Virial Relations for Nonlinear Wave Equations and Nonexistence of Almost-Periodic Solutions *

Randall Pyke†

Department of Mathematics, University of Toronto

Toronto, Ontario, Canada M5S 3G3

May, 1996

Abstract

We present a formalism for constructing integral identities involving time almost-periodic solutions of nonlinear wave equations. As an application we prove a nonexistence theorem that is applicable to a large class of nonlinearities.

1 Introduction

A virial relation is an integral identity involving the solution of a differential equation. The identity can be derived from the equation itself or, if the equation can be formulated as a variational problem, from infinitesimal variations of the action functional associated to the equation. A well known example from physics relates the time-average kinetic and potential energies of an n-particle system under the influence of central forces (usually referred to as *the* virial theorem; see for example [LL]). In mathematics virial relations have been used extensively, for example, in deriving necessary conditions for the existence of solutions of differential equations beginning with the work of Pohozaev [Po].

*To appear in Reviews in Mathematical Physics

†Research in partial fulfillment of the PhD degree at the University of Toronto

In this article we present a systematic approach to deriving virial relations for almost periodic solutions of nonlinear wave equations (NLW's) of the form

$$\partial_t^2\varphi - \Delta\varphi + f(\varphi) = 0. \tag{1.1}$$

Here $\varphi : \mathrm{R}_x^N \times \mathrm{R}_t \to \mathrm{R}$, $\partial_t^2\varphi = \partial^2\varphi/\partial t^2$, $\Delta\varphi = \sum_{i=1}^N \partial^2\varphi/\partial x_i^2$, and $f : \mathrm{R} \to \mathrm{R}$ with $f(0) = 0$. By an almost periodic solution we understand a solution that is almost periodic in time t and $L^2(\mathrm{R}^N)$ in space x. Almost periodic solutions have special significance in field theories because they represent bound states, i.e., solutions that are localized in space uniformly in time (cf. below). Our objective is twofold. First, we want to illuminate a method that has been known and used in various guises for many years in mathematics and physics and is well-suited for problems arising in nonlinear differential equations. Second, we apply this method to derive necessary conditions for the existence of almost periodic solutions of NLW.

We remark about the notion of an almost periodic solution. Due to the invariance of NLW under the Poincaré group, an almost periodic solution may not appear to be almost periodic in another Lorentz frame, although it still satisfies NLW in this frame. For example, if the second frame is moving at a constant nonzero velocity with respect to the first frame (in which the solution is almost periodic) then the solution is not almost periodic in this moving frame. This means that when we are considering an almost periodic solution we are tacitly referring to a particular Lorentz frame (it is not unique). Thus, for us an almost periodic solution of NLW is a solution that is almost periodic in *some* Lorentz frame.

In this paper we are concerned solely with almost periodic solutions and do not address the question of the existence of general solutions. In particular, our results are that NLW's with certain nonlinearities do not possess almost periodic solutions. Never the less, global (in time) existence theory does apply to some of these equations (for a survey see [GV] or [Str2]). Thus, together these two results imply that there are NLW's that possess global solutions but that none of these solutions are almost periodic - that the set of solutions of these NLW's do not contain functions with a certain temporal behavior (almost periodicity). An example of such a NLW is given below (cf. Example 1 in Section 3).

When NLW is viewed as an evolution equation it is natural to attempt to characterize solutions by their temporal behavior. The work of this article is a start in that direction. In deciding what sorts of behavior are appropriate for such a characterization we are motivated by two observations. The first is that $L^2(\mathrm{R}^N)$ valued almost periodic functions have the property of being uniformly bounded with respect to time. That is, for any $\varepsilon > 0$ there exists a ball $B(\rho) \subset \mathrm{R}^N$ of radius $\rho = \rho(\varepsilon)$ such that

$$\int_{B^c(\rho)} |\varphi(x,t)|^2 \, dx \; < \; \varepsilon \tag{1.2}$$

for all t (cf. Definition 4.5 and Lemma 4.7 below). We refer to functions satisfying this condition as being *bound states*. Secondly, an important theorem in quantum mechanics due to Ruelle [R] asserts that bound state solutions of the Schrödinger equation are almost periodic in time. The converse is true as we have just seen. Thus, for the Schrödinger equation bound states are characterized by their temporal behavior and by this alone. We believe that a similar statement can be made for some nonlinear equations such as NLW and the nonlinear Schrödinger equation [Py]. That is, solutions of these equations that are bound states, as defined above, are almost periodic in time (we may have to first transform to an appropriate coordinate system via a symmetry of the equation). Therefore, from this perspective the work presented in this article aims to contribute towards a classification of nonlinear wave equations according to whether they possess bound states or not.

One of the few NLW's known to possess periodic solutions that are localized in x (often referred to as "breathers") is the sine-Gordon equation $\partial_t^2\varphi - \partial_x^2\varphi + \sin(\varphi) = 0$. Here the solutions are given by the formula

$$\varphi(x,t;\omega) \;=\; 4\tan^{-1}\left(\frac{\varepsilon \sin \omega t}{\omega \cosh \varepsilon x}\right), \qquad \varepsilon^2 + \omega^2 = 1. \tag{1.3}$$

Being an integrable system, these solutions can be found using techniques related to the inverse scattering transform [NMPZ]. However, many NLW's of interest are not integrable, for example, the "φ^4" equation $\partial_t^2\varphi - \partial_x^2\varphi + \varphi - \varphi^3 = 0$, which is an important model in particle physics. It is not known whether this latter equation possesses periodic solutions. The method we describe in this article does not depend on the integrability properties of the particular equation in question.

We briefly review previous studies concerning periodic solutions of nonlinear wave equations. The structural stability of the sine-Gordon breather has been an object of study for a number of years (for recent work see [BMW] and [D], for an earlier study see [MS]). Here one looks for periodic solutions of the perturbed sine-Gordon equation $\partial_t^2\varphi - \partial_x^2\varphi + \sin(\varphi) + \varepsilon g(\varphi) = 0$ that are close to the sine-Gordon breather. Because the sine-Gordon breather is known explicitly, a detailed analysis can be carried out. These references reveal a complex interaction between the breather and nonlinearity $\sin(\varphi)$ that is easily upset by a perturbation. The conclusion is that typically the perturbed equation does not have a periodic solution close to the sine-Gordon breather. This result is corroborated by the work [Ki] in which the sine-Gordon equation is singled-out as essentially the only NLW on R^{1+1} having (analytic) breathers. In the same vein one can look at how periodic solutions of *linear* equations behave under nonlinear perturbations. This was investigated in [Si] where it was shown that these solutions, like the sine-Gordon breather, are generically unstable under nonlinear structural perturbations. A necessary condition for the existence of periodic solutions on R^1 follows from a result of Coron's [Co] which states that if $\varphi \in C^2(\mathrm{R}^2)$ is a $2\pi/\omega$ -periodic solution of NLW, then $\omega^2 \le f'(0)$. Recently, we have been able to extend this result to multi-spatial dimensions [PS].

The situation is very different on a bounded or semi-bounded spatial domain; here periodic and almost periodic solutions are abundant. In [SV],[V2],[V3], and [We], periodic, quasiperiodic, or almost periodic solutions of NLW on the half-line R_+ were constructed using methods from centre manifold theory. Here the one dimensionality of the spatial variable is essential as NLW is treated as a dynamical system in a phase space of periodic functions with x playing the role of the dynamical variable. The (one-sided) exponentially decaying periodic solutions are points in the stable or unstable manifolds of the zero solution. The failure of these manifolds to intersect prohibits extending these existence results to the entire real line [V3]. [Sc] and [Sm] apply a similar analysis for radially symmetric periodic solutions on R^N. Other existence results using a variety of methods from KAM theory, Lyapunov-Schmidt bifurcation theory, and variational methods, can be found in the references [Wa], [CW], [BCN] and [H].

Although the above results indicate that almost periodic solutions of nonlinear wave equations on R^{N+1} are rare, the question of existence is still very much open. With this in mind we set out to investigate what conditions the nonlinearity must satisfy in order for NLW on the infinite spatial domain R^N to support solutions that are time periodic or, more generally, time almost periodic. Although in this paper we will be considering the unbounded spatial domain, we could also carry out our analysis on bounded or semi-bounded spatial domains.

Our approach to this problem was motivated by the study [V1] where NLW on R^{1+1} was formulated as a variational problem. Since periodic solutions are critical points of the action functional, the Fréchet derivative of the functional is zero at these points. Evaluating the Fréchet derivative on certain functions derived from the solution itself results in integral identities involving the solution and nonlinearity. From these identities several nonexistence theorems for classical (i.e., $C^2(\mathrm{R}^2)$) periodic solutions was proven. Subsequently, the development of our ideas in the direction of describing virial relations as infinitesimal variations of the action functional (cf. the appendix to the present paper) was aided by a recent article [M] which surveys the role of virial relations in physical theories that are based on an action principle.

The work presented in this paper extends the results of [V1] in several directions. At first we work with more general quasiperiodic solutions (multiple, but finitely many, frequencies instead of one), and in several spatial dimensions. Our admissible set of solutions is also larger, being of class H^1 in space and time (weak solutions). After deriving a class of virial relations for quasiperiodic solutions and using these to state necessary conditions for the existence of such solutions, we will see that by using ideas from ergodic theory we can express our results in a way that makes sense for the larger class of almost periodic solutions (i.e., allowing for infinitely many frequencies). In a latter section of this paper we show that these formulae are in fact valid for almost periodic solutions.

Historically, the idea of formulating a nonlinear equation as a minimization problem to obtain necessary conditions for the existence of solutions has its origins in a work of Pohozaev [Po] who used the method to study solutions of a nonlinear elliptic boundary problem. Subsequent applications of this technique can be found, for example, in [Str1] and [B] where it was used to study stationary states of some nonlinear wave equations, and in [BL] and [BrLi] where it was used in a study of a class of nonlinear elliptic variational problems.

The main results of this article are the following. We first derive a class of integral identities that must be satisfied by almost periodic solutions of NLW (Theorems 2.7 and 4.10). Then, by choosing a particular subclass of these we are able to prove the following theorem, which we state here without all the technical details (cf. Theorems 3.5 and 5.1).

Theorem *Suppose φ is an almost periodic solution of NLW such that $\varphi(\cdot,t) \in H^1(\mathrm{R}^N)$. Let $F(z) = \int_0^z f(w)\,dw$. If $F(z) - czf(z) \le 0$ for some $c \in [\frac{N-2}{2N}, \frac{1}{2}]$ and for all z such that $|z| \le \|\varphi\|_{L^\infty(\mathrm{R}^{N+1})}$, then φ is independent of time.*

Application of this theorem can be widened by considering small amplitude solutions since then only properties of the nonlinearity near zero enters. For such solutions we present two nonexistence results (Corollaries 3.6 and 3.7). The first has relevance, in particular, to nonlinear Klein-Gordon equations (i.e., $f'(0) > 0$) with odd nonlinearity. For the second we combine Wirtinger's inequality with virial relations to obtain a result similar to Coron's [Co] in multi-spatial dimensions. By taking advantage of the decay of solutions as $|x| \to \infty$ we can, by "localizing" the virial relation in a neighborhood of spatial infinity, extend some of the results about small amplitude solutions to solutions of arbitrary amplitude (Corollary 3.8).

Before concluding this introduction we present some heuristics to help motivate and describe our approach. We begin with the observation that a function $\varphi(x,t) \in H^1(\mathrm{R}^{N+1})$ can be viewed as a function from R to $H^1(\mathrm{R}^N)$ via the map $t \mapsto \varphi(\cdot,t)$. Furthermore, if φ is $2\pi/\omega$ -periodic then we can write $\varphi(x,t) = \gamma(x,\omega t)$ where $\gamma : S^1 \to H^1(\mathrm{R}^N)$ is a function defined on the unit circle S^1. From this it is clear how to define quasiperiodic functions: φ is quasiperiodic if $\varphi(x,t) = \gamma(x,\omega t)$ for some $\gamma : \mathrm{T}^l \to H^1(\mathrm{R}^N)$ where T^l is the l-torus $S^1 \times \cdots \times S^1$, and for some $\omega \in \mathrm{R}^l$. We call γ the *generating function* of φ and ω the *frequency* of φ. Given a quasiperiodic solution φ of NLW with frequency $\omega \in \mathrm{R}^l$, by the chain rule we derive an equation satisfied by its generating function γ;

$$\mathcal{D}_\omega^2\gamma - \Delta\gamma + f(\gamma) = 0.$$

Here $\mathcal{D}_\omega^2 = \sum_{i,j=1}^l \omega_i\omega_j\mathcal{D}_i\mathcal{D}_j$, $\mathcal{D}_i = \partial/\partial\theta_i$, and $\theta_1, \ldots, \theta_l$ are coordinates on T^l. We call this equation the nonlinear wave equation on $\mathrm{R}^N \times \mathrm{T}^l \equiv \Omega_{N,l}$ with frequency ω. To derive virial relations for quasiperiodic solutions φ then, we derive virial relations for solutions γ of NLW on $\Omega_{N,l}$ and transfer these back to φ using the identification $\varphi(x,t) = \gamma(x,\omega t)$.

In general, an almost periodic function is a quasiperiodic function with infinitely many (independent) frequencies. Almost periodic functions φ can be characterized by generating functions γ that are defined on the infinite dimensional torus T^∞; $\gamma: \mathrm{T}^\infty \to H^1(\mathrm{R}^N)$. Then, there is a dense embedding $\Gamma: \mathrm{R} \to \mathrm{T}^\infty$ such that $\varphi(t) = \gamma(\Gamma(t))$. For general almost periodic solutions though, we will work directly with the function itself rather than with its generating function. This is facilitated by passing from T^∞ to R via the Bohr compactification of the real line which is realized by the formula

$$\int_{\mathrm{T}^\infty} \gamma(\theta)\, d\theta = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Big(\gamma \circ \Gamma\Big)(t)\, dt. \tag{1.4}$$

Here $d\theta$ is the (normalized) Haar measure on T^∞. That is, instead of integrating γ over the torus T^∞, which is its "space" average, we take its time average along the curve $\Gamma(\mathrm{R})$.

