mathcal{H}_d$ *if and only if for each* $\varepsilon > 0$ *there is an* $R_\varepsilon > 0$ *such that*

$$\sup_t \;\|\bar{\chi}_{R_\varepsilon}\varphi(t)\|_{L^2(\mathbf{R}^n)} \;<\; \epsilon$$

(b) $\varphi \in \mathcal{H}_c$ *if and only if for each* $R > 0$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \|\chi_R\varphi(t)\|_{L^2(\mathbf{R}^n)}\, dt \;=\; 0$$

Here, χ_R *is the characteristic function of the ball of radius* R *in* $\mathbf{R}^n$.

Remark: With a slight strengthening of the assumptions on H, part (b) can be replaced with

$$\|\chi_R\varphi(t)\|_{L^p(\mathbf{R}^n)} \;\to\; 0 \quad \text{as} \quad t \;\to\infty \quad \forall\, p \geq 2 \qquad \text{(local decay)}$$

or even,

$$\|\varphi(t)\|_{L^p} \leq ct^{-\frac{n}{2}+\frac{n}{p}}\|\varphi\|_{L^q}, \quad \frac{1}{p}+\frac{1}{q} = 1, \;\; 2 \leq p \leq \infty \quad \text{(see for example [JSS])}$$

If $\varphi \in \mathcal{H}_{ac}$, the absolutely continuous spectral subspace of H, then it is easy to see that $\varphi(t)$ converges weakly to zero in $L^2(\mathbf{R}^n)$ (via spectral theory and the Riemann-Lebesgue Lemma; see [RSIII p.24 for instance). $\mathcal{H}_{ac} \subset \mathcal{H}_c$ so part (b) of the RAGE Theorem is a more general statement. ($\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_c$, but it is not always true that $\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_{ac}$.)

H being locally compact means that $\chi_B(H+i)^{-1}$ is a compact operator where χ_B is the characteristic function of any bounded set B in $\mathbf{R}^n$, i.e., the resolvent $R_H(z) = (H-z)^{-1}$ is a compact operator when restricted to bounded regions of $\mathbf{R}^n$ ("H has only discrete spectrum on a bounded domain"). For example, $-\Delta$ is locally compact, and if V is continuous (or if $V \in L^2_{loc}(\mathbf{R}^n), V \geq 0$ and vanishing at ∞, then $H = -\Delta + V$ is locally compact. For more information on locally compact operators and their uses, see [P], [CFKS], and [HisS].

Part (b) of the RAGE Theorem is a direct consequence of Weiner's (classical) Theorem on the time-mean of the Fourier transform $F(t)$ of a finite, continuous, Borel measure on $\mathbf{R}$:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |F(t)|^2\, dt \;=\; 0.$$

This is proved by explicit calculations (see [P] p.13 or [RSIII] p. 340). In the present context $F(t) = (\varphi,\, e^{-iHt}\varphi)$, which is the Fourier transform of the spectral measure μ_φ of φ; $\mu_\varphi(B) = (\varphi,\, E_B(H)\varphi)$ for all Borel sets B of $\mathbf{R}$, and where $E_B(H)$ is the spectral projection of H onto B (i.e., $E_B(H)$ is the operator associated to χ_B, the characteristic function of B, via the functional calculus: $\Phi(H)$ is defined by $(\varphi,\, \Phi(H)\varphi) = \int \Phi(H)\, d\mu_\varphi(\lambda)$ for all Borel functions Φ.) Conversely, if the Fourier transform of a finite Borel measure has time mean zero as above, then it is continuous.

Note that if $\varphi \in \mathcal{H}_d$, then $\varphi = \sum c_j\psi_j$ where $H\psi_j = \lambda_j\psi_j$, so $\varphi(t) = \sum c_j e^{-i\lambda_j t}\psi_j$. Thus, the 'only if' part of (a) is obvious. The (nontrivial) 'if' part of (a) states that a space-time boundstate is almost periodic in phase space (periodic if $\varphi = \psi_k$ for some k, quasiperiodic if φ is a finite sum of eigenvectors of H.)

A related result concerning bound states in space-time is,

Exponential Bounds ('Froese-Herbst Theory' via postive commutator estimates; see [CFKS] or [HunS]) *Under similar hypothesis on H as above, if $H\psi = \lambda\psi$, then*

$$e^{\sqrt{|\lambda|}\;|x|}\psi \;\in\; L^2(\mathbf{R}^n),$$

and so $e^{\sqrt{|\lambda|}\;|x|}\psi(x,t) = e^{\sqrt{|\lambda|}\;|x|}(e^{-iHt}\psi)(x) \;\in\; L^2(\mathbf{R}^n)$. (These L^2 bounds can be strengthened to point wise bounds, i.e., $\psi(x) \leq ce^{-\sqrt{|\lambda|}\;|x|}$.)

Combining this with the RAGE Theorem we conclude that bound states (in phase-space) are in fact exponentially localized (with exponent that can be computed explicitly). Or, quasiperiodic solutions of the linear Schrödinger equation are exponentially localized in space (uniformly in time). The same result holds for periodic solutions of nonlinear wave and Schrödinger equations [PS].

The RAGE Theorem states that we can identify the spectral subspaces $\mathcal{H}_d$ and $\mathcal{H}_c$ of H by observing the space-time behavior of solutions of $i\partial_t\varphi = H\varphi$. Solutions that are dispersive come from the continuous spectral subspace while solutions that are boundstates ($\equiv$ almost periodic in time) come from the discrete spectral subspace. Since $\mathcal{H}$ is a direct sum of these two spectral subspaces, all solutions of the Schrödinger equation can be described by these two types of solutions (in fact, as a sum of these two types of solutions). Furthermore, for scattering theory (the long-time behavior of solutions), we have that for any solution $\varphi(t)$, there exists a bound state solution $\varphi_b(t)$ such that

$$\varphi(t) \overset{locally}{\rightarrow} \varphi_b(t) \quad \text{as} \quad t \to \infty$$

That is, the bound states capture the long-time behavior of solutions. For integrable nonlinear equations (such as KdV, cubic NLS, and sine-Gordon), the same is true (at least formally); all solutions converge (locally) to a bound state, the rest of the solution is dispersive ([FT], [NMPZ], and the recent work [CVZ] and [DZ] which is a step in the direction of making this rigorous).