To illustrate the general strategy as well as to describe some aspects of our approach, we present an example of a virial relation. Suppose γ is a solution of NLW on $\Omega_{N,l}$ with frequency ω and let $\beta: \mathrm{T}^l \to H^1(\mathrm{R}^N)$ be some function on T^l. Then, multiplying NLW by β and integrating we have,

$$\int_{\mathrm{T}^l}\int_{\mathrm{R}^N} \Big(\mathcal{D}_\omega^2\gamma - \Delta\gamma + f(\gamma)\Big)\beta = 0.$$

Integrating by parts we obtain

$$\int_{\mathrm{T}^l}\int_{\mathrm{R}^N} \Big\{ -\mathcal{D}_\omega\gamma\mathcal{D}_\omega\beta + \nabla\gamma\cdot\nabla\beta + f(\gamma)\beta\Big\} = 0,$$

which expresses that γ is a weak solution. Now take for the multiplier β some combination of γ and its derivatives, for example, $\beta = x \cdot \nabla\gamma$. Using this β in the latter formula and removing a divergence term we arrive at the virial relation

$$\int_{\mathrm{T}^l} \int_{\mathrm{R}^N} \Big\{ \frac{1}{2}(\mathcal{D}_\omega \gamma)^2 + \Big(\frac{2-N}{2N}\Big) \mid \nabla\gamma \mid^2 - F(\gamma) \Big\} \;=\; 0, \quad F' = f. \tag{1.5}$$

We see right away that in spatial dimensions 1 or 2, F cannot be a nonpositive function. Thus, we have a necessary condition on the nonlinearity for the existence of solutions of NLW on $\Omega_{N,l}$. The virial relation for the quasiperiodic solution φ can be recovered from this with the identification $\varphi(x,t) = \gamma(x, \omega t)$ and replacing the integral over T^l by the time-mean over R (cf. (1.4)).

What is required to make this argument rigorous is that γ be sufficiently regular and integrable. How regular and integrable depends on how β is defined. One could, at the start, consider only those solutions for which the formal manipulations remain valid. However, the final formula (the virial relation) that one obtains requires less of the solution than the rigorous analysis asks for. In the above example we see that the virial relation makes sense (i.e., the integral in (1.5) is finite) if $\gamma \in H^1(\Omega_{N,l})$ and $F(\gamma) \in L^1(\Omega_{N,l})$. In our analysis we assume from a solution and nonlinearity only what is needed to make the (formally derived) virial relation a well-defined formula. This defines then the largest possible class of solutions (for a given nonlinearity) for which the virial relation is valid. To make the derivation rigorous we regularize the multiplier $\beta \mapsto \beta_\varepsilon$, $\beta_0 = \beta$, derive the virial relation corresponding to β_ε, and then regain the virial relation corresponding to β in the limit $\varepsilon \to 0$.

Because we allow for infinitely many frequencies in the almost periodic case, the construction of virial relations for almost periodic solutions differs from that for periodic and quasiperiodic solutions. But since periodic and quasiperiodic solutions are also almost periodic, the latter results include the former. However, the method of proof for quasiperiodic solutions is different than the proof for almost periodic solutions. We have included both proofs here because the method for quasiperiodic solutions could be adapted to handle other partial differential equations; those defined on open subsets Ω of R^m (bounded or unbounded) where the solutions are in $L^2(\Omega)$. The technical tools for this are a regularization of the solution and Lebesgue's dominated convergence theorem. The dominated convergence theorem cannot be applied to almost periodic solutions because they are not integrable on R^{N+1}. Instead, we utilize the uniform boundedness of almost periodic functions (cf. (1.2)) in conjunction with the uniform (in t) convergence of β_ε on compact subsets of R^N.

In the above example we used the function $\beta = x \cdot \nabla\gamma$ to derive a virial relation. One can view x as a vector field v on R^N: $v(x) = x$. Conversely, for any vector field v we may derive a virial relation by setting $\beta = v \cdot \nabla\gamma$. In our study we use this identification to characterize a class of virial relations. Furthermore, these vector fields generate flows $\Phi_\lambda : \mathrm{R}^N \to \mathrm{R}^N$ which in turn define transformation groups T_λ acting on functions via the formula $T_\lambda\varphi \equiv \varphi \circ \Phi_\lambda$. In general, every transformation group will lead to a virial relation via the infinitesimal variations of the action associated to NLW under this group. If T_λ is a symmetry of the action then there is a corresponding differential identity (a conservation law: Noether's theorem) which can be thought of as a "trivial" virial relation. This is discussed more thoroughly in the appendix.

We outline the contents of this paper. The next section deals with quasiperiodic solutions of NLW. We define quasiperiodic functions and the types of quasiperiodic solutions we will be concerned with, as well as discuss our hypothesis on the nonlinearity. Then we state and prove the main result of that section (Theorem 2.7): a class of virial relations for quasiperiodic solutions of NLW that are characterized by vector fields on R^N. In the following section, Section 3, we use these virial relations to derive several nonexistence results. Sections 4 and 5 deal with more general almost periodic solutions. The main results there, Theorems 4.10 and 5.1, characterize a class of virial relations for almost periodic solutions by vector fields on R^N and state a necessary condition for the existence of such solutions. Finally, in the appendix we show how virial relations may be derived by formulating NLW as a variational problem. This will elucidate the relationship between virial relations and conservation laws.

Notation: ∇ denotes the gradient operator on R^N: $\nabla = (\partial_1, ..., \partial_N)$, $\partial_i = \partial/\partial x_i$, and for multi-index $a \in \mathrm{Z}^N \geq 0$, $\partial^a = \partial_1^{a_1}, ...\partial_N^{a_N}$. By the vector $\omega \in \mathrm{R}^l$ we will always mean an l-tuple of incommensurate numbers ("frequencies"). $\mathbf{1}$ will stand for the identity matrix. T^l

will denote the l-torus $S^1 \times \cdots \times S^l$, and $\Omega_{N,l} \equiv \mathrm{R}^N \times \mathrm{T}^l$. Given a frequency vector $\omega \in \mathrm{R}^l$ we define the differential operator $\mathcal{D}_\omega \equiv \omega \cdot \mathcal{D}$ on T^l where $\mathcal{D} = (\mathcal{D}_1, \ldots, \mathcal{D}_l)$, $\mathcal{D}_i = \partial/\partial\theta_i$, is the gradient operator on T^l. $\|\psi\|_p$ is the $L^p(\mathrm{R}^N)$ norm and $H^{s,p}$ the L^p Sobolev space of order s which for $p = 2$ we write simply as H^s. The space $L^p\Big(\mathrm{R},\, L^q(\mathrm{R}^N)\Big)$ denotes the set of functions $\varphi :\ \mathrm{R} \to L^q(\mathrm{R}^N)$ such that $\int_{\mathrm{R}} \|\varphi(t)\|^p_{L^q(\mathrm{R}^N)} \equiv \|\varphi\|^p_{p,q} < \infty$. $C_c^\infty(\mathrm{R}^N)$ denotes smooth functions on R^N with compact support. $(L^q(\mathrm{R}^N))^n$ is the set of n-dimensional vector valued functions on R^N, each component being an element of $L^q(\mathrm{R}^N)$. For $v : \mathrm{R}^N \to \mathrm{R}^N$, $dv = dv(x)$ denotes the matrix-valued function with entries $[dv]_{ij} = \partial_i v^j$; then, $\nabla \cdot v = tr\, dv$ where $tr\, A$ is the trace of the matrix A.

Acknowledgements: I would like to thank Professors I.M. Sigal and I. Kupka for stimulating and helpful discussions. Financial support from NSERC of Canada, the Ontario Ministry of Education, and the University of Toronto is gratefully acknowledged.

2 Virial relations for quasiperiodic solutions

2.1 Quasiperiodic solutions

Definition 2.1 *Let* $\Omega_{N,l} \equiv \mathrm{R}^N \times \mathrm{T}^l$. $\varphi :\ \mathrm{R}^{N+1} \to \mathrm{R}$ *is an* l-quasiperiodic function with frequency $\omega \in \mathrm{R}^l$ *if* $\varphi(x,t) = \gamma(x, \omega t) = \Big(\gamma \circ \Gamma_\omega\Big)(t)$ *for some function* $\gamma \in C(\Omega_{N,l})$ *where* $\Gamma_\omega : \mathrm{R} \to \mathrm{T}^l$ *is the continuous dense embedding of* R *into* T^l *given by* $t \mapsto (\omega t)\ mod\, 2\pi$. *We call* γ *the* generating function *of* φ.

To fix what we mean by a classical quasiperiodic solution of NLW we state the following definition.

Definition 2.2 *An l-quasiperiodic function* $\varphi :\ \mathrm{R}^{N+1} \to \mathrm{R}$ *is a* classical quasiperiodic solution of NLW on R^{N+1} *(equation (1.1)) if* φ *solves NLW and its generating function is of class* $C^2(\Omega_{N,l}) \cap H^2(\Omega_{N,l})$.

To motivate our notion of a weak quasiperiodic solution of NLW we first derive an equation satisfied by the generating function of a classical quasiperiodic solution.

Lemma 2.3 *Suppose φ is a classical l-quasiperiodic solution of NLW with frequency ω. Then its generating function γ solves the equation*

$$\mathcal{D}_\omega^2\gamma - \Delta\gamma + f(\gamma) \;=\; 0 \tag{2.1}$$

on $\Omega_{N,l}$. Here $\mathcal{D}_\omega^2\gamma = \sum_{i,j=1}^{l}\omega_i\omega_j\mathcal{D}_{i,j}^2\gamma$. Conversely, if γ is a member of $C^2(\Omega_{N,l})\cap H^2(\Omega_{N,l})$ and solves (2.1), then the function $\varphi(x,t)=\gamma(x,\omega t)$ is a classical l-quasiperiodic solution of NLW with frequency ω.

Proof:

The curve $\Gamma = \{(x,(\omega t)\, mod\, 2\pi) \;\mid\; (x,t)\in \mathrm{R}^{N+1}\}$ is dense in $\Omega_{N,l}$ and by the chain rule γ solves (2.1) along Γ. By the continuity of $\gamma, \Delta\gamma, \mathcal{D}_{ij}^2\gamma$ and f we conclude that γ solves (2.1) on $\Omega_{N,l}$. The converse is just the chain rule. $\square$

Definition 2.4 *Equation (2.1) is the* nonlinear wave equation (NLW) on $\Omega_{N,l}$ with frequency ω.

Definition 2.5 $\gamma : \Omega_{N,l} \to \mathrm{R}$ *is a* weak solution of NLW on $\Omega_{N,l}$ with frequency ω *if* $\gamma \in H^1(\Omega_{N,l})$ *and*

$$\int_{\mathrm{T}^l}\int_{\mathrm{R}^n}\Big\{-\mathcal{D}_\omega\gamma\mathcal{D}_\omega\beta \;+\; \nabla\gamma\cdot\nabla\beta \;+\; f(\gamma)\beta\Big\} \quad=\quad 0 \qquad \forall\beta\in H^1(\Omega_{N,l}). \tag{2.2}$$

Definition 2.6 *A quasiperiodic function $\varphi : \mathrm{R}^{N+1} \to \mathrm{R}$ is a* weak l-quasiperiodic solution of NLW on R^{N+1} with frequency ω *if its generating function γ is a weak solution of NLW on $\Omega_{N,l}$ with frequency ω.*

Using standard arguments from the calculus of variations, if $\gamma \in C^2(\Omega_{N,l})\cap H^2(\Omega_{N,l})$ then (2.1)$\Leftrightarrow$(2.2).

We will work with the generating function γ from now on in this section, deriving virial relations for solutions of NLW on $\Omega_{N,l}$. The virial relations for solutions φ of NLW on R^{N+1} will be recovered from these through the identification $\varphi(x,t)=\gamma(x,\omega t)$.

Conditions on the nonlinearity

For the derivation of virial relations for solutions γ of NLW on $\Omega_{N,l}$ we require that f be continuous with $f(0) = 0$, and that $f(\gamma) \in L^2(\Omega_{N,l})$ and $F(\gamma) \in L^1(\Omega_{N,l})$ where $F(z) \equiv \int_0^z f(w)\,dw$. The latter two conditions will depend on two factors, the growth of f and the integrability of γ. If $\mid f(z) \mid \leq c(\mid z \mid + \mid z \mid^{q/2})$ for some $q \in [2, \infty)$, then $f : L^2(\Omega_{N,l}) \cap L^q(\Omega_{N,l}) \to L^2(\Omega_{N,l})$ and $F : L^2(\Omega_{N,l}) \cap L^q(\Omega_{N,l}) \to L^1(\Omega_{N,l})$. This is true by virtue of the inequalities

$$\begin{aligned} \mid f(z) \mid^2 &\leq c'\Big(\mid z \mid^2 + \mid z \mid^{(q+2)/2} + \mid z \mid^q \Big), \\ \mid F(z) \mid &\leq c'\Big(\mid z \mid^2 + \mid z \mid^{(q+2)/2} \Big) \end{aligned}$$

and the fact that $L^2(\Omega_{N,l}) \cap L^q(\Omega_{N,l}) \subset L^{q'}(\Omega_{N,l})$ for all $q' \in [2,\ q]$. If f satisfies a Lipschitz condition at the origin, then $f : L^2(\Omega_{N,l}) \cap L^\infty(\Omega_{N,l}) \to L^2(\Omega_{N,l})$ and $F : L^2(\Omega_{N,l}) \cap L^\infty(\Omega_{N,l}) \to L^1(\Omega_{N,l})$. Indeed, let c and d be such that $\mid f(z) \mid < c \mid z \mid$ for $\mid z \mid < d$. Then,

$$\begin{aligned} \|f(\gamma)\|^2_{L^2(\Omega_{N,l})} = \int_{\Omega_{N,l}} \mid f(\gamma) \mid^2 &= \int_{\{|\gamma|<d\}} \mid f(\gamma) \mid^2 + \int_{\{|\gamma|\geq d\}} \mid f(\gamma) \mid^2 \\ &\leq c^2 \|\gamma\|^2_{L^2(\Omega_{N,l})} + b^2\, meas\,(\{\mid \gamma \mid \geq d\}) \quad = M, \end{aligned}$$

where $b = sup\{\mid f(z) \mid\ ;\quad \mid z \mid \leq \|\gamma\|_{L^\infty(\Omega_{N,l})}\}$. If $\|\gamma\|_{L^\infty(\Omega_{N,l})} < \infty$ the continuity of f implies that b is finite and so M is finite. A similar argument applies to F since in this case $\mid F(z) \mid < \frac{c}{2} \mid z \mid^2$ for $\mid z \mid < d$.

After these considerations we now state our hypothesis on f which will guarantee that $f : L^2(\Omega_{N,l}) \cap L^q(\Omega_{N,l}) \to L^2(\Omega_{N,l})$ and $F : L^2(\Omega_{N,l}) \cap L^q(\Omega_{N,l}) \to L^1(\Omega_{N,l})$ where $q \in [2, \infty]$.

Hypothesis (H) $f \in C(\mathrm{R}, \mathrm{R})$ with $f(0) = 0$. Either there exists a $q \in [2, \infty)$ such that $\mid f(z) \mid \leq c(\mid z \mid + \mid z \mid^{q/2})$, or f satisfies a Lipschitz condition at the origin, a property which we designate by setting $q = \infty$.

Recall the Sobolev embeddings [A]: $H^1(\Omega_{1,1}) \subset L^q(\Omega_{1,1})$ for $q \in [2, \infty)$, and $H^1(\Omega_{N,l}) \subset L^q(\Omega_{N,l})$ for $q \in [2,\ 2(N+l)/(N+l-2)]$, where $N + l > 2$. Therefore, because we are assuming that $\gamma \in H^1(\Omega_{N,l})$, a priori the parameter q in (H) can be any number in the interval $[2, \infty)$ in the case $N = l = 1$, or any number in the interval $[2,\ 2(N+l)/(N+l-2)]$ in the case $N + l > 2$. We include the parameter q in the hypothesis (H) so as to make special consideration when the solution γ possesses additional integrability than that given by the Sobolev embedding.

2.2 Virial relations for quasiperiodic solutions

A virial relation involving the solution γ of NLW on $\Omega_{N,l}$ results when in equation (2.2) we take for β some combination of γ and its derivatives. A class of such virial relations can be characterized by vector fields on R^N (the spatial domain): given a vector field v, set $\beta = v \cdot \nabla\gamma$. The resulting virial relation is the content of the following theorem.

Theorem 2.7 *Let N and l be positive integers, $\omega \in \mathrm{R}^l$ any vector of incommensurate frequencies, and v a smooth vector field on R^N that satisfies*

$$\sup_{x \in \mathrm{R}^N} \mid \partial^a v(x) \mid\mid x \mid^{|a|-1} < \infty \qquad \mid a \mid = 0, 1, 2. \tag{2.3}$$

If

(i) $N = l = 1$ with $\gamma \in H^1(\Omega_{1,1})$ satisfying (2.2) and f satisfying (H) for some (any) $q \in [2, \infty)$, or

(ii) if $N + l > 2$ and if $\gamma \in H^1(\Omega_{N,l}) \cap L^q(\Omega_{N,l})$, for some $q \in [2, \infty]$, satisfies (2.2) and f satisfies (H) with this same q,

then

$$\int_{\mathrm{T}^l} \int_{\mathrm{R}^N} \left\{ tr\, dv \left(\frac{1}{2} (\mathcal{D}_\omega \gamma)^2 - F(\gamma) \right) \ + \ \nabla\gamma \cdot [dv - \frac{1}{2} tr\, dv\, \mathbf{1}] \nabla\gamma \right\} \ = \ 0, \tag{2.4}$$

where $F(z) \equiv \int_0^z f(w)\, dw$.

Remarks: Equation (2.4) is the virial relation for the solution γ of NLW on $\Omega_{N,l}$ associated to the vector field v. The corresponding virial relation for φ, where $\varphi(x,t) = \gamma(x, \omega t)$, is

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{\mathrm{R}^N} \left\{ tr\, dv \left(\frac{1}{2}(\partial_t \varphi)^2 - F(\varphi) \right) \;+\; \nabla\varphi \cdot [dv - \frac{1}{2} tr\, dv \mathbf{1}] \nabla\varphi \right\} \;=\; 0.$$

The derivation of this formula from equation (2.4) follows from some results of ergodic theory and will be presented in Section 4.

During the proof of Theorem 2.7 we will require the following Lemma.

Lemma 2.8 *If* $h \in (H^{1,1}(\mathrm{R}^N))^N$ *then* $\int_{\mathrm{R}^N} \nabla \cdot h \;=\; 0.$

Proof of Lemma 2.8:

There exists a family $\{h_\varepsilon\} \subset (C_c^\infty(\mathrm{R}^N))^N$ that converges to h in $(H^{1,1}(\mathrm{R}^N))^N$ as $\varepsilon \to 0$. In particular this means that $\lim_{\varepsilon\to 0} \int_{\mathrm{R}^N} \nabla \cdot h_\varepsilon \;=\; \int_{\mathrm{R}^N} \nabla \cdot h$. For each h_ε we have, via the divergence theorem, that $\int_{\mathrm{R}^N} \nabla \cdot h_\varepsilon = 0$. Therefore $\int_{\mathrm{R}^N} \nabla \cdot h = 0$ $\quad\Box$

Proof of Theorem 2.7:

Our motivation in deriving equation (2.4) comes from proceeding formally: take $\beta = v \cdot \nabla\gamma$ in equation (2.2) and apply the standard rules of calculus (without justification) to arrive at equation (2.4). In the proof below we show that this procedure can be made rigorous after a suitable regularization of γ and v.

To this end we introduce the family of operators $R_\varepsilon = R_\varepsilon(-\varepsilon\, i\nabla)$ on $L^2(\mathrm{R}^N)$ defined through the Fourier transform: $\widehat{R_\varepsilon\psi} = (1 + \varepsilon^2 p^2)^{-1}\hat{\psi}(p)$. Because R_ε dampens the high frequency components of $\hat{\psi}$, it acts as a smoothing operator. We have that $R_\varepsilon\psi \to \psi$ in $L^2(\mathrm{R}^N)$ as $\varepsilon \to 0$ with $\|R_\varepsilon\psi\|_2 \le \|\psi\|_2$, and that R_ε commutes with ∇. Using the Fourier transform we see that R_ε acts as a bounded self-adjoint operator from $H^s(\mathrm{R}^N)$ to $H^{s+2}(\mathrm{R}^N)$.

We begin by regularizing $v \cdot \nabla\gamma$. For $\varepsilon, \delta > 0$ define

$$\beta_{\varepsilon,\delta} \equiv v_\delta \cdot \nabla_\varepsilon \gamma \tag{2.5}$$

where

$$v_\delta(x) = \frac{v(x)}{1+\delta \mid x \mid^2} \tag{2.6}$$

and

$$\nabla_\varepsilon \equiv \nabla R_\varepsilon^2. \tag{2.7}$$

By condition (2.3), $v_\delta \in L^\infty(\mathrm{R}^n)$ for $\delta > 0$ while for $|a| > 0$, $\partial^a v^i_\delta \in L^\infty(\mathrm{R}^N)$ with $\|\partial^a v^i_\delta\|_\infty$ bounded uniformly in δ.

The end result of the regularization (2.5) is that $\nabla\gamma$ gains an additional derivative while retaining its integrability properties even under multiplication by v_δ. This is precisely what is required to make the formal manipulations rigorous.

Since $\beta_{\varepsilon,\delta} \in H^1(\Omega_{N,l})$ for $\varepsilon, \delta > 0$, we have (cf. equation (2.2))

$$\int_{\mathrm{T}^l}\int_{\mathrm{R}^N} \Big\{ -\mathcal{D}_\omega\gamma\mathcal{D}_\omega\beta_{\varepsilon,\delta} \; + \; \nabla\gamma\cdot\nabla\beta_{\varepsilon,\delta} \; + \; f(\gamma)\beta_{\varepsilon,\delta}\Big\} \quad = \quad 0, \qquad \forall\varepsilon,\delta > 0. \tag{2.8}$$

Formally, we derive equation (2.4) from equation (2.2) (with $\beta = v\cdot\nabla\gamma$) by writing the integrand in equation (2.2) as $g + div\,(h)$ for some R^{N+l} valued function h on $\Omega_{N,l}$. Here $div = (\nabla_x, \nabla_\theta)$ is the divergence operator on $\Omega_{N,l}$. The divergence term vanishes after performing the integration and we are left with $\int_{\Omega_{N,l}} g = 0$; this is equation (2.4). But since we are dealing with weak solutions we begin with the regularized equation (2.8) instead of (2.2). To derive equation (2.4) from this, then, we will write the integrand in (2.8) as $\tilde{g} = g + g_{\varepsilon,\delta} + div\,(h) + div\,(h_{\varepsilon,\delta})$ where g and h are as before. We will show that $\lim_{\varepsilon,\delta\to 0}\int_{\Omega_{N,l}} g_{\varepsilon,\delta} = 0$ and that $\int_{\Omega_{N,l}} div\,(h_{\varepsilon,\delta}) = 0$ for all $\varepsilon,\delta > 0$. Therefore, $\lim_{\varepsilon,\delta\to 0}\int_{\Omega_{N,l}}\tilde{g} = \int_{\Omega_{N,l}} g$. To prove that $\lim_{\varepsilon,\delta\to 0}\int_{\Omega_{N,l}} g_{\varepsilon,\delta} = 0$ we first show that when this term is integrated over R^N the resulting function of $\theta \in \mathrm{T}^l$ converges pointwise to zero and can be dominated by an $L^1(\mathrm{T}^l)$ function. Then we apply the dominated convergence theorem to estimate the integral over T^l.

Consider the term (we suppress the dot product initially)

$$\nabla\gamma\,\nabla\beta_{\varepsilon,\delta} \quad = \quad \nabla\gamma\,\nabla\, v_\delta\cdot\nabla R_\varepsilon^2\gamma$$

$$= \quad \nabla\gamma\, v_\delta \cdot \nabla\mathbf{1}\, \nabla R_\varepsilon^2 \gamma \quad + \quad \nabla\gamma\, dv_\delta\, \nabla R_\varepsilon^2 \gamma. \tag{2.9}$$

In this formula, and for similar formulae in the sequel, the expression $v_\delta \cdot \nabla\mathbf{1}$, to the immediate left of the vector $\nabla R_\varepsilon^2 \gamma$, denotes the $n \times n$ diagonal matrix with diagonal entries $v_\delta \cdot \nabla$, while for adjacent vectors the dot product is implied.

We will show that

$$\lim_{\varepsilon,\delta\to 0} \int_{\mathrm{T}} \int_{\mathrm{R}^N} \nabla\gamma \cdot \nabla\beta_{\varepsilon,\delta} \quad = \quad \int_{\mathrm{T}} \int_{\mathrm{R}^N} \nabla\gamma \cdot [dv - \frac{1}{2} tr\, dv\mathbf{1}]\nabla\gamma. \tag{2.10}$$

Commuting R_ε through the operator ∇ and commuting v_δ with R_ε, we write the first term on the right side of equation (2.9) as

$$\nabla\gamma\, v_\delta \cdot R_\varepsilon \nabla\mathbf{1}\, \nabla R_\varepsilon\gamma \quad = \quad \nabla\gamma\, R_\varepsilon v_\delta \cdot \nabla\mathbf{1}\, \nabla R_\varepsilon \gamma \quad - \quad \nabla\gamma\, [R_\varepsilon, v_\delta] \cdot \nabla\mathbf{1}\, R_\varepsilon \nabla\gamma. \tag{2.11}$$

Using the self-adjointness of R_ε,

$$\int_{\mathrm{R}^N} \nabla\gamma\, R_\varepsilon v_\delta \cdot \nabla\mathbf{1}\, \nabla R_\varepsilon\gamma \quad = \quad \int_{\mathrm{R}^N} (\nabla R_\varepsilon \gamma)\, v_\delta \cdot \nabla\mathbf{1}\, \nabla R_\varepsilon\gamma \tag{2.12}$$

(where we recognize the integral on the left as the inner product $\langle \nabla\gamma,\ R_\varepsilon v_\delta \cdot \nabla\mathbf{1}\, \nabla R_\varepsilon\gamma\rangle$ in $(L^2(\mathrm{R}^N))^N$). By the chain rule,

$$\begin{aligned} \int_{\mathrm{R}^N} (\nabla R_\varepsilon\gamma)\, v_\delta \cdot \nabla\mathbf{1}\, \nabla R_\varepsilon\gamma \quad &= \quad \int_{\mathrm{R}^N} \left\{ \frac{1}{2} \nabla \cdot (v_\delta \mid \nabla R_\varepsilon \gamma \mid^2) - \frac{1}{2} \nabla \cdot v_\delta \mid \nabla R_\varepsilon\gamma \mid^2 \right\} \\ &= \quad -\frac{1}{2} \int_{\mathrm{R}^N} \nabla \cdot v_\delta \mid \nabla R_\varepsilon \gamma \mid^2 . \end{aligned} \tag{2.13}$$

Here we've used Lemma 2.8 to conclude that $\int_{\mathrm{R}^N} \nabla \cdot (v_\delta \mid \nabla R_\varepsilon\gamma \mid^2) = 0$.

Furthermore,

$$\begin{aligned} \nabla \cdot v_\delta \mid \nabla R_\varepsilon \gamma \mid^2 \quad &= \quad \nabla \cdot v \mid \nabla\gamma \mid^2 \quad + \quad [\nabla \cdot v_\delta - \nabla \cdot v] \mid \nabla\gamma \mid^2 \\ &\quad + 2\, \nabla \cdot v_\delta [(R_\varepsilon - 1)\nabla\gamma]\nabla\gamma \quad + \quad \nabla \cdot v_\delta [(R_\varepsilon - 1)\nabla\gamma]^2. \end{aligned} \tag{2.14}$$

Note that for almost all $\theta \in \mathrm{T}^l$, $\nabla\gamma = \nabla\gamma(x,\theta) \in (L^2(\mathrm{R}^N))^n$. Since $\nabla \cdot v_\delta \to \nabla \cdot v$ pointwise as $\delta \to 0$ and $\|\nabla \cdot v_\delta - \nabla \cdot v\|_\infty$ can be bounded uniformly in δ, $J_\delta(x,\theta) \equiv [\nabla \cdot v_\delta - \nabla \cdot v] \mid \nabla\gamma \mid^2 \to 0$ in $L^1(\mathrm{R}^N)$ as $\delta \to 0$ for those θ in which $\nabla\gamma \in (L^2(\mathrm{R}^N))^n$. Thus the function $\bar{J}_\delta(\theta) \equiv \int_{\mathrm{R}^N} J_\delta(x,\theta)$ converges pointwise to zero almost everywhere in T^l as $\delta \to 0$ and is dominated pointwise by $\bar{J}(\theta) = c\|\nabla\gamma\|_2^2 \in L^1(\mathrm{T}^l)$. Here c is the uniform bound on $\|\nabla \cdot v_\delta - \nabla \cdot v\|_\infty$. Lebesgue's dominated convergence theorem then implies that $\int_{\mathrm{T}} \bar{J}_\delta \to 0$ as $\delta \to 0$. That is, $J_\delta \to 0$ in $L^1(\Omega_{N,l})$.