This scenario is becoming a paradigm in the scattering theory of general nonlinear dispersive wave equations. That is, one expects that a general solution will converge locally to bound states as $t \to \infty$ while radiating energy, the convergence being driven by dispersion. In linear theory the bound states come from the eigenfunctions of the linear operator (as stated above in the RAGE Theorem). For nonlinear equations there is no spectral theory. Here the bound states are the solitary waves (including solitons). Rigorous results corroborating this view for nonlinear equations include [Cu, SW, BP] where it is proven that solutions of NLS initially near a soliton converge (locally) to a soliton, while in [MM] it is proven that solutions starting nearby solitons of modified KdV converge to a soliton.

Interesting numerical experiments with interacting (colliding) solitons and the resulting metastable states can be found in [CP].

Ideas of Proof of RAGE (following the excellent exposition [P])

Local compactness as stated above is equivalent to the property that the operators $P_{R,E} = \chi_R F(|H| \leq E)$ are compact, where $F(|H| \leq E)$ denotes the spectral projection onto the subspace of $\mathcal{H}$ where $|H| \leq E$ (eg., if $H = -\Delta$, $F(|H| \leq E) = \{\psi \in \mathcal{H} \mid \text{support } (\hat{\psi}) \in [-E, E]\}$).

Notation: $\mathcal{H}_d$ and $\mathcal{H}_c$ are the discrete and continuous spectral subspaces of H respectively. Recall that $\mathcal{H} = \mathcal{H}_d \oplus \mathcal{H}_c$.

An intermediate result is that for *any* self-adjoint H, $\mathcal{M}_{bd} = \mathcal{H}_d$ and

$\mathcal{M}_{lv} = \mathcal{H}_c$, where $\mathcal{M}_{bd}$ is the set of φ such that $\varphi(t) = e^{-iHt}\varphi$ remains inside a compact subset of $\mathcal{H}$ for all t (i.e., the orbit $\mathcal{O}(\varphi)$ of φ has compact closure), and $\mathcal{M}_{lv}$ is the set of ψ such that $e^{-iHt}\varphi$ leaves any compact set of $\mathcal{H}$ (at least in the time-mean). $\mathcal{M}_{bd}$ and $\mathcal{M}_{lv}$ are closed subspaces of $\mathcal{H}$, and $\mathcal{M}_{bd} \perp \mathcal{M}_{lv}$. One shows that $\mathcal{M}_{lv} = \mathcal{H}_c$ by arguments using Weiner's Theorem (as mentioned above), and that $\mathcal{M}_{bd} = \mathcal{H}_d$ by first noting that any $\varphi \in \mathcal{M}_{bd}$ must be orthogonal to $\mathcal{H}_{lv}$ so $\mathcal{M}_{bd} \subset \mathcal{H}_d$; the reverse inclusion $\mathcal{H}_d \subset \mathcal{M}_{bd}$ is easy. (Details can be found in [P] Section 1.1.)

Then (a) is proven as follows (see [P] Section 1.2). We want to transfer the results in the previous paragraph on the behavior of solutions in *phase space* to behavior in *space-time*. For this we need the extra assumption on H of local compactness. The (easy) 'only if' part was addressed above. So suppose that for any ε' there is an R such that $\sup_t \|\bar{\chi}_R\varphi(t)\|_{L^2} < \varepsilon'$. Let $\varepsilon > 0$ be given. Choose R and E such that $\sup_t \|\bar{\chi}_R\varphi(t)\|_{L^2} < \varepsilon/6$ and $\sup_t \|F(|H| > E)\varphi(t)\|_{L^2} < \varepsilon/6$ (note that $F(|H| \leq E) \xrightarrow{s} \mathbf{1}$ as $E \to \infty$, and that e^{-iHt} and $F(|H| \leq E)$ commute so that $\|F(|H| > E)\varphi(t)\| = \|e^{-iHt}F(|H| > E)\varphi\|$). Writing $\mathbf{1} - P_{R,E} = \mathbf{1} - (\mathbf{1} - \bar{\chi}_R)(\mathbf{1} - F(|H| > E)) = \bar{\chi}_R + F(|H| > E) - \bar{\chi}_R F(|H| > E)$, we have that $\sup_t \|(\mathbf{1} - P_{R,E})\varphi(t)\|_{L^2} < \varepsilon/2$. Since $P_{R,E}$ is compact, $S = \{P_{R,E}\varphi(t), t \in \mathbf{R}\}$ is a compact subset of $\mathcal{H}$, and so there is a finite rank orthogonal projection Q_ε such that $\sup_t \|(\mathbf{1} - Q_\varepsilon)P_{R,E}\varphi(t)\|_{L^2} < \varepsilon/2$ v('S is almost finite dimensional'), and so by writing $(\mathbf{1} - Q_\varepsilon) = (\mathbf{1} - Q_\varepsilon)P_{R,E} + (\mathbf{1} - Q_\varepsilon)(\mathbf{1} - P_{R,E})$, we conclude that $\sup_t \|(1 - Q_\varepsilon)\varphi(t)\|_{L^2} < \varepsilon$ (for any ε!), which implies that the orbit $\mathcal{O}(\varphi)$ of φ is contained inside a compact set . By the intermediate result, this implies that $\varphi \in \mathcal{H}_d$ $\square$

A result analogous to the RAGE Theorem holds for those (very) special *integrable* nonlinear wave equations (such as the cubic NLS, KdV, sine-Gordon). One can apply - at least on a formal level - the Inverse Scattering (IST) method to deduce that general solutions of these equations will converge (locally) to bound states. The IST is a kind of 'nonlinear Fourier transform'. Think about the linear Schrödinger equation. Here spectral theory tells us from the initial data what the evolution of the solution will be. That is, the 'generalized Fourier transform' (i.e., decomposition of a general $L^2(\mathbf{R}^n)$ function into components along the discrete and continuous spectral subspaces of the linear operator H; the component along the continuous spectral subspace being computed via the *generalized eigenfunctions* of H - see [RS §XI.6] for example) measures the 'content' of bound states and scattering states of a general $L^2(\mathbf{R}^n)$ function. Furthermore, these two 'modes' evolve independently so for example (in the case of the IST), the future soliton 'content' of the solution is the same as the soliton 'content' of the initial data.

Schematically, the analogy can be seen as follows.

Generalized Fourier Transform for SE :

$$
\begin{array}{llcl}
 & \varphi(x) \quad \overset{\mathcal{F}}{\longrightarrow} & (c(\lambda),\, \mu_j) & \mathcal{F}\,:\, L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n) \times \mathbf{C}^m \\
\text{SE} & \qquad \downarrow & \downarrow & \text{Lin} \left\{ \begin{array}{l} \mu_j(t) = e^{-i\lambda_j t}\mu_j \\ c(\lambda, t) = e^{-i\lambda^2 t} c(\lambda) \end{array} \right. \\
 & \varphi(x,t) \quad \overset{\mathcal{F}^{-1}}{\longleftarrow} & (c(\lambda,t),\, \mu_j(t)) & \text{SE} = \mathcal{F}^{-1} \circ \text{ Lin } \circ \mathcal{F}
\end{array}
$$

$$
\begin{aligned}
\varphi(x) \;&=\; \varphi_c(x) \;+\; \varphi_b(x) \;\in\; \mathcal{H}_c \oplus \mathcal{H}_d \;=\; L^2(\mathbf{R}^n) \\
\varphi_c(x) \;&=\; \int c(\lambda)\, e(x,\lambda)\, d\lambda, \qquad (c(\lambda) \sim \hat{\varphi}(\lambda)) \\
\varphi_b(x) \;&=\; \sum_j \mu_j \zeta_j(x); \qquad H\zeta_j = \lambda_j \zeta_j, \qquad \mu_j = \langle \varphi_b,\, \zeta_j \rangle \\
\varphi(x,t) \;&=\; \int e^{-i\lambda^2 t} c(\lambda)\, e(x,\lambda)\, d\lambda \;+\; \sum_j e^{-i\lambda_j t} \mu_j \zeta_j(x)
\end{aligned}
$$

Inverse Scattering Transform for cubic NLS :

$$
\begin{array}{llcl}
 & \varphi(x) \quad \overset{\mathcal{F}_{NL}}{\longrightarrow} & (b(\lambda),\, \gamma_j) & \mathcal{F}_{NL} :\; \mathcal{S}(\mathbf{R}) \to \mathcal{S}(\mathbf{R}) \times \mathbf{C}^m \\
\text{NLS} & \qquad \downarrow & \downarrow & \text{Lin} \left\{ \begin{array}{l} \gamma_j(t) = e^{-i\lambda_j t}\gamma_j \\ b(\lambda, t) = e^{-i\lambda^2 t} b(\lambda) \end{array} \right. \\
 & \varphi(x,t) \quad \overset{\mathcal{F}_{NL}^{-1}}{\longleftarrow} & (b(\lambda,t),\, \gamma_j(t)) & \text{NLS } = \mathcal{F}_{NL}^{-1} \circ \text{Lin} \circ \mathcal{F}_{NL}
\end{array}
$$

$$
\begin{aligned}
\{\gamma_j\} \;\sim\; & \text{ solitons (bound states)} \\
b(\lambda) \;\sim\; & \text{ radiation}
\end{aligned}
$$

The ‘independent modes’ $(c(\lambda), \mu_j)$ and $(b(\lambda), \gamma_j)$ of the linear Schrödinger equation (SE) and of the cubic nonlinear Schrödinger equation (NLS) respectively satisfy linear (Lin) equations (the IST is a change of coordinates which linearizes the flow of NLS). As with the generalized Fourier transform for SE, one can show that the component of the solution of NLS corresponding to the ‘radiation’ modes $b(\lambda, t)$ are dispersive, and as a result (also in a similar way as for SE) a general solution to NLS will converge locally to bound states. See [FT] and [NMPZ] for more discussion about this.

We now return to the investigation of the asymptotic behavior of solutions of NLS. Our modest goal is not to prove that a general solution converges to bound states, but a very particular instance where this occurs. We will study the case when the initial data is close to a bound state, i.e., close to a soliton. We wish to show that this solution will converge to a nearby soliton as it evolves in time.