To treat the third term on the right in equation (2.14) we note that $\|(R_\varepsilon - 1)\nabla\gamma\|_2 \to 0$ for almost all θ while $\|\nabla \cdot v_\delta\|_\infty$ is uniformly bounded in δ. Using the Schwarz inequality on $(L^2(\mathrm{R}^N))^N$ we see that this term, when integrated over R^N, converges to zero in $L^1(\mathrm{R}^N)$ pointwise θ-almost everywhere as $\varepsilon \to 0$ uniformly in δ. Since $\|\nabla \cdot v_\delta\|_\infty \leq \|\nabla \cdot v\|_\infty + c$, we can dominate this term by $4(\|\nabla \cdot v\|_\infty + c)\|\nabla\gamma\|_2^2 \in L^1(\mathrm{T}^l)$ and hence conclude that it converges to zero in $L^1(\Omega_{N,l})$ as $\varepsilon \to 0$, uniformly in δ. A similar argument with the same conclusion applies to the last term on the right hand side of equation (2.14).

We expand the commutator on the right hand side of equation (2.11);

$$\begin{aligned}\int_{\mathrm{R}^n} \nabla\gamma[R_\varepsilon, v_\delta] \cdot \nabla\mathbf{1}\, R_\varepsilon\nabla\gamma &= \int_{\mathrm{R}^N} \nabla\gamma R_\varepsilon \varepsilon^2[\Delta, v_\delta]R_\varepsilon \cdot \nabla\mathbf{1}\, R_\varepsilon\nabla\gamma \\ &= \int_{\mathrm{R}^N} \nabla\gamma R_\varepsilon \varepsilon^2(\Delta v_\delta \cdot R_\varepsilon\nabla\mathbf{1} + 2\nabla v_\delta\nabla \cdot R_\varepsilon\nabla\mathbf{1})R_\varepsilon\nabla\gamma. \quad (2.15)\end{aligned}$$

Here we have written $[R_\varepsilon, v_\delta]$ as $-R_\varepsilon[(1-\varepsilon^2\Delta), v_\delta]R_\varepsilon$. The expressions Δv_δ and $\nabla v_\delta \nabla$ denote the vectors $(\Delta v_\delta^1, ..., \Delta v_\delta^N)$ and $(\nabla v_\delta^1 \cdot \nabla, ..., \nabla v_\delta^N \cdot \nabla)$ respectively.

Before estimating the two terms in this equation we first note that for $\mid a \mid = 1,2$ the function $\mid p^a \mid^2 \; (1+\varepsilon^2 p^2)^{-2}$ is bounded by $c\varepsilon^{-2|a|}$, $c = c(a)$, so that, using the Fourier transform, for $\psi \in L^2(\mathrm{R}^N)$

$$\|\partial^a R_\varepsilon \psi\|_2 \leq c\varepsilon^{-|a|}\|\psi\|_2, \quad (2.16)$$

and

$$\lim_{\varepsilon\to 0} \|\varepsilon^{|a|}\partial^a R_\varepsilon\psi\|_2^2 \;=\; \lim_{\varepsilon\to 0}\int_{\mathrm{R}^N} \frac{\varepsilon^{2|a|} \mid p^a \mid^2}{(1+\varepsilon^2p^2)^2} \mid \hat{\psi}(p) \mid^2 \;=\; 0 \quad (2.17)$$

by dominated convergence. Using equation (2.16) we estimate

$$\begin{aligned}
&\Big|\int_{\mathrm{R}^N}\varepsilon^2\nabla\gamma R_\varepsilon\Delta v_\delta\cdot R_\varepsilon\nabla\mathbf{1}\,R_\varepsilon\nabla\gamma\Big| \\
&\quad\le \varepsilon^2\|\nabla\gamma\|_2\;\|\Delta v_\delta\cdot R_\varepsilon\nabla\mathbf{1}\,R_\varepsilon\nabla\gamma\|_2 \\
&\quad\le c\varepsilon^2\|\nabla\gamma\|_2\;\|\nabla\mathbf{1}\,R_\varepsilon\nabla\gamma\|_2 \\
&\quad\le c\varepsilon\|\nabla\gamma\|_2^2. \qquad (2.18)
\end{aligned}$$

Since $\|\nabla\gamma\|_2^2\in L^1(\mathrm{T}^l)$ we see immediately that the first term on the right in the last line of equation (2.15) converges to zero in $L^1(\mathrm{T}^l)$ as $\varepsilon\to 0$. The constant c here is derived from $\|\partial^a v_\delta\|_\infty$ and is independent of δ because these norms are uniform in δ.

For the second term on the right of equation (2.15) we have,

$$\begin{aligned}
&\Big|\int_{\mathrm{R}^N}\varepsilon^2\nabla\gamma R_\varepsilon\nabla v_\delta\nabla\cdot R_\varepsilon\nabla\mathbf{1}\,R_\varepsilon\nabla\gamma\Big| \\
&\quad\le \|\nabla\gamma\|_2\;\|\varepsilon^2\nabla v_\delta\nabla\cdot R_\varepsilon\nabla\mathbf{1}\,R_\varepsilon\nabla\gamma\|_2 \\
&\quad\le c\|\nabla\gamma\|_2\;\|\varepsilon^2\nabla^2\mathbf{1}\,R_\varepsilon\nabla\gamma\|_2 \quad\to 0 \qquad (2.19)
\end{aligned}$$

for almost all θ as $\varepsilon\to 0$ by (2.17), where c is independent of δ (for the same reason as before) and $\nabla^2\mathbf{1}$ denotes the matrix $diag(\sum_{i,j=1}^N\partial_{ij}^2)$. Since $\|\varepsilon^2\nabla^2 R_\varepsilon\nabla\gamma\|_2\le c\|\nabla\gamma\|_2$ for all $\varepsilon>0$ (cf. equation (2.16)) we can dominate the left hand side of (2.19) by $c\|\nabla\gamma\|_2^2\in L^1(\mathrm{T}^l)$ to conclude that it converges to zero in $L^1(\mathrm{T}^l)$ as $\varepsilon\to 0$ uniformly in δ.

Combining this with the preceding result (cf. equation (2.14)), we have shown that

$$\lim_{\varepsilon,\delta\to 0}\int_{\mathrm{T}^l}\int_{\mathrm{R}^N}\nabla\gamma\,v_\delta\cdot\nabla\mathbf{1}\,\nabla R_\varepsilon^2\gamma \;=\; \int_{\mathrm{T}^l}\int_{\mathrm{R}^N}-\frac{1}{2}tr\,dv\;|\,\nabla\gamma\,|^2\,. \qquad (2.20)$$

The term $\nabla\gamma\,dv_\delta\,\nabla R_\varepsilon^2\gamma$ in equation (2.9) we write as

$$\begin{aligned}
\nabla\gamma\,dv_\delta\,\nabla R_\varepsilon^2\gamma \;&=\; \nabla\gamma\,dv\,\nabla\gamma \;+\; \nabla\gamma(dv_\delta-dv)\nabla\gamma \\
&\quad+\; \nabla(R_\varepsilon^2-1)\gamma\,dv_\delta\,\nabla\gamma. \qquad (2.21)
\end{aligned}$$

We have : $dv_\delta \to dv$ pointwise with $\|dv_\delta - dv\|_\infty$ and $\|dv_\delta\|_\infty$ (matrix norm) bounded uniformly in δ, while $\nabla(R_\varepsilon^2 - 1)\gamma \to 0$ in $(L^2(\mathrm{R}^N))^N$ for almost all θ. Therefore, by arguments similar to those made above, the last two terms on the right hand side of equation (2.21) converge to zero in $L^1(\Omega_{N,l})$ as $\varepsilon \to 0$ uniformly in δ. We conclude then that

$$\lim_{\varepsilon,\delta\to 0} \int_{\mathrm{T}^l}\int_{\mathrm{R}^N} \nabla\gamma \, dv_\delta \, \nabla R_\varepsilon^2 \gamma \;=\; \int_{\mathrm{T}^l}\int_{\mathrm{R}^N} \nabla\gamma \, dv \, \nabla\gamma \tag{2.22}$$

and equation (2.10) is shown.

Considering now the first term in equation (2.8), commuting $\mathcal{D}_\omega$ with R_ε and ∇ we obtain

$$\begin{aligned} \mathcal{D}_\omega\gamma\mathcal{D}_\omega\beta_{\varepsilon,\delta} &= \mathcal{D}_\omega\gamma\mathcal{D}_\omega\, v_\delta\cdot\nabla R_\varepsilon^2\gamma \;=\; \mathcal{D}_\omega\gamma\, v_\delta\cdot\nabla R_\varepsilon^2\mathcal{D}_\omega\gamma \\ &= \mathcal{D}_\omega\gamma R_\varepsilon v_\delta\cdot\nabla R_\varepsilon\mathcal{D}_\omega\gamma \;-\; \mathcal{D}_\omega\gamma[R_\varepsilon, v_\delta]\cdot\nabla R_\varepsilon\mathcal{D}_\omega\gamma. \end{aligned} \tag{2.23}$$

Integrating the first term on the right hand side over R^N yields

$$\begin{aligned} \int_{\mathrm{R}^N} \mathcal{D}_\omega\gamma R_\varepsilon v_\delta\cdot\nabla R_\varepsilon\mathcal{D}_\omega\gamma &= \int_{\mathrm{R}^N} R_\varepsilon\mathcal{D}_\omega\gamma\, v_\delta\cdot\nabla R_\varepsilon\mathcal{D}_\omega\gamma \\ &= \int_{\mathrm{R}^N} \Big\{\frac{1}{2}\nabla\cdot\Big(v_\delta \mid R_\varepsilon\mathcal{D}_\omega\gamma\mid^2\Big) - \frac{1}{2}\nabla\cdot v_\delta \mid R_\varepsilon\mathcal{D}_\omega\gamma\mid^2\Big\} \\ &= -\frac{1}{2}\int_{\mathrm{R}^N} \nabla\cdot v_\delta|R_\varepsilon\mathcal{D}_\omega\gamma|^2. \end{aligned} \tag{2.24}$$

Treating this and the second term on the right hand side of equation (2.23) as above (cf. equations (2.13) and (2.15) respectively) we conclude that

$$\lim_{\varepsilon,\delta\to 0} \int_{\mathrm{T}^l}\int_{\mathrm{R}^N} -\mathcal{D}_\omega\gamma\mathcal{D}_\omega\beta_{\varepsilon,\delta} \;=\; \frac{1}{2}\int_{\mathrm{T}^l}\int_{\mathrm{R}^N} tr\, dv\,(\mathcal{D}_\omega\gamma)^2. \tag{2.25}$$

Finally, we consider the nonlinear term $f(\gamma)\beta_{\varepsilon,\delta}$ in equation (2.8) which we write as

$$\begin{aligned} f(\gamma)\beta_{\varepsilon,\delta} &= f(\gamma)v_\delta\cdot R_\varepsilon^2\nabla\gamma \\ &= f(\gamma)v_\delta\cdot\nabla\gamma + \Big[f(\gamma)v_\delta\cdot R_\varepsilon^2\nabla\gamma - f(\gamma)v_\delta\cdot\nabla\gamma\Big]. \end{aligned} \tag{2.26}$$

Since $\nabla\gamma$ exists (in the classical sense) almost everywhere with $\nabla\gamma f(\gamma) \in L^1(\mathrm{R}^N)$ and $F \in C^1(\mathrm{R}, \mathrm{R})$, we can write $\nabla F(\gamma) = \nabla\gamma f(\gamma)$. That is, $F(\gamma) \in H^{1,1}(\mathrm{R}^N)$. Thus,

$$\begin{aligned}
\int_{\mathrm{R}^N} f(\gamma) v_\delta \cdot \nabla\gamma &= \int_{\mathrm{R}^N} v_\delta \cdot \nabla F(\gamma) \\
&= \int_{\mathrm{R}^N} \Big\{ \nabla \cdot (v_\delta F(\gamma)) - \nabla \cdot v_\delta F(\gamma) \Big\} \\
&= \int_{\mathrm{R}^N} -\nabla \cdot v_\delta F(\gamma) \\
&= \int_{\mathrm{R}^N} -\nabla \cdot v F(\gamma) \;+\; \int_{\mathrm{R}^N} (\nabla \cdot v_\delta - \nabla \cdot v) F(\gamma). \qquad (2.27)
\end{aligned}$$

In going from the second to the third equality we have used the fact that $v_\delta F(\gamma) \in (H^{1,1}(\mathrm{R}^N))^N$ (Lemma 2.8). The last integrand on the right tends to zero in $L^1(\mathrm{R}^N)$ as $\delta \to 0$ and by dominated convergence tends to zero in $L^1(\Omega_{N,l})$.

For the remaining terms on the right hand side of equation (2.26) we have

$$\int_{\mathrm{R}^N} \Big[f(\gamma) v_\delta \cdot R_\varepsilon^2 \nabla\gamma - f(\gamma) v_\delta \cdot \nabla\gamma \Big] = \int_{\mathrm{R}^N} f(\gamma) v_\delta \cdot (R_\varepsilon^2 \nabla\gamma - \nabla\gamma). \qquad (2.28)$$

Because $R_\varepsilon^2 \nabla\gamma \to \nabla\gamma$ in $L^2(\mathrm{R}^N)$ with $\|R_\varepsilon^2 \nabla\gamma - \nabla\gamma\|_2 \leq 2\|\nabla\gamma\|_2$ and $f(\gamma) \in L^2(\mathrm{R}^N)$, the Schwarz inequality and the boundedness of v_δ for $\delta > 0$ imply that this last integral converges to zero as $\varepsilon \to 0$ for almost all θ and for each $\delta > 0$, and hence by dominated convergence converges to zero in $L^1(\mathrm{T}^l)$ as $\varepsilon \to 0$ for each $\delta > 0$.

We conclude that

$$\int_{\mathrm{T}^l} \int_{\mathrm{R}^N} f(\gamma) \beta_{\varepsilon,\delta} = -\int_{\mathrm{T}^l} \int_{\mathrm{R}^N} tr\, dv\, F(\gamma) \quad + \quad E_1(\varepsilon, \delta) \qquad (2.29)$$

where $E_1(\varepsilon, \delta)$ can be made arbitrarily small first by choosing δ sufficiently small and then ε sufficiently small.