We begin with the weaker result of orbital stability. Let φ_ω be a particular ground state of (3). We call $\mathcal{O}(\varphi_\omega)$ the *orbit of* φ_ω;

$$\mathcal{O}(\varphi_\omega) \;=\; \{e^{ib}\varphi_\omega(x-c) \mid (b,c) \in \mathbf{R}\times\mathbf{R}^n\} \tag{7}$$

Let $d(u,\mathcal{O}(\varphi_\omega))$ be the distance from u to $\mathcal{O}(\varphi_\omega)$;

$$d(u,\mathcal{O}(\varphi_\omega)) \;\equiv\; \inf_{\psi\in\mathcal{O}(\varphi_\omega)} \|u-\psi\|_{H^1} \tag{8}$$

Then orbital stability (Lyapunov stability) of soliton states means that for any $\epsilon > 0$, $\exists\,\delta$ such that

$$d(u(0),\mathcal{O}(\varphi_\omega)) \;<\; \delta \;\;\Longrightarrow\;\; d(u(t),\mathcal{O}(\varphi_\omega)) \;<\; \epsilon \;\;\forall t>0 \tag{9}$$

We recall the argument (see for example [W2] or [SS]). Write $u(0) = u_o = \psi_{\sigma(0)} + w_o$ and define $w(t) = u(t) - \psi_{\sigma(t)}$ where for each $t \geq 0$, $\psi_{\sigma(t)}$ is the minimizer of $\inf_{\psi\in\mathcal{O}(\varphi_\omega)} \|u(t)-\psi\|_{H^1}$ (i.e., $\psi_{\sigma(t)}$ is the closest point in $\mathcal{O}(\varphi_\omega)$ to the solution $u(t)$). Note that $\psi_{\sigma(t)} = e^{ib(t)}\varphi_\omega(x-c(t))$ for some functions $b(t)$ and $c(t)$.

We define the energy functional $\mathcal{E}(\psi)$;

$$\mathcal{E}(\psi) \;=\; \int_{\mathbf{R}^n}\left(\frac{1}{2}|\nabla\psi|^2 + \frac{1}{2}F(|\psi|^2) + \frac{\omega}{2}|\psi|^2\right), \qquad \text{where } F' = f.$$

One can show that for any $\varepsilon' > 0$, if δ is sufficiently small (δ given in (9)),

$$0 \;<\; \mathcal{E}(u_o) - \mathcal{E}(\psi_o) \;<\; \varepsilon'.$$

Now,

$$\begin{aligned}
\epsilon \;>\; & \mathcal{E}(u_o) \;-\; \mathcal{E}(\psi_o) \\
=\;& \mathcal{E}(u(t)) \;-\; \mathcal{E}(\psi_o), && \text{conservation of energy}\\
=\;& \mathcal{E}(e^{-ib}u(\bullet + c,t)) \;-\; \mathcal{E}(\psi_o), && \text{scale invariance}\\
=\;& \mathcal{E}(\psi_o + \tilde{w}(t)) \;-\; \mathcal{E}(\psi_o), && u=\psi+w,\;\; \tilde{w} = e^{-ib}w(\bullet + c)\\
=\;& \mathcal{E}'(\psi_o)[\tilde{w}(t)] \;+\; \frac{1}{2}\mathcal{E}''(\psi_o)[\tilde{w}(t),\tilde{w}(t)] \;+\; r, && \text{Taylor exp about } \psi_o\\
=\;& \frac{1}{2}\mathcal{E}''(\psi_o)[\tilde{w}(t),\tilde{w}(t)] \;+\; r, && \mathcal{E}'(\psi_o)=0\\
=\;& \langle L\tilde{w}(t),\tilde{w}(t)\rangle \;+\; r, && L = \mathcal{E}''(\psi_o)\\
\geq\;& c_2\|\tilde{w}(t)\|_{H^1}, && \tilde{w}(t)\in N_g^{\perp}(L) \quad \text{via } b(t)\,,c(t) \qquad *\\
=\;& c_2 d(u(t),\mathcal{O}(\varphi_\omega)) \qquad \square
\end{aligned}$$

To obtain the inequality in the second last line we used the hypothesis that $\|w_o\|_{H^1}$, and hence $\|w(t)\|_{H^1}$ is sufficiently small. $N_g^{\perp}(L)$ denotes the orthogonal complement (in H^1) of the generalized null space of L (see below).

Orbital stability does not imply that the solution converges to a particular (travelling) soliton. This stronger notion of stability is asymptotic stability. To prove this we need the extra degree of freedom offered by varying the parameter ω. So let

$$\mathcal{M} \;=\; \{e^{ib}\varphi_\omega(x-c) \mid (\omega, b, c) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}^n\} \tag{10}$$

Then asymptotic stability of soliton states means that

$$d(u(0),\mathcal{M}) < \delta \implies d(u(t),\mathcal{M}) \to 0 \text{ as } t\to\infty \tag{11}$$

and in fact that the solution $u(t)$ converges to a travelling soliton. Note that we don't expect the solution to converge to a translate of the soliton it was near initially (i.e., the parameter ω will typically change).

Outline of Proof of Asymptotic Stability

Notation:

$\sigma \;=\; (\omega,\, \gamma,\, v,\, D), \quad \sigma(t) \;=\; (\omega(t),\, \gamma(t),\, v(t),\, D(t))$
$\varphi_\omega(x)$ is a solution of time-dependent NLS; $\Delta\varphi_\omega + f(|\varphi_\omega|^2)\varphi_\omega = \omega\varphi_\omega$
$\psi_\sigma(x,t) \;=\; e^{i(\omega t+\frac{1}{2}v\cdot x-\frac{1}{4}v^2t+\gamma)}\varphi_\omega(x-vt-D)$
$\varphi_\sigma = \varphi_{\sigma(t)} = \varphi_{\omega(t)}(y(x,t))$, so $\varphi_\sigma = \varphi_\omega(y)$ and $\varphi_\omega = \varphi_\omega(x)$.
$\varphi_\sigma' = (\partial_\omega\varphi_\omega)(y)$
$y(x,t) = x - \int_0^t v(s)\,ds \;-\; D(t)$

Theorem 1.5 [Cu1] (Asymptotic Stability)