Combining all of the above results, we have shown that

$$\begin{aligned}
&\int_{\mathrm{T}^l} \int_{\mathrm{R}^N} \Big\{ -\mathcal{D}_\omega \gamma \mathcal{D}_\omega \beta_{\varepsilon,\delta} + \nabla\gamma \cdot \nabla\beta_{\varepsilon,\delta} + f(\gamma)\beta_{\varepsilon,\delta} \Big\} \\
&\quad = \int_{\mathrm{T}^l} \int_{\mathrm{R}^N} \Big\{ tr\, dv\, \Big(\frac{1}{2} (\mathcal{D}_\omega \gamma)^2 - F(\gamma) \Big) + \nabla\gamma \cdot [dv - \frac{1}{2} tr\, dv \mathbf{1}] \nabla\gamma \rangle \Big\}
\end{aligned}$$

$$+ \ E(\varepsilon, \delta), \tag{2.30}$$

where $E(\varepsilon, \delta)$ can be made arbitrarily small by choosing δ and ε sufficiently small. This, together with equation (2.8), completes the proof of Theorem 2.7 □

3 Nonexistence of quasiperiodic solutions

The dilation group on R^N is the one-parameter group of diffeomorphisms $\Phi_\lambda(x) = \lambda x, \ x \in \mathrm{R}^N, \ \lambda \geq 1$. Its infinitesimal generator, the vector field $v(x) = x$, satisfies the hypothesis of Theorem 2.7. Here $dv = \mathbf{1}$ and $tr\, dv = N$. The following corollary then follows from equation (2.4).

Corollary 3.1 *Let $v(x) = x$ be the generator of dilations on R^N. If γ and f satisfy the hypothesis of Theorem 2.7, then*

$$\int_{\mathrm{T}^l} \int_{\mathrm{R}^N} \left\{ \frac{1}{2}(\mathcal{D}_\omega \gamma)^2 + (\frac{2-N}{2N}) \mid \nabla\gamma \mid^2 - F(\gamma) \right\} \ = \ 0. \tag{3.1}$$

<u>Remark:</u> In the case when γ is independent of θ, so that γ solves the nonlinear elliptic equation $\Delta\gamma = f(\gamma)$, equation (3.1) is the well known Pohozaev identity [Po].

The next proposition describes a class of virial relations that are not derived from a vector field on R^N.

Proposition 3.2 *(Gauge transformations) Let $\gamma \in H^1(\Omega_{N,l})$ satisfy (2.2). If h is a smooth function on R^N that is bounded along with its derivatives, then*

$$\int_{\mathrm{T}^l} \int_{\mathrm{R}^N} \left\{ h \left(-(\mathcal{D}_\omega\gamma)^2 + \mid \nabla\gamma \mid^2 + f(\gamma)\gamma \right) \ - \ \frac{1}{2}(\Delta h)\gamma^2 \right\} \ = \ 0. \tag{3.2}$$

<u>Proof</u>:

Since $h\gamma \in H^1(\Omega_{N,l})$, from equation (2.2)

$$\int_{\mathrm{T}^l} \int_{\mathrm{R}^N} \left\{ -\mathcal{D}_\omega\gamma\mathcal{D}_\omega(h\gamma) \ + \ \nabla\gamma \cdot \nabla(h\gamma) + f(\gamma)h\gamma \right\} \ = 0. \tag{3.3}$$

Expanding the second term in the integrand,

$$\begin{aligned}\nabla\gamma\cdot\nabla(h\gamma) &= (\nabla\gamma\cdot\nabla h)\gamma + h\mid\nabla\gamma\mid^2\\ &= \frac{1}{2}\nabla\cdot(\gamma^2\nabla h) + h\mid\nabla\gamma\mid^2 - \frac{1}{2}(\Delta h)\gamma^2.\end{aligned}$$

The first term on the right vanishes when integrated over R^N and equation (3.2) follows □

By combining the previous two virial relations, take $h\equiv c$ in equation (3.2) and add this to equation (3.1), we obtain the following identity that allows us to prove several nonexistence theorems for quasiperiodic solutions. A special case of this identity was obtained in [V1] for periodic solutions on R^1.

Proposition 3.3 *Let γ and f satisfy the hypothesis of Theorem 2.7. Then for any $c\in\mathrm{R}$,*

$$\int_{\mathrm{T}^l}\int_{\mathrm{R}^N}\left\{(\frac{1}{2}-c)(\mathcal{D}_\omega\gamma)^2 + (c+\frac{2-N}{2N})\mid\nabla\gamma\mid^2\right\} = \int_{\mathrm{T}^l}\int_{\mathrm{R}^N}\{F(\gamma)-c\gamma f(\gamma)\}. \qquad (3.4)$$

Proposition 3.4 *Let γ be a weak solution of NLW on $\Omega_{N,l}$ (cf. Definition 2.5) with γ and f satisfying the hypothesis of Theorem 2.7. If for some $c\in[\frac{N-2}{2N},\frac{1}{2}]$, $F(z)-czf(z)\le 0$ for all z such that $\mid z\mid\le\|\gamma\|_{L^\infty(\Omega_{N,l})}$, then γ is independent of θ.*

Proof:

By hypothesis (on c) the left hand side of equation (3.4) is nonnegative while the right hand side is nonpositive. Hence, both sides must be zero. For $c\in(\frac{N-2}{2N},\frac{1}{2}]$ this implies that $\int_{\mathrm{T}^l}\|\nabla\gamma\|_2^2=0$ so that for almost all θ, $\nabla\gamma=0$ almost everywhere on R^N. That is, $\gamma(x,\theta)$ is constant almost everywhere on R^N for almost all θ. Because γ is continuous, γ is therefore constant on $\Omega_{N,l}$. Since $\gamma\in L^2(\Omega_{N,l})$, $\gamma(\theta)\in L^2(\mathrm{R}^N)$ for almost all θ so that this constant must be zero.

In the case $c=\frac{N-2}{2N}$ equation (3.4) implies that $\mathcal{D}_\omega\gamma=0$ almost everywhere on $\Omega_{N,l}$. This implies that for almost all $x\in\mathrm{R}^N$, γ is invariant under the flow $\theta\mapsto\theta+\omega t$. Since this flow is ergodic (ω is incommensurate), γ is constant on T^l for these x ([Pe], Prop 2.4.1). Therefore, γ is independent of θ almost everywhere on R^N □

Remark: Let $\mathcal{Z} = \{z \in \mathrm{R}\ ;\quad F(z) - czf(z) = 0\}$. Then, under the hypothesis that $F(z) - czf(z) \leq 0$ for all z such that $|z| \leq \|\gamma\|_{L^\infty(\Omega_{N,l})}$, $\int_{\Omega_{N,l}}\{F(\gamma) - c\gamma f(\gamma)\} = 0$ implies that $\gamma(x,\theta) \in \mathcal{Z}$ for almost all $(x,\theta) \in \Omega_{N,l}$. If $\mathcal{Z}$ is composed of isolated points then by continuity γ must be constant on $\Omega_{N,l}$. If $\mathcal{Z}$ contains an interval then this argument does not imply that γ is constant.

If φ is a weak quasiperiodic solution of NLW (Definition 2.6), then applying Proposition 3.4 to its generating function we obtain the following theorem concerning the nonexistence of quasiperiodic solutions.

Theorem 3.5 *(Nonexistence of quasiperiodic solutions of NLW) Suppose φ is a weak l-quasiperiodic solution of NLW on R^{N+1} with frequency ω (cf. Definition 2.6). Let γ be the generating function of φ and assume that γ and f satisfy the hypothesis of Theorem 2.7. If for some $c \in [\frac{N-2}{2N}, \frac{1}{2}]$,*

$$F(z) - czf(z) \leq 0 \text{ for all } z \text{ such that } |z| \leq \|\varphi\|_{L^\infty(\mathrm{R}^{n+1})} \tag{3.5}$$

where $F(z) \equiv \int_0^z f(w)\,dw$, then φ is independent of time.

Remarks:

We point out that condition (3.5) is compatible with our hypothesis (H) on the nonlinearity. For instance, if we are considering bounded solutions then any polynomial is an admissible nonlinearity, i.e., bounded solutions and polynomial nonlinearities satisfy the hypothesis of Theorem 2.7. In Example 1 below we exhibit a polynomial satisfying (3.5) for all z.

A sufficient (but not necessary) condition such that the nonlinearity satisfies the inequality $F(z) - czf(z) \leq 0$ is as follows. Suppose $F(z) = H(z^a)$ for some convex function H. Then

$$F'(z) = f(z) = az^{a-1}H'(z^a).$$

Since H is convex, $H'(w) \leq H'(y)$ when $w \leq y$. Integrating this inequality with respect to w from 0 to y gives $H(y) \leq yH'(y)$. Setting $y = z^a$; $F(z) - z^aH'(z^a) \leq 0$, and hence $F(z) - a^{-1}zf(z) \le 0$. In particular, for a pure power law nonlinearity $f(z) = az^{\alpha}$, $F(z) - \frac{1}{\alpha+1}zf(z) = 0$. The condition $\frac{1}{\alpha+1} \in [\frac{N-2}{2N}, \frac{1}{2}]$ implies then that NLW with this nonlinearity has no quasiperiodic solutions for any $\alpha \ge 1$ in the case $N = 1, 2$ or for $1 \le \alpha \le \frac{N+2}{N-2}$ in the case $N > 2$.

The proof of Theorem 3.5 makes no use of the frequency ω of the solution: If the nonlinearity satisfies condition (3.5) then NLW will not support any kind of quasiperiodic solution, where by "any kind" we mean quasiperiodic functions with any frequency. From the point of view of virial relations the nonlinearity does not distinguish between different frequencies or dimensions in the time variable of quasiperiodic solutions.

Examples and Applications

1) Let $f(z) = a_1 z + a_3 z^3 + \cdots + a_{2m+1} z^{2m+1}$ be a polynomial containing only odd powers of z. Then

$$F(z) - czf(z) = a_1(\frac{1}{2} - c)z^2 + a_3(\frac{1}{4} - c)z^4 + \cdots + a_{2m+1}(\frac{1}{2m+2} - c)z^{2m+2}.$$

Here $\mid f(z) \mid \le c'(\mid z \mid + \mid z \mid^{2m+2})$ so that, referring to the hypothesis of Theorem 2.7, the generating functions that are covered in our analysis for this particular nonlinearity are those that are of class $L^{4(m+1)}(\Omega_{N,l})$ if $N + l > 2$. If $N = l = 1$ then any generating function is admissible. If $a_{2k+1} \ge 0$, $k = 0, \ldots, m$, then by choosing $c = 1/2$ we conclude, using Theorem 3.5, that NLW with this nonlinearity has no weak l-quasiperiodic solution, for any l, whose generating function is of class $H^1(\Omega_{N,l}) \cap L^{4(m+1)}(\Omega_{N,l})$ in spatial dimension $N > 1$, or whose generating function is of class $H^1(\Omega_{1,1})$ if $N = l = 1$. In particular, generating functions that are bounded satisfy these criteria.

Remarks:

Other possibilities for the choice of c to derive necessary conditions for existence may arise if the coefficients are not all positive. For example, if $a_1 \le 0$ and $a_{2k+1} \ge 0$, $k = 1, \ldots, m$, then we reach the same conclusion with any $c \in [1/4, 1/2]$, which is a valid interval for c if $N \le 4$ (that is, we can apply Theorem 3.5).

Global (in time) existence for some equations of this form is proven in [GV] (see also [Str2]). The class of solutions considered there is somewhat different than that considered here but our results suggest that these equations possess no quasiperiodic solutions even though the Cauchy problem leads to global solutions. In Section 5 we will extend Theorem 3.5 to almost periodic solutions (Theorem 5.1) so that in fact these equations may not have almost periodic solutions either. Consequently, these nonlinear wave equations may have no bound states (cf. the discussion in the introduction).

2) Small Amplitude Solutions

If a solution has small amplitude, i.e., if it has small $L^\infty(\mathrm{R}^{N+1})$ norm, then only the properties of f in a neighborhood of the origin contribute to the dynamics. In particular, referring to equation (3.5), if $F(z) - czf(z) \leq 0$ in a neighborhood of zero for some $c \in [\frac{N-2}{2N}, \frac{1}{2}]$, then NLW has no small amplitude quasiperiodic solutions. A particular case where small amplitude solutions arise is when NLW has a family of localized, periodic solutions that originate from the zero solution (see, for example, [BMW],[SK]). An example of such a family is provided by the sine-Gordon breather (equation (1.3)).

We now present two nonexistence results for small amplitude quasiperiodic solutions based on Theorem 3.5 by using a more detailed description of the nonlinearity. Here we assume that f can be expanded in a Taylor series about the origin as

$$f(z) \;=\; f'(0)z + \frac{f^{(2k+1)}(0)}{(2k+1)!}z^{2k+1} + R(z) \tag{3.6}$$

where $f^{(2k+1)}(0) \neq 0$, $k \geq 1$ and $R(z) = O(|z|^{2k+2})$. To state the next two corollaries we will require the following definition.

Condition A With reference to (3.6), the three numbers $(f^{(2k+1)}(0), k, N)$ satisfy the following conditions (N is the spatial dimension). If $f^{(2k+1)}(0) < 0$, then $N \leq 3$; in the case $N = 1$ or 2, k can be any positive integer, in the case $N = 3$, $k = 1$. If $f^{(2k+1)}(0) > 0$ then k and N can be any positive integers.

Corollary 3.6 *Let f have a Taylor series at the origin of the form (3.6). If either (i) $f'(0) < 0$, or (ii) $f'(0) > 0$ and $f^{(2k+1)}(0) > 0$, or (iii) $f'(0) = 0$ with $(f^{(2k+1)}(0), k, N)$*

satisfying Condition A, then NLW has no weak l-quasiperiodic solutions φ of sufficiently small amplitude for any $l \in \mathrm{N}$ and for any frequency $\omega \in \mathrm{R}^l$. This result holds in any spatial dimension in the cases (i) and (ii) and in those spatial dimensions determined by Condition A in the case (iii).