There exists a $\delta > 0$ *such that if*

$$d(u(0),\mathcal{M}) < \delta$$

then

$$u(x,t) = e^{i\Theta(x,t)}\Big[\varphi_{\omega(t)}(y(x,t)) \;+\; R(y(x,t),t)\Big]$$

where

$$\Theta(x,t) \;=\; \frac{1}{2}v(t)\cdot x - \frac{1}{4}\int_0^t |v(s)|^2\,ds \;+\; \int_0^t \omega(s)\,ds \;+\gamma(t) \tag{12}$$

$$y(x,t) \;=\; x - \int_0^t v(s)\,ds \;-D(t) \tag{13}$$

$$\|R(t)\|_{W^{m,\infty}} \;\le\; ct^{-n/2} \tag{14}$$

$$|\dot\sigma(t)| \;\le\; ct^{-n} \tag{15}$$

That is, the solution $u(x,t)$ converges (in $W^{m,\infty}$) to a particular soliton state $\psi_{\sigma_\infty}(x,t) \in \mathcal{M}$. $d(u(0),\mathcal{M}) < \delta$ means $\exists\,\sigma_o = (\omega_o,\gamma_o,v_o,D_o)$ such that,

$$\|u(x,0) - e^{i(\frac{1}{2}v_o\cdot x+\gamma_o)}\varphi_{\omega_o}(x-D_o)\|_{H_1^{2m}} + \|\cdots\|_{H_1^{2m+1}} + \|\cdots\|_{W^{2m_1,1}} \;<\; \delta, \tag{16}$$

where H^s_a is the $(1+|x|^2)^{a/2}$ weighted Sobolev space of order s and $W^{m,p}$ is the L^p Sobolev space of order m. So, $u(x,t)$ starts near the soliton ψ_{σ_o} and asymptotically approaches the soliton ψ_{σ_∞}.

Theorem 1.9 [Cu1] (Scattering)

For any sufficiently smooth $R_\infty(x)$ and any $\sigma = (\omega, \gamma, v, D) \in \mathcal{O} = \mathbf{R}^{2n+2}$, there is a unique solution $u(t)$ of NLS which has the decomposition as in Theorem 1.5 and

$$\lim_{t\to+\infty} \sigma(t) \;=\; \sigma \quad \text{and} \quad \lim_{t\to+\infty} \|R(t) - (e^{J(-\Delta+\omega)t} R_\infty)(x - vt - D)\|_{H^{[\frac{n}{2}]+1}} \;=\; 0$$

Remark: $R_\infty(t) = e^{J(-\Delta+\omega)} R_\infty$ is the solution of $i\partial_t R = -\Delta R + \omega R$ with $R_\infty(0) = R_\infty$. As we'll see below, the asymptotic (in time) behavior of solutions of $i\partial_t \psi = L_\pm \psi$ is governed by the 'free' equation $\partial_t \psi = -\Delta\psi + \omega\psi$. $L_\pm$ are the linearized operators about the soliton, so Theorem 1.9 states that, for solutions that are initially close to a soliton, the asymptotic behavior of the 'radiation' part of the solution is governed by the 'free' part of the linearized equation about the (asymptotic) soliton.

Modulation Equations

Modulation means that the parameters describing a soliton state are allowed to vary in time; $\sigma(t) = (\omega(t), \gamma(t), v(t), D(t))$. The task is to determine the function $\sigma(t)$, $\sigma(0) = \sigma_o$, so that $u(x,t) - \psi_{\sigma(t)}$ converges to zero (in the appropriate sense.

Plugging the ansatz $u(x,t) = e^{\Theta}[\varphi_\omega + R]$ into NLS we obtain,

$$\begin{aligned} &i\partial_t R - i\dot{D}\cdot\nabla R = \qquad (17)\\ &\quad -\Delta R + \omega(t) R - f(\varphi^2_{\sigma(t)})R - f'(\varphi^2_{\sigma(t)})\varphi^2_{\sigma(t)} R - f'(\varphi^2_{\sigma(t)})\varphi^2_{\sigma(t)}\bar{R} + \frac{\dot{v}\cdot x}{2} R \\ &\quad + \left(\frac{\dot{v}\cdot x}{2} + \dot{\gamma}(t)\right)\varphi_{\sigma(t)} - i\dot{\omega}(t)\varphi'_{\sigma(t)} + i\dot{D}\cdot\nabla\varphi_{\sigma(t)} + \dot{\gamma}(t)R + e^{-i\Theta}N(e^{i\Theta}R) \end{aligned}$$

where $\varphi'_\sigma = (\partial_\omega \varphi_\omega)(y)$ and $R = R(y,t)$ are evaluated at $y = y(x,t)$, and $N(s)$ satisfies $|N(s)| \le c(\varphi_\sigma |s|^2 + |s|^p)$, $p \ge 3$, for $|s|$ small.

Define

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad H_\sigma = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix} \tag{18}$$

where

$$L_+ \;=\; -\Delta + \omega - f(\varphi^2_\sigma) - 2\varphi^2_\sigma f'(\varphi^2_\sigma), \tag{19}$$

$$L_- \;=\; -\Delta + \omega - f(\varphi^2_\sigma). \tag{20}$$

Hypothesis (for Theorem 1.5): *If $\sigma \in \mathcal{O} = \mathbf{R}^{2n+2}$,* (ND) the only eigenvalue of JH_σ and $H_\sigma J$ is 0

(NR) neither of $\pm i\omega$ is a resonance for JH_σ or $H_\sigma J$.

So the spectrum of JH_σ and $H_\sigma J$ is $i(-\infty, -\omega] \cup \{0\} \cup i[\omega, \infty)$.

Theorem 1.3 [Cu1]

$$\begin{aligned}\text{Ker } L_+ &= \text{span } \{\partial_j \phi_\sigma\}_{j=1,\ldots,n} \\ \text{Ker } L_- &= \text{span } \{\phi_\sigma\}\end{aligned}$$

(Note that $L_+\partial_j\phi_\sigma = 0$ can be seen by differentiating (3) w.r.t. x_j, $L_+\phi_\sigma^{'} = \phi_\sigma$ (differentiate (3) w.r.t. ω), $L_-\phi_\sigma = 0$ (differentiate (3) w.r.t. γ), and so $L_-L_+\phi_\sigma^{'} = 0$.)

Theorem 1.4 [Cu1]

$$\begin{aligned}\text{span } \{\phi_\sigma,\ J\phi_\sigma^{'},\ J\nabla\phi_\sigma,\ y\phi_\sigma\} &= N_g(H_\sigma J) \\ H_\sigma \ \geq\ c\ >\ 0 \quad \text{on} &\ \ N_g^{\perp}(H_\sigma J)\end{aligned}$$

(Here, $N_g(H)$ denotes the generalized eigenspace of H; $N_g(H) = \cup_{j=1}^{\infty} N(H^j)$.)

Let $R = R_1 + iR_2$ where $R_1 = \Re R$ and $R_2 = \Im R$ and interpret R as the column vector $(R_1, R_2)^t$. Re-write (18) as

$$\begin{aligned}\partial_t R \ &= \ JH_{\sigma(t)}R + \dot{D}\cdot\nabla R + J\frac{\dot{v}\cdot y}{2}R + J\dot{\tilde{\gamma}}(t)R \qquad (21)\\ &+J\left(\frac{\dot{v}(t)\cdot y}{2} + \dot{\tilde{\gamma}}(t)\right)\phi_{\sigma(t)} - \dot{\omega}(t)\phi_{\sigma(t)}^{'} + \dot{D}\cdot\nabla\phi_{\sigma(t)} + Je^{J\Theta}N(e^{-J\Theta}R).\end{aligned}$$

Here,

$$\begin{aligned}\dot{\tilde{\gamma}}(t) \ &= \ \dot{\gamma}(t) + \frac{1}{2}\dot{v}(t)\cdot\int_0^t v(s)ds \ + \frac{\dot{v}(t)\cdot D(t)}{2} \\ \Theta(x,t) \ &= \ \frac{1}{2}v(t)\cdot x - \frac{1}{4}\int_0^t |v(s)|^2\, ds \ + \ \int_0^t \omega(s)\, ds \ + \ \gamma(t)\end{aligned}$$

To determine equations describing the evolution of $\sigma(t)$ and $R(t)$ we impose the orthogonality condition $R(t) \in N_g^{\perp}(H_{\sigma(t)}J)\ \ \forall t \geq 0$. As can be seen from (22), the evolution of $R(t)$ is (primarily) governed by JH_σ. The orthogonality condition assures that $R(t)$ lies in the continuous spectral subspace of JH_σ where the evolution $e^{JH_\sigma t}$ is dispersive (the solution of $\partial_t\psi = JA\psi$ is $\psi(t) = e^{JAt}\psi_o$.)

As can be seen from Theorem 1.4, the orthogonality condition is

$$\begin{aligned}\langle R, \varphi_\sigma\rangle &= 0 \quad \forall t \geq 0 \\ \langle R, J\varphi_\sigma^{'}\rangle &= 0 \quad \forall t \geq 0 \\ \langle R, J\nabla\varphi_\sigma\rangle &= 0 \quad \forall t \geq 0 \\ \langle R, y\varphi_\sigma\rangle &= 0 \quad \forall t \geq 0\end{aligned}$$

Let ξ be any one of $\varphi_\sigma,\ J\varphi_\sigma^{'},\ J\nabla\varphi_\sigma,\ y\varphi_\sigma$. Note that

$$\langle R,\xi\rangle = 0\ \ \forall t\geq 0 \ \ \Longrightarrow\ \ \partial_t\langle R,\xi\rangle = 0 \Longleftrightarrow \langle\partial_t R,\xi\rangle\ +\ \langle R,\partial_t\xi\rangle = 0 \quad \forall t\geq 0.$$

Plugging (22) in for $\partial_t R$ in these inner products, we obtain the **modulation equations**;

$$\begin{aligned}
\dot{\omega}\langle\varphi_\sigma,\varphi_\sigma^{'}\rangle &= \langle Je^{J\Theta}N(e^{-J\Theta}R),\, e^{-J\Theta}\varphi_\sigma\rangle\ +\ O(\dot{\tilde{\sigma}}\|R\|_{W^{1,\infty}})\\
\dot{\tilde{\gamma}}\langle\varphi_\sigma,\varphi_\sigma^{'}\rangle &= -\langle e^{J\Theta}N(e^{-J\Theta}R),\, e^{-J\Theta}\varphi_\sigma^{'}\rangle\ +\ O(\dot{\tilde{\sigma}}\|R\|_{W^{1,\infty}})\\
\dot{v}_j\langle y_j\varphi_\sigma,\,\partial_j\varphi_\sigma\rangle &= -\langle e^{J\Theta}N(e^{-J\Theta}R),\, e^{-J\Theta}\partial_j\varphi_\sigma\rangle\ +\ O(\dot{\tilde{\sigma}}\|R\|_{W^{1,\infty}})\\
\dot{D}_j\langle y_j\varphi_\sigma,\,\partial_j\varphi_\sigma\rangle &= -\langle Je^{J\Theta}N(e^{-J\Theta}R),\, e^{-J\Theta}y_j\varphi_\sigma\rangle\ +\ O(\dot{\tilde{\sigma}}\|R\|_{W^{1,\infty}}) \qquad (22)
\end{aligned}$$