<u>Proof</u>:

We work with the generating function γ of φ. Since φ, and hence γ, is of small amplitude it is, in particular, bounded. Therefore, as we will apply Proposition 3.3 it is enough that f satisfy a Lipschitz condition at the origin (cf. statement (ii) in Theorem 2.7 with $q = \infty$), which it clearly does. We have that

$$F(z) - czf(z) = f'(0)(\frac{1}{2} - c)z^2 + \frac{f^{(2k+1)}(0)}{(2k+1)!}\left(\frac{1}{2k+2} - c\right) z^{2k+2} + \tilde{R}(z) \qquad (3.7)$$

where $\tilde{R}(z) = \int_0^z R(w)\,dw - czR(z)$. We have expanded the integrand on the right hand side of (3.4). Basing our analysis on (3.7), our goal is to adjust the parameter c, within the interval $[\frac{N-2}{2N}, \frac{1}{2}]$, according to the properties of the Talor series of f so that the right hand side of (3.4) is strictly negative for sufficiently small solutions γ, $\gamma \neq 0$. Then, because the left hand side of (3.4) will be nonnegative, this contradiction will imply that γ must in fact be zero.

We first consider the case $f'(0) < 0$. Thus, the lowest order term on the right hand side of (3.7) is negative. In this case we set $c = \frac{1}{2k+2}$ (any $c \in [\frac{N-2}{2N}, \frac{1}{2})$ will do, though).

If $f'(0) > 0$ then we set $c = \frac{1}{2}$. Then since $\frac{1}{2k+2} - \frac{1}{2} < 0$, if $f^{(2k+1)}(0) > 0$, regardless of k, the lowest order term on the right hand side of (3.7) will be negative.

If $f'(0) = 0$, then if $f^{(2k+1)}(0) < 0$ we set $c = \frac{N-2}{2N}$ which is less than $\frac{1}{2k+2}$ for those k and N specified in Condition A. If $f^{(2k+1)}(0) > 0$ we set $c = \frac{1}{2}$. In either case the second term on the right hand side of (3.7) is negative for $z \neq 0$.

With c specified in this way the lowest order term on the right hand side of (3.7) is strictly negative for $z \neq 0$. Now we show that the remainder term $\tilde{R}(z)$ does not upset this for z sufficiently small. Applying the mean value theorem to the remainder term in the Taylor series of F, we see that

$$\tilde{R}(z) = \frac{f^{(2k+2)}(u)}{(2k+3)!} z^{2k+3}$$

for some $u = u(z)$, $|u| < |z|$. Therefore, in the case when $c = \frac{1}{2k+2}$ there is a $d_1 > 0$ such that for $|z| \leq d_1$, $z \neq 0$,

$$| \tilde{R}(z) | < \left| f'(0)(\frac{1}{2} - \frac{1}{2k+2}) \right| z^2.$$

In the case when $c = \frac{1}{2}$ or $c = \frac{N-2}{2N}$, there exists a $d_2 > 0$ (which depends on c) such that for $|z| \leq d_2$, $z \neq 0$,

$$| \tilde{R}(z) | < \left| \frac{f^{(2k+1)}(0)}{(2k+1)!} \left(\frac{1}{2k+2} - c \right) \right| z^{2k+2}.$$

Therefore, for $|z| \leq d = \min\{d_1, d_2\}$, $z \neq 0$, and with c specified as above, the right hand side of equation (3.7) is strictly negative. This implies that if γ is a nonzero solution of NLW on $\Omega_{N,l}$ such that $\|\gamma\|_{L^\infty(\Omega_{N,l})} \leq d$, then for this c the right hand side of (3.4) is strictly negative. However the left hand side of (3.4) is nonnegative. Therefore, there cannot be such a solution □

Remark: If one is considering nonlinear Klein-Gordon equations (i.e., $f'(0) > 0$) with odd nonlinearity, which is typical in physical applications, then by statement (ii) of the corollary a necessary condition for the existence of small amplitude quasiperiodic solutions in any spatial dimension is that the next highest term in the Taylor series after the linear term have a negative coefficient.

We now demonstrate how a priori information about the solution can be used in conjunction with virial relations to derive a nonexistence result for periodic solutions on R^N. We will obtain a result similar to Coron's [Co], that is, that ω^2 is bounded above by $f'(0)$, in multi-spatial dimensions, but only for small amplitude periodic solutions with zero mean.

We assume here that $f'(0) > 0$; the case $f'(0) \leq 0$ could be covered in a way analogous to how it was treated in the previous corollary. Referring to (3.7), the virial relation of Proposition 3.3 (equation (3.4)) can be written as

$$(\frac{1}{2} - c)\Big(\|\mathcal{D}_\omega \gamma\|^2_{L^2(\Omega_{N,1})} - f'(0)\|\gamma\|^2_{L^2(\Omega_{N,1})}\Big) + (c + \frac{2-N}{2N})\|\nabla\gamma\|^2_{L^2(\Omega_{N,1})}$$
$$= \int_{\mathrm{T}^1} \int_{\mathrm{R}^N} \left\{ \frac{f^{(2k+1)}(0)}{(2k+1)!} \left(\frac{1}{2k+2} - c \right) \gamma^{2k+2} + \tilde{R}(\gamma) \right\}. \tag{3.8}$$

From this we see that if we had some a priori estimate on how $\|\mathcal{D}_\omega\gamma\|^2_{L^2(\Omega_{N,1})}$ compares to $\|\gamma\|^2_{L^2(\Omega_{N,1})}$ so as to make definite the sign of $(\|\mathcal{D}_\omega\gamma\|^2_{L^2(\Omega_{N,1})} - f'(0)\|\gamma\|^2_{L^2(\Omega_{N,1})})$, then we would not be forced to set $c = 1/2$ to obtain a necessary condition for existence (as in the previous corollary). An instance of when this is possible is provided by Wirtinger's inequality for periodic functions along with an additional hypothesis on the frequency of γ. We first recall this inequality. γ has a Fourier expansion $\gamma = \sum_k \gamma_k(x)e^{ikt}$ (we are writing t instead of θ). Then, since $\gamma \in H^1(\Omega_{N,1})$, $\partial_t\gamma$ has the Fourier expansion $\partial_t\gamma = \sum_k ik\gamma_k(x)e^{ikt}$. Therefore, $\|\partial_t\gamma\|^2_{L^2(\Omega_{N,1})} = \sum_k k^2\|\gamma_k\|^2_{L^2(\mathrm{R}^N)}$. If in addition $\gamma_0 = 0$ (i.e., if φ has zero mean) we derive the inequality $\|\partial_t\gamma\|^2_{L^2(\Omega_{N,1})} \geq \|\gamma\|^2_{L^2(\Omega_{N,1})}$, which implies that $\|\mathcal{D}_\omega\gamma\|^2_{L^2(\Omega_{N,1})} = \omega^2\|\partial_t\gamma\|^2_{L^2(\Omega_{N,1})} \geq \omega^2\|\gamma\|^2_{L^2(\Omega_{N,1})}$ (recall that $\mathcal{D}_\omega = \omega\partial_t$ in this case).

Substituting this inequality into (3.8), if $\omega^2 > f'(0)$ then $(\|\mathcal{D}_\omega\gamma\|^2_{L^2(\Omega_{N,1})} - f'(0)\|\gamma\|^2_{L^2(\Omega_{N,1})}) \geq 0$ with equality holding if and only if $\gamma = 0$. Thus, if $\omega^2 > f'(0)$ the left hand side of (3.8) is nonnegative for $c \in [\frac{N-2}{2N}, \frac{1}{2}]$. Actually, the left hand side of (3.8) is strictly positive for $c \in [\frac{N-2}{2N}, \frac{1}{2}]$ and $\gamma \neq 0$; that this is true when $c \in [\frac{N-2}{2N}, \frac{1}{2})$ was just pointed out, while if $c = \frac{1}{2}$, then $\|\nabla\gamma\|^2_{L^2(\Omega_{N,l})} = 0$ implies that $\gamma = 0$, as explained in the proof of Proposition 3.4.

Now we determine conditions under which the right hand side of (3.8) is nonpositive. If these conditions are met, then the assumptions $\omega^2 > f'(0)$ and φ having zero mean, together which we have just seen leads to the conclusion that the left hand side of (3.8) is strictly positive if $\varphi \neq 0$, will imply that φ must be zero.

First note that because $\tilde{R}(\gamma) = O(|\gamma|^{2k+3})$, if γ is of sufficiently small amplitude the right hand side of (3.8) will be dominated by the first term in the integrand (as described in the previous corollary). We choose c so as to make $f^{(2k+1)}(0)\left(\frac{1}{2k+2} - c\right) < 0$. If $f^{(2k+1)}(0) > 0$ then set $c = \frac{1}{2}$. If $f^{(2k+1)}(0) < 0$ then set $c = \frac{N-2}{2N}$. In either case the left hand side of (3.8) is strictly positive if $\gamma \neq 0$, while if $(f^{(2k+1)}(0), k, N)$ satisfies Condition A, the right hand side of (3.8) is nonpositive for sufficiently small γ. Therefore, there cannot be such solutions γ. This completes the proof of the following corollary.

Corollary 3.7 *Let f have a Taylor series at zero of the form (3.6). Assume that $f'(0) > 0$ and that $(f^{(2k+1)}(0), k, N)$ satisfies Condition A. Let φ be a small amplitude weak $2\pi/\omega$ - periodic solution of NLW on R^{N+1}. In addition suppose that φ has zero mean, i.e.,*

$\int_0^{2\pi/\omega} \varphi(x,t)\, dt \;=\; 0$ *for all* $x \in \mathrm{R}^N$*. Then* $\omega^2 \le f'(0)$.

Remarks:

The proof of this corollary was carried out by contradiction and was outlined above. We required φ to have zero mean so as to be able to use Wirtinger's inequality, which provided us with an a priori estimate.

By a small amplitude solution we mean that φ has sufficiently small $L^\infty(\mathrm{R}^{N+1})$ norm, sufficiency being determined by the values of $f'(0), f^{(2k+1)}(0)$ and the parameter c whose value was assigned during the proof above. To illustrate this precisely let $\{\varepsilon_n\}$ be a sequence of positive numbers converging to zero and suppose that for each $n_o \in \mathrm{N}$ there is an $\bar{n} \ge n_o$ such that there exists a (nontrivial) $2\pi/\omega$-periodic solution $\varphi_{\bar{n}}$ of NLW with zero mean and with $\|\varphi_{\bar{n}}\|_{L^\infty(\mathrm{R}^{N+1})} \le \varepsilon_{\bar{n}}$. If $f'(0) > 0$ and if $(f^{(2k+1)}(0), k, N)$ satisfies Condition A, then $\omega^2 \le f'(0)$.

One may try to prove a similar result for small amplitude *quasiperiodic* breathers. However, a Wirtinger-type inequality for quasiperiodic functions is generally not possible for the following reason. Let $\gamma \in H^1(\Omega_{N,l})$, $l > 1$, be the generating function of φ with Fourier series $\sum_{k\in Z^l} \gamma_k(x) e^{ik\cdot\theta}$ (so that $\varphi(x,t) = \sum_{k\in Z^l} \gamma_k(x) e^{ik\cdot\omega t}$). Then, $\|\gamma\|^2_{L^2(\Omega_{N,l})} \;=\; \sum_{k\in Z^l} \|\gamma_k\|_2^2$ and $\|\mathcal{D}_\omega \gamma\|^2_{L^2(\Omega_{N,l})} = \sum_{k\in Z^l} \mid k\cdot\omega \mid^2 \|\gamma_k\|_2^2$. Suppose that $\gamma_0 = 0$. Because ω is incommensurate, $k\cdot\omega$ becomes arbitrarily small infinitely often as k ranges over Z^l. Thus it is impossible to bound $\mid k\cdot\omega \mid^2$ from below by a strictly positive number c and obtain an inequality of the form $\|\mathcal{D}_\omega \gamma\|^2_{L^2(\Omega_{N,l})} \ge c\|\gamma\|^2_{L^2(\Omega_{N,l})}$.

3) Local vector fields

The preceding examples assumed that the solutions were of small amplitude. Since solutions are a priori in $H^1(\Omega_{N,l})$, they decay in an average sense as $\mid x \mid \to \infty$. Making the additional assumption that this decay is pointwise and uniform in t, we could obtain some of the previous results without any assumptions of small amplitude provided that the virial relation was localized in a neighborhood of infinity. We illustrate this idea with an example concerning quasiperiodic solutions on R^{1+1}.

Corollary 3.8 *Suppose* φ *is a weak* l*-quasiperiodic solution of NLW on* R^{1+1} *with frequency*

ω *that converges to zero as* $|x| \to \infty$ *uniformly in* t. *If* $F(z) \leq 0$ *in a neighborhood of zero, then* $\varphi \equiv 0$.

Proof:

Let $\rho > 1$ and define $g : \mathrm{R} \to \mathrm{R}$ by

$$\begin{aligned} g(x) &= 6x^2 + (4 - 12\rho)x + (6\rho^2 - 4\rho + 1) & \rho \leq x & \\ &= (x - \rho + 1)^4 & \rho - 1 \leq x &\leq \rho \\ &= 0 & -\rho + 1 \leq x &\leq \rho - 1 \\ &= (x + \rho - 1)^4 & -\rho \leq x &\leq -\rho + 1 \\ &= 6x^2 - (4 - 12\rho)x + (6\rho^2 - 4\rho + 1) & x &\leq -\rho \end{aligned}$$

On R define the vector field $v(x) = g'(x)$. Then v satisfies the hypothesis of Theorem 2.7 except that Δv has a finite discontinuity at $| x |= \rho$. This does not affect the validity of the theorem, however (see equations (2.15) and (2.23): this is the only place in the proof where derivatives of v of order greater than one are encountered). Here we have $v = 0$ on $\{| x |\leq \rho - 1\}$ with $dv = g'' \geq 0$. For this vector field equation (2.4) reads

$$\int_{\mathrm{T}^l} \int_{|x|>\rho-1} dv \left\{ \frac{1}{2}(\mathcal{D}_\omega \gamma)^2 + \frac{1}{2} \mid \nabla \gamma \mid^2 - F(\gamma) \right\} \;=\; 0. \tag{3.9}$$

If $F(z) \leq 0$ in a neighborhood of zero then, by taking ρ sufficiently large, equation (3.9) implies that $\gamma = 0$ on the support of v. The conclusion of the corollary then follows from the next Proposition, the proof of which we leave as an open problem (see, however, Theorem 5.5 of [PS] which states that periodic solutions of NLW with compact spatial support are identically zero - we expect the same to be true for quasiperiodic solutions).