The system of modulation equations and equation (22) are solved together (for $R(t)$ and $\sigma(t)$) to obtain the desired result, namely, that for sufficiently small δ,

$$|\sigma(t)-\sigma_o| \ <\ \delta \tag{23}$$
$$\|R(t)\|_{H^{2m+1}} \ <\ \delta \tag{24}$$
$$\|R(t)\|_{W^{m,\infty}} \ <\ \delta(1+|t|)^{-\frac{n}{2}} \tag{25}$$
$$|\dot{\tilde{\sigma}}(t)| \ <\ c\delta^2(1+|t|)^{-n} \tag{26}$$
$$\|\langle y(t)\rangle R(t)\|_{H^{2m}} \ <\ \delta(1+t) \tag{27}$$

Idea of Proof:

Write

$$\partial_t R \ =\ JH_\sigma R\ +\ E(t) \tag{28}$$

where

$$E(t)\ =\ E(\dot{\sigma}\varphi_\omega,\,\dot{\sigma}R,\,\varphi_\omega R,\,R^p) \tag{29}$$

The form of $E(t)$ suggests the norms

$$\begin{aligned}
M_1(t) &= \sup_{0<s<t}(1+|s|)^n|\dot{\sigma}(s)|\\
M_2(t) &= \sup_{0<s<t}(1+|s|)\|\langle x\rangle R(s)\|_{H^1}\\
M_3(t) &= \sup_{0<s<t}(1+|s|)^{n/2}\|R(s)\|_{L^\infty}
\end{aligned}$$

The modulation equations can also be estimated using these norms. Our goal is to show these norms remain bounded for all time.

By variation of parameters (or Duhamel's formula),

$$R(t)\ =\ e^{JH_\sigma t}R_o\ +\ \int_0^t e^{JH_\sigma(t-s)}E(s)\,ds, \tag{30}$$

so that

$$\begin{aligned}
\|R(t)\|_X &\leq \|e^{JH_\sigma t}R_o\|_X\ +\ \int_0^t\|e^{JH_\sigma(t-s)}E(s)\|_X\,ds\\
&\leq \|e^{JH_\sigma t}R_o\|_X\ +\ \int_0^t d(t-s)\|E(s)\|_Y\,ds \qquad (31)
\end{aligned}$$

where

$$\|e^{JH_\sigma t}u\|_X \;\leq\; d(t)\|u\|_Y \qquad \text{(dispersive decay estimate for } e^{JH_\sigma t}\text{)} \tag{32}$$

Recall the dispersive estimates for the free Schrödinger operator;

$$\|e^{-\Delta+\omega}u\|_{L^p} \;\leq\; d(t)\|u\|_{L^q} \;=\; ct^{-\frac{n}{2}+\frac{n}{p}}\|u\|_{L^q}, \quad \frac{1}{p}+\frac{1}{q}=1,\;\; 2\leq p\leq\infty \tag{33}$$

From this follow the dispersive estimates for the free matrix Schrödinger operator e^{JH_o}, $H_o = \begin{bmatrix} -\Delta+\omega & 0 \\ 0 & -\Delta+\omega \end{bmatrix}$;

$$\|e^{JH_o t}u\|_{L^p} \;\leq\; ct^{-\frac{n}{2}+\frac{n}{p}}\|u\|_{L^p} \tag{34}$$

However, what we require are dispersive estimates on the matrix Schrödinger operator $e^{JH_\sigma t}$. The wave operator, W_+, provides this:

Theorem 2.1 [Cu1]

$$W_+u \;=\; \lim_{t\to+\infty} e^{JH_\sigma t}e^{-JH_o t} \tag{35}$$

The wave operator is a bounded operator from $L^p(\mathbf{R}^n)$ *to* $L^p(\mathbf{R}^n)\cap N_g^\perp(H_\sigma J)$ *for all* $p\in[1,\infty]$ *along with its inverse. It satisfies the intertwining property;*

$$e^{JH_\sigma t}W_+ \;=\; W_+e^{JH_o t}.$$

Thus we have,

$$\begin{aligned} \|e^{JH_\sigma t}u\|_{L^p} \;=\; \|e^{JH_\sigma t}W_+v\|_{L^p} \;&=\; \|W_+e^{JH_o t}v\|_{L^p} \\ &\leq\; c\|e^{JH_o t}v\|_{L^p} \\ &\leq\; ct^{-\frac{n}{2}+\frac{n}{p}}\|v\|_{L^q} \\ &\leq\; ct^{-\frac{n}{2}+\frac{n}{p}}\|u\|_{L^q} \end{aligned} \tag{36}$$

Using these estimates in (32) we obtain an inequality of the form,

$$M_2(t) \;\leq\; C(M)[(M_2(t)+M_2^2(t)+M_1(t)+M_1(t)^2+M_3(t)+M_3(t)^2] \tag{37}$$

where $C(M)$ depends on $M_1(0), M_2(0), M_3(0)$ with $C(0)=0$. There are similar inequalities for $M_1(t)$ and $M_3(t)$. Thus it follows that for sufficiently small initial data, the $M_i(t)$ remain bounded for all time.

Remark: Notice that the operators JH_σ and $H_\sigma J$ are time-*dependent* through $\sigma=\sigma(t)$. To get rid of this time dependence, $\sigma(t)$ is fixed at some time T; $\sigma(T)$, and the analysis is done with these time-independent operators. Thus, we need to know apriori that $|\sigma(t)-\sigma_o|$ remains sufficiently small for all time (so the operators $JH_{\sigma(t)}$ and $H_{\sigma(t)}J$ do not differ significantly from $JH_{\sigma(T)}$ and $H_{\sigma(T)}J$).

Resonant Case

Here we assume the existence of an eigenvalue $\pm\lambda$ of JH_σ between 0 and the edges of the continuous spectrum $\pm i\omega$; $JH_\sigma\zeta = \lambda\zeta$. To insure the resonant interaction between the bound state and the continuous spectrum, we assume that $2\lambda > \omega$ and that the Fermi Golden Rule holds (these conditions are described below).