Proposition 3.9 *Suppose* γ *satisfies (2.2) and vanishes in a neighborhood of infinity on* R^N *uniformly in* θ. *Then* $\gamma \equiv 0$.

4 Virial relations for almost periodic solutions

4.1 Almost periodic solutions

To motivate our approach to almost periodic solutions we return for a moment to the quasi-periodic case. Recall that our definition of a weak solution γ of NLW on $\Omega_{N,l}$ with frequency ω was that $\gamma \in H^1(\Omega_{N,l})$ and for all $\beta \in H^1(\Omega_{N,l})$,

$$\int_{\mathrm{T}^l} \int_{\mathrm{R}^N} \Big\{ -\mathcal{D}_\omega \gamma \mathcal{D}_\omega \beta + \nabla\gamma \cdot \nabla\beta + f(\gamma)\beta \Big\} = 0. \tag{4.1}$$

We set

$$h(\theta) \quad = \quad \int_{\mathrm{R}^N} \Big\{ -\mathcal{D}_\omega \gamma \mathcal{D}_\omega \beta + \nabla\gamma \cdot \nabla\beta + f(\gamma)\beta \Big\} \quad \in L^1(\mathrm{T}^l).$$

Because ω is incommensurate the flow $\theta \mapsto \theta + \omega t$ on T^l is ergodic. It follows, therefore, by a result from ergodic theory (see for example [Pe] Thm. 2.2.3) that

$$(2\pi)^{-l} \int_{\mathrm{T}^l} h(\theta)\, d\theta \quad = \quad \lim_{T\to\infty} \frac{1}{T} \int_0^T h(\omega t)\, dt. \tag{4.2}$$

Defining $\varphi(t) \equiv \gamma(\omega t)$ and $\psi(t) \equiv \beta(\omega t)$ we have that $\mathcal{D}_\omega\gamma(\omega t)\mathcal{D}_\omega\beta(\omega t) = \partial_t\varphi(t)\partial_t\psi(t)$. Combining equations (4.1) and (4.2) we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{\mathrm{R}^N} \Big\{ -\partial_t\varphi\partial_t\psi + \nabla\varphi \cdot \nabla\psi + f(\varphi)\psi \Big\} = 0. \tag{4.3}$$

Similarly, we can also cast the virial relation of Theorem 2.7, equation (2.4), in terms of φ ;

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{\mathrm{R}^N} \Big\{ tr\, dv\, (\frac{1}{2}(\partial_t\varphi)^2 - F(\varphi)) \ + \ \nabla\varphi \cdot [dv - \frac{1}{2} tr\, dv \mathbf{1}]\, \nabla\varphi \Big\} \quad = \quad 0. \tag{4.4}$$

The property that $\lim_{T\to\infty} \frac{1}{T} \int_0^T h(t)$ exists is shared by functions from a larger class than the quasiperiodic functions - the almost periodic functions (for an introduction to almost periodic functions see [C] or [LZ]). We will see that formula (4.4) holds for solutions from this larger class. For any Banach space $\mathcal{B}$ we denote by $\mathcal{AP}(\mathrm{R}, \mathcal{B})$ the Banach space of $\mathcal{B}$ valued almost periodic functions on R. $\mathcal{AP}(\mathrm{R}, \mathcal{B})$ is a closed subspace of the space of bounded continuous functions from R to $\mathcal{B}$ with the uniform norm $\|h\| = \sup_{t\in\mathrm{R}} \|h(t)\|_{\mathcal{B}}$. The class of almost periodic solutions we consider is described in the following definition.

Definition 4.1 *For any Banach space $\mathcal{B}$ let $\mathcal{AP}(\mathrm{R}, \mathcal{B})$ denote the space of $\mathcal{B}$ valued almost periodic functions on* R. *Let $\mathcal{AP} \equiv \{\varphi \in \mathcal{AP}(\mathrm{R}, H^1(\mathrm{R}^N))$ such that $\partial_t\varphi$ exists in the strong sense as a uniformly continuous map $\mathrm{R} \to L^2(\mathrm{R}^N)\}$, and for $q \in [2,\infty]$ let $\mathcal{AP}_q \equiv \{\varphi \in \mathcal{AP}(\mathrm{R}, H^1(\mathrm{R}^N) \cap L^q(\mathrm{R}^N))$ such that $\partial_t\varphi$ exists in the strong sense as a uniformly continuous map $\mathrm{R} \to L^2(\mathrm{R}^N)\}$.*

That $\partial_t\varphi$ is uniformly continuous as indicated implies that $\partial_t\varphi \in \mathcal{AP}(\mathrm{R}, L^2(\mathrm{R}^N))$ ([LZ] pp3). Equation (4.3) motivates our definition of weak almost periodic solution to NLW, but first we state what we mean by a classical almost periodic solution.

Definition 4.2 *$\varphi : \mathrm{R}^{N+1} \to \mathrm{R}$ is a* classical almost periodic solution of NLW *if φ solves NLW and such that $\partial_t^2\varphi \in L^2_{loc}(\mathrm{R}, L^2(\mathrm{R}^N))$, $\varphi \in C^2(\mathrm{R}^{N+1}) \cap \mathcal{AP}_q$ for some $q \in [2,\infty]$.*

Definition 4.3 *$\varphi : \mathrm{R}^{N+1} \to \mathrm{R}$ is a* weak almost periodic solution of NLW *if $\varphi \in \mathcal{AP}$ and*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathrm{R}^N} \Big\{ -\partial_t\varphi\partial_t\psi + \nabla\varphi\cdot\nabla\psi + f(\varphi)\psi \Big\} \;=\; 0 \qquad \forall\, \psi \in \mathcal{AP}. \tag{4.5}$$

Our definitions of weak and strong almost periodic solutions are compatible, that is, a strong almost periodic solution is a weak almost periodic solution.

Proposition 4.4 *If φ is a classical almost periodic solution of NLW then φ is a weak almost periodic solution of NLW.*

Proof:

Since φ is a solution of NLW, for any $\psi \in \mathcal{AP}$ and any $T > 0$,

$$\frac{1}{T}\int_0^T \int_{\mathrm{R}^N} \Big(\partial_t^2\varphi - \Delta\varphi + f(\varphi)\Big)\,\psi \;=\; 0.$$

Interchanging the order of integration for the first term (we can use Fubini's Theorem since $\|(\partial_t^2\varphi)\psi\|_{L^1(\mathrm{R}^N)} \in L^1_{loc}(\mathrm{R})$),

$$\frac{1}{T}\int_{\mathrm{R}^N}\int_0^T (\partial_t^2\varphi)\psi = \frac{1}{T}\int_{\mathrm{R}^N}\int_0^T \{\partial_t(\partial_t\varphi\,\psi) - \partial_t\varphi\partial_t\psi\}\,.$$

Since $(\int_{\mathrm{R}^N}(\partial_t\varphi)\psi)|_0^T$ is bounded independently of T ($\partial_t\varphi$ and ψ are both in $L^\infty(\mathrm{R}, L^2(\mathrm{R}^N))$ so that $(\partial_t\varphi)\psi \in L^\infty(\mathrm{R}, L^1(\mathrm{R}^N)))$, $\lim_{T\to\infty}\frac{1}{T}\int_{\mathrm{R}^N}\int_0^T \partial_t(\partial_t\varphi\psi) = 0$. Integrating the next term by parts,

$$\int_{\mathrm{R}^N} -\Delta\varphi\psi = \int_{\mathrm{R}^N} \nabla\varphi\cdot\nabla\psi,$$

we conclude that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}\{-\partial_t\varphi\partial_t\psi + \nabla\varphi\cdot\nabla\psi + f(\varphi)\psi\} = 0 \qquad \square$$

<u>Remark</u>:

In the references [C] and [LZ], $\mathcal{AP}(\mathrm{R},\mathcal{B})$ is defined as the uniform closure of the set of trigonometric polynomials of the form $p(t) = \sum_{j=1}^n a_j e^{i\lambda_j t}$ where $n \in \mathrm{N}$, $a_j \in \mathcal{B}$ and $\lambda_j \in \mathrm{R}$. There is another way of viewing almost periodic functions which is in keeping with the point of view we have adopted in treating quasiperiodic functions. For any almost periodic function $\varphi \in \mathcal{AP}(\mathrm{R},\mathcal{B})$ there is a curve $\Gamma : \mathrm{R} \to \mathrm{T}^\infty$, dense in T^∞ (T^∞ is the compact abelian group $\Pi_{j=1}^\infty S^1$), and a continuous function $\gamma : \mathrm{T}^\infty \to \mathcal{B}$ (the *generating function* of φ) such that $\varphi(t) = (\gamma\circ\Gamma)(t)$. Then, $\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi\, dt = \int_{\mathrm{T}^\infty}\gamma\, d\mu$ where $d\mu$ is the (normalized) Haar measure on T^∞ (see [HR] or [DS]). When almost periodic functions are looked at this way, from the point of view of generating functions, their special nature becomes apparent. We are referring to their "compactness" in the time variable: Although an almost periodic function is defined on the entire real line, the generating function is defined on the compact space T^∞, in which the real line is embedded: $\Gamma(\mathrm{R}) \subset \mathrm{T}^\infty$. This is the *Bohr compactification* of the real line [HR]. Thus, almost periodic functions are "effectively" defined on the compact space T^∞ (through their generating function). It is this fact which makes almost periodic functions a generalization of periodic functions and is responsible for them sharing many properties, an example of which is their uniform boundedness as described below in Lemma 4.7. This is a key property that we use to derive virial relations for almost periodic solutions (Theorem 4.10).

Definition 4.5 *A set $K \subset L^p(\mathrm{R}^N)$ is* uniformly bounded *if for any $\varepsilon > 0$ there exists a ball*

$B_\varepsilon \subset \mathrm{R}^N$ *such that*

$$\int_{B_\varepsilon^c} |\, \varphi \,|^p \; < \; \varepsilon \qquad \forall\, \varphi \in K. \tag{4.6}$$

Definition 4.6 *For a function $\varphi : \mathrm{R} \to L^p(\mathrm{R}^N)$, the set $\{\varphi(t)\,;\ t \in \mathrm{R}\} \subset L^p(\mathrm{R}^N)$ is the* orbit *of φ.*

Lemma 4.7 *If $\varphi \in \mathcal{AP}(\mathrm{R}; L^p(\mathrm{R}^N))$, then the orbit of φ is uniformly bounded.*

Proof:

A basic property of almost periodic functions is that their orbits are relatively compact ([LZ] pp.2). Relatively compact subsets of $L^p(\mathrm{R}^N)$ are uniformly bounded ([DS] Thm. IV.8.21) □

Remark: That is, if $\varphi \in \mathcal{AP}(\mathrm{R}, L^p(\mathrm{R}^N))$, then for any $\varepsilon > 0$ there exists a ball $B_\varepsilon \subset \mathrm{R}^N$ such that

$$\int_{B_\varepsilon^c} |\varphi(x,t)|^p \;\; < \;\; \varepsilon$$

for all t.

In this section we are treating the almost periodic function φ as the primary object rather than its generating function γ. Consequently, we have lost the ability to apply Lebesgue's dominated convergence theorem which was the main tool in the proof of Theorem 2.7. This is because φ is defined only on the set $\Gamma(\mathrm{R})$ which has measure zero in T^∞ so that pointwise convergence of φ on $\Gamma(\mathrm{R})$ (that is, pointwise convergence in t) does not imply the convergence of the integral of its generating function over T^∞. For the proof of Theorem 4.10, below, in place of dominated convergence we use the fact that the orbit of φ is uniformly bounded.

Definition 4.8 *For $\varphi \in \mathcal{AP}(\mathrm{R}, \mathcal{B})$ the mean value of φ is denoted by $\mathcal{M}(\varphi) \;\equiv\; \lim_{T\to\infty} \frac{1}{T}\int_0^T \varphi(t)$.*

Remark: Because almost periodic functions are bounded, $\mathcal{M}(\varphi)$ is finite.

Lemma 4.9 *Let $\varphi \in \mathcal{AP}_q$. Then*

(i) $\varphi,\ \nabla\varphi \;\in \mathcal{AP}(\mathrm{R}, L^2(\mathrm{R}^N))$,

(ii) $\hat{\varphi},\ \widehat{\nabla\varphi} \;\in \mathcal{AP}(\mathrm{R}, L^2(\mathrm{R}^N))$, *where* ˆ *denotes the Fourier transform,*

(iii) $R_\varepsilon\varphi \in \mathcal{AP}(\mathrm{R}, H^1(\mathrm{R}^N))$, *where* R_ε *is as in Theorem 2.7,*

(iv) $\|\partial_t\varphi\|_2^m$, $\|\nabla\varphi\|_2^m \in \mathcal{AP}(\mathrm{R},\mathrm{R})$ *for* $m \in \mathrm{Z} > 0$,

(v) if h *is a differentiable function on* R^N *that is bounded along with its derivatives then* $h\varphi \in \mathcal{AP}(\mathrm{R}, H^1(\mathrm{R}^N))$,

(vi) if f *satisfies (H) then* $f(\varphi) \in \mathcal{AP}(\mathrm{R}, L^2(\mathrm{R}^N))$ *and* $F(\varphi) \in \mathcal{AP}(\mathrm{R}, L^1(\mathrm{R}^N))$.

Proof:

If $L : \mathcal{B}_1 \to \mathcal{B}_2$ is a continuous map between Banach spaces and $\varphi \in \mathcal{AP}(\mathrm{R}, \mathcal{B}_1)$, then $L(\varphi) \in \mathcal{AP}(\mathrm{R}, \mathcal{B}_2)$ ([LZ] pp. 3). Parts $(i) - (v)$ are a straightforward consequence of this. Similarly for part (vi) after realizing that if f satisfies (H) then the maps $f : L^2(\mathrm{R}^N) \cap L^q(\mathrm{R}^N) \to L^2(\mathrm{R}^N)$ and $F : L^2(\mathrm{R}^N) \cap L^q(\mathrm{R}^N) \to L^1(\mathrm{R}^N)$ are continuous (this follows from the proof of Theorem I.2.1 in [K]).