Now the ansatz is

$$\begin{aligned} u(x,t) &= e^{i\Theta(x,t)}[\varphi_\omega(x) + R(x,t)], \quad \text{with} && (38)\\ R(x,t) &= a(t)\zeta(x) \ + \ \eta(x,t) && (39) \end{aligned}$$

where $\zeta(x)$ is the eigenvector associated to λ and $\eta(x,t)$ is in the continuous spectrum for all t (orthogonality condition). Actually, we should write $\zeta(x,t)$ since $\lambda = \lambda(t)$, but we will fix the parameters at some time $t = T$ and thus reduce to a time independent problem.

Theorem [Cu2] *(Dim = 3). For initial data $u(x,0)$ close enough to a soliton state, we have that*

$$u(x,t) \ = \ e^{i\Theta(x,t)}[\varphi_{\omega(t)} + R(x,t)]$$

where $\sigma(t)$ has limiting value σ_∞,

$$\begin{aligned} R(x,t) &= a(t)\zeta(x) \ + \ \eta(x,t), \quad \text{with} && (40)\\ \|a(t)\zeta(x)\|_{L^\infty} &\le c(1+t)^{-1/2}, && (41)\\ \|\eta(x,t)\|_{L^p} &\le c(1+t)^{-3/2+3/q}\|\eta(x,0)\|_{L^q}. && (42) \end{aligned}$$

Thus, the solution u converges to a travelling soliton.

The estimates on the parameters $\omega(t), \gamma(t), v(t)$ and $D(t)$ are as before, as is the estimate for $\eta(x,t)$. The new feature in this problem is proving decay of the localized part $a(t)\zeta(x)$. Thus, we want to show that $a(t)$ decays.

Sketch of proof:

For simplicity we will consider a 'toy' problem (full details - and there are lots of them! - can be found in [SW2] and [SW3]). Let's assume the nonlinearity has Taylor expansion $f(x) = x + \cdots$, and let's work with the NLS equation $i\partial_t u = -\Delta u - f(|u^2|)u$. Let $H = -\Delta + \omega - f(|\varphi_\omega^2|)$ with $H\zeta = \lambda\zeta$ and let $\mathbf{P}_c$ and $\mathbf{P}_d$ be projections onto the continuous and discrete spectral subspaces of H respectively (note that $\mathbf{P}_d(g) = \langle g, \zeta\rangle\zeta$). Set $\mathbf{P}_c\eta = \eta$ (orthogonality condition) and write $a(t) = A(t)e^{-i\lambda t}$. Plugging the ansatz $u = e^{i\omega t}[\varphi_\omega + R]$ into NLS and using the fact that $e^{i\omega t}\varphi_\omega$ is a solution, we obtain equations describing the evolution of η and A;

$$\begin{aligned} i\partial_t\eta &= H\eta - A^2 e^{-2i\lambda t}\mathbf{P}_c\zeta^2\varphi_\omega + \cdots \ \text{(applying } \mathbf{P}_c \text{ to the equation)} && (43)\\ \dot{A} &= ie^{2i\lambda}\bar{A}\langle\bar{\zeta}\eta\varphi_\omega, \zeta\rangle \ + \ \cdots \quad \text{(applying } \langle \bullet , \zeta\rangle \text{ to the equation)} && (44) \end{aligned}$$

(these obtained by looking at the terms linear in φ_ω and quadratic in R on the right hand side of NLS). Duhamel's formula gives,

$$\begin{aligned}
\eta &= e^{-iHt}\eta_o \; - \; i\int_0^t e^{-iH(t-s)}A^2(s)e^{-2i\lambda s}\mathbf{P}_c\zeta^2\varphi_\omega\, ds \; + \; \cdots \\
&= e^{-iHt}\eta_o \; - \; ie^{-iHt}\int_0^t e^{i(H-2\lambda)s}A^2(s)\mathbf{P}_c\zeta^2\varphi_\omega\, ds \; + \; \cdots \qquad (45)
\end{aligned}$$

Regularizing the integrand before integrating by parts leads to,

$$\eta \; = \; e^{-iHt}\eta_o + ie^{-2i\lambda t}\delta(H-2\lambda)A^2(t)\mathbf{P}_c\zeta^2\varphi_\omega \; + \; \cdots \qquad (46)$$

where we've used the formula (interpreted in the sense of distributions);

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{1}{x-x_o+i\epsilon} &= \lim_{\epsilon\to 0}\frac{x-x_o}{(x-x_0)^2+\epsilon^2} - i\lim_{\epsilon\to 0}\frac{\epsilon}{(x-x_0)^2+\epsilon^2} \\
&= \text{P.V.}\left(\frac{1}{x-x_0}\right) \; - \; i\pi\delta(x-x_0). \qquad (47)
\end{aligned}$$

* * Note that if $2\lambda > \omega$ then $\delta(H-2\lambda) \neq 0$ * * . Plugging expression (47) for η into the A equation (45) and using that $\mathbf{P}^2 = \mathbf{P}, \mathbf{P}^* = \mathbf{P}$, and $\varphi_\omega \in \mathbf{R}$ gives,

$$\dot{A} \; = \; -\Gamma|A|^2A + \; \cdots \qquad (48)$$

where

$$\Gamma \; \equiv \; \langle\delta(H-2\lambda)\mathbf{P}_c\zeta^2\varphi_\omega,\, \mathbf{P}_c\zeta^2\varphi_\omega\rangle \; \geq \; 0 \qquad (49)$$

If $\Gamma > 0$, equation (49) implies that $A(t) \sim t^{-1/2}$. The condition $\Gamma > 0$ is called the *resonance condition*, or *nonvanishing of the Fermi Golden Rule* (FGR). (See [RSIV] and [SW2] for discussion of time-dependent resonance theory and FGR.)

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