4.2 Virial relations for almost periodic solutions

Theorem 4.10 *Let* $\varphi \in \mathcal{AP}_q$, *for some* $q \in [2, \infty]$, *satisfy (4.5),* f *satisfy (H) with this same* q, *and* v *be a smooth vector field on* R^N *satisfying the hypothesis of Theorem 2.7. Then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{\mathrm{R}^N} \Big\{ tr\, dv \left(\frac{1}{2}(\partial_t\varphi)^2 - F(\varphi)\right) \;+\; \nabla\varphi \cdot [dv - \frac{1}{2} tr\, dv \mathbf{1}]\, \nabla\varphi \Big\} \;=\; 0. \qquad (4.7)$$

Proof:

Our approach begins as in Theorem 2.7. Set

$$\psi_{\varepsilon,\delta} \equiv v_\delta \cdot \nabla R_\varepsilon^2 \varphi, \qquad (4.8)$$

where v_δ and R_ε are as in Theorem 2.7. Then $\psi_{\varepsilon,\delta} \in \mathcal{AP}$ $\forall \varepsilon, \delta > 0$ and so (cf. equation (4.5))

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{\mathrm{R}^N} \big\{ -\partial_t\varphi \partial_t \psi_{\varepsilon,\delta} + \nabla\varphi \cdot \nabla\psi_{\varepsilon,\delta} + f(\varphi)\psi_{\varepsilon,\delta} \big\} \;=\; 0 \quad \forall \varepsilon, \delta > 0. \qquad (4.9)$$

We write

$$\partial_t\varphi \partial_t \psi_{\varepsilon,\delta} = \partial_t\varphi \partial_t v_\delta \cdot \nabla R_\varepsilon^2 \varphi \;=\; \partial_t\varphi v_\delta \cdot \nabla R_\varepsilon^2 \partial_t\varphi$$

$$= \quad \partial_t\varphi R_\varepsilon v_\delta\cdot\nabla R_\varepsilon\partial_t\varphi \quad - \quad \partial_t\varphi[R_\varepsilon, v_\delta]\cdot\nabla R_\varepsilon\partial_t\varphi. \quad (4.10)$$

For the first term on the right,

$$\begin{aligned}
\int_{\mathrm{R}^N}\partial_t\varphi R_\varepsilon v_\delta\cdot\nabla R_\varepsilon\partial_t\varphi &= \int_{\mathrm{R}^N} R_\varepsilon\partial_t\varphi v_\delta\cdot\nabla R_\varepsilon\partial_t\varphi \qquad \text{(self-adjointness of } R_\varepsilon\text{)}\\
&= \frac{1}{2}\int_{\mathrm{R}^N}\nabla\cdot(v_\delta(R_\varepsilon\partial_t\varphi)^2) \;-\frac{1}{2}\int_{\mathrm{R}^N}\nabla\cdot v_\delta(R_\varepsilon\partial_t\varphi)^2\\
&= -\frac{1}{2}\int_{\mathrm{R}^N}\nabla\cdot v(\partial_t\varphi)^2 \;-\frac{1}{2}\int_{\mathrm{R}^N}(\nabla\cdot v_\delta-\nabla\cdot v)(\partial_t\varphi)^2\\
&\quad -\int_{\mathrm{R}^N}\nabla\cdot v_\delta[(R_\varepsilon-1)\partial_t\varphi]\partial_t\varphi\\
&\quad -\frac{1}{2}\int_{\mathrm{R}^N}\nabla\cdot v_\delta[(R_\varepsilon-1)\partial_t\varphi]^2. \qquad (4.11)
\end{aligned}$$

Let $B(\rho)\subset \mathrm{R}^N$ denote the ball of radius ρ. By Lemma 4.7, for any $\eta>0$ there exists a $\rho>0$ such that $\int_{B^c(\rho)}(\partial_t\varphi)^2 \;<\; \eta/\|\nabla\cdot v_\delta-\nabla\cdot v\|_\infty$ for all t. Because $\nabla\cdot v_\delta\to\nabla\cdot v$ uniformly on compact sets there is a $\delta_o(\eta)>0$ such that $|\nabla\cdot v_\delta-\nabla\cdot v| <\; \eta$ on $B(\rho)$ for $\delta<\delta_o$. Then, for any such δ and any T,

$$\begin{aligned}
\Big|\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}(\nabla\cdot v_\delta-\nabla\cdot v)(\partial_t\varphi)^2\Big| \quad &\le \quad \frac{1}{T}\int_0^T\int_{B(\rho)}|\,\nabla\cdot v_\delta-\nabla\cdot v\,|\,(\partial_t\varphi)^2\\
&\quad + \quad \frac{1}{T}\int_0^T\int_{B^c(\rho)}|\,\nabla\cdot v_\delta-\nabla\cdot v\,|\,(\partial_t\varphi)^2\\
&\le \quad \eta\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}(\partial_t\varphi)^2 \quad + \quad \eta. \qquad (4.12)
\end{aligned}$$

Taking the limit $T\to\infty$,

$$\Big|\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}(\nabla\cdot v_\delta-\nabla\cdot v)(\partial_t\varphi)^2\Big| \;\le\; \eta\mathcal{M}(\,\|\partial_t\varphi\|_2^2)+\eta. \qquad (4.13)$$

Since η was arbitrary we conclude that

$$\lim_{\delta\to 0}\;\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}(\nabla\cdot v_\delta-\nabla\cdot v)(\partial_t\varphi)^2=0. \qquad (4.14)$$

Bounding $\|\nabla \cdot v_\delta\|_\infty \leq c$ uniformly in δ,

$$\Big|\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathrm{R}^N} \nabla \cdot v_\delta [(R_\varepsilon - 1)\partial_t\varphi]\partial_t\varphi\Big| \;\leq\; \lim_{T\to\infty} \frac{1}{T}\int_0^T c\|(R_\varepsilon - 1)\partial_t\varphi\|_2\|\partial_t\varphi\|_2. \quad (4.15)$$

Using the Fourier transform,

$$\|(R_\varepsilon - 1)\partial_t\varphi\|_2^2 = \int_{\mathrm{R}^N} \frac{(\varepsilon^2 p^2)^2}{(1+\varepsilon^2 p^2)^2}(\partial_t\hat{\varphi}(p))^2 = \int_{\mathrm{R}^N} h_\varepsilon(p)(\partial_t\hat{\varphi}(p))^2. \quad (4.16)$$

Since $h_\varepsilon \to 0$ as $\varepsilon \to 0$ uniformly on compact sets with $h_\varepsilon(p) \leq 1$, and since $\{\partial_t\hat{\varphi};\ t \in \mathrm{R}\}$ is uniformly bounded in $L^2(\mathrm{R}^N)$, applying the same argument as above we conclude that, for any $\eta^2 > 0$ and ε sufficiently small,

$$\begin{aligned}\|(R_\varepsilon - 1)\partial_t\varphi\|_2^2 &\leq \eta^2\|\partial_t\varphi\|_2^2 + \eta^2 \\ &\leq \eta^2(\|\partial_t\varphi\|_2 + 1)^2 \quad \forall t. \end{aligned} \quad (4.17)$$

Therefore,

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T}\int_0^T \|(R_\varepsilon - 1)\partial_t\varphi\|_2\|\partial_t\varphi\|_2 &\leq \lim_{T\to\infty} \frac{1}{T}\int_0^T \eta(\|\partial_t\varphi\|_2 + 1)\|\partial_t\varphi\|_2 \\ &= \eta\left(\mathcal{M}(\|\partial_t\varphi\|_2^2) + \mathcal{M}(\|\partial_t\varphi\|_2)\right), \end{aligned} \quad (4.18)$$

and so

$$\lim_{\varepsilon\to 0} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathrm{R}^N} \nabla \cdot v_\delta [(R_\varepsilon - 1)\partial_t\varphi]\partial_t\varphi \;=\; 0 \quad (4.19)$$

uniformly in δ. The last term in equation (4.11) is treated in a similar way while the second term on the right hand side of equation (4.10) can be treated as below (see equation (4.25)); the result being that both of these terms converge to zero. Thus,

$$\lim_{\varepsilon,\delta\to 0} \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathrm{R}^N} \partial_t\varphi\partial_t\psi_{\varepsilon,\delta} \;=\; \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathrm{R}^N} -\frac{1}{2} tr\, dv\, (\partial_t\varphi)^2. \quad (4.20)$$

Considering the next term in equation (4.9),

$$\nabla\varphi \cdot \nabla\psi_{\varepsilon,\delta} \;=\; \nabla\varphi\nabla v_\delta \cdot \nabla R_\varepsilon^2\varphi$$

$$= \quad \nabla\varphi v_\delta \cdot \nabla\mathbf{1}\,\nabla R_\varepsilon^2\varphi + \nabla\varphi\, dv_\delta\, \nabla R_\varepsilon^2\varphi. \tag{4.21}$$

We write the first term on the right hand side as

$$\nabla\varphi v_\delta \cdot \nabla\mathbf{1}\,\nabla R_\varepsilon^2\varphi = \nabla\varphi R_\varepsilon v_\delta \cdot \nabla\mathbf{1}\,\nabla R_\varepsilon\varphi - \nabla\varphi[R_\varepsilon, v_\delta]\cdot\nabla\mathbf{1}\,R_\varepsilon\nabla\varphi. \tag{4.22}$$

Furthermore,

$$\begin{aligned}
\nabla\varphi R_\varepsilon v_\delta \cdot \nabla\mathbf{1}\,\nabla R_\varepsilon\varphi \quad &= \quad -\frac{1}{2}\nabla\cdot v_\delta \mid \nabla R_\varepsilon\varphi \mid^2 + \frac{1}{2}\nabla\cdot(v_\delta|\nabla R_\varepsilon\varphi|^2) \\
&= \quad -\frac{1}{2}\nabla\cdot v \mid \nabla\varphi \mid^2 - \frac{1}{2}[\nabla\cdot v_\delta - \nabla\cdot v] \mid \nabla\varphi \mid^2 \\
&- \quad \nabla\cdot v_\delta[(R_\varepsilon - 1)\nabla\varphi]\nabla\varphi - \frac{1}{2}\nabla\cdot v_\delta[(R_\varepsilon-1)\nabla\varphi]^2 \\
&+ \quad \frac{1}{2}\nabla\cdot(v_\delta|\nabla R_\varepsilon\varphi|^2).
\end{aligned} \tag{4.23}$$

Ignoring the divergence term, the terms on the right can be treated in the same way as was done for equation (4.11) whence we conclude that

$$\lim_{\varepsilon,\delta\to 0}\ \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}\nabla\varphi R_\varepsilon v_\delta\cdot\nabla\mathbf{1}\,\nabla R_\varepsilon\varphi = \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N} -\frac{1}{2}tr\, dv \ \mid\nabla\varphi\mid^2. \tag{4.24}$$

Considering now the second term on the right hand side of equation (4.22), we write

$$\int_{\mathrm{R}^N}\nabla\varphi[R_\varepsilon, v_\delta]\cdot\nabla\mathbf{1}\,R_\varepsilon\nabla\varphi = \int_{\mathrm{R}^N}\nabla\varphi R_\varepsilon\varepsilon^2(\Delta v_\delta\cdot R_\varepsilon\nabla\mathbf{1} + 2\nabla v_\delta\nabla\cdot R_\varepsilon\nabla\mathbf{1})R_\varepsilon\nabla\varphi \tag{4.25}$$

(cf. equation (2.15)). We have

$$\Big|\int_{\mathrm{R}^N}\varepsilon^2\nabla\varphi R_\varepsilon\Delta v_\delta\cdot R_\varepsilon\nabla\mathbf{1}R_\varepsilon\nabla\varphi\Big| \ \le\ c\varepsilon\|\nabla\varphi\|_2^2 \tag{4.26}$$

(cf. equation (2.18)) so that this term will converge to zero in the time mean as $\varepsilon\to 0$, uniformly in δ.

The other term in equation (4.25) we treat in a similar way as in equation (2.19);

$$\Big|\int_{\mathrm{R}^N}\varepsilon^2\nabla\varphi R_\varepsilon\nabla v_\delta\nabla\cdot R_\varepsilon\nabla\mathbf{1}\,R_\varepsilon\nabla\varphi\Big| \ \le\ c\|\nabla\varphi\|_2\|\varepsilon^2\nabla^2\mathbf{1}\,R_\varepsilon\nabla\varphi\|_2$$

$$\leq \; c\|\nabla\varphi\|_{\infty,2}\|\varepsilon^2\nabla^2\mathbf{1}\, R_\varepsilon\nabla\varphi\|_2 \tag{4.27}$$

where c is independent of δ. Note that $\|\nabla\varphi\|_{\infty,2}$ is a constant. Passing to the Fourier transform, for $|a|=2$ and $\psi \in \mathcal{AP}(\mathrm{R}, L^2(\mathrm{R}^N))$,

$$\|\varepsilon^2\partial^a R_\varepsilon\psi\|_2^2 = \int_{\mathrm{R}^N} \frac{\varepsilon^4 \mid p^a \mid^2}{(1+\varepsilon^2 p^2)^2} \mid \hat{\psi}(p,t) \mid^2 . \tag{4.28}$$

The function $\varepsilon^4 \mid p^a \mid^2 (1+\varepsilon^2p^2)^{-2}$ converges to zero uniformly on compact sets and is bounded by a constant $c^{'}$ that depends only on a while $\hat{\psi}(t)$ is uniformly bounded (in $L^2(\mathrm{R}^N)$) so that, by the same arguments as made before (cf. equation (4.17)), for any $\eta > 0$ and sufficiently small ε,

$$\|\varepsilon^2\partial^a R_\varepsilon\psi\|_2^2 \;\leq\; \eta^2(\|\psi\|_2 + c^{'})^2 \quad \forall\, t. \tag{4.29}$$

Thus,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T c\|\nabla\varphi\|_{\infty,2}\|\varepsilon^2\nabla^2\mathbf{1}\, R_\varepsilon\nabla\varphi\|_2 \;\leq\; \eta c\|\nabla\varphi\|_{\infty,2}\mathcal{M}(\|\nabla\varphi\|_2^2 + c^{'}) \tag{4.30}$$

and so

$$\lim_{\varepsilon,\delta\to 0}\quad \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N} \varepsilon^2\nabla\varphi R_\varepsilon\nabla v_\delta\nabla\cdot R_\varepsilon\nabla\mathbf{1}\, R_\varepsilon\nabla\varphi \;=\; 0. \tag{4.31}$$

The second term in equation (4.21) we write as

$$\nabla\varphi dv_\delta\nabla R_\varepsilon^2\varphi = \nabla\varphi\, dv\, \nabla\varphi + \nabla\varphi(dv_\delta - dv)\nabla\varphi + \nabla\varphi\, dv_\delta\, (R_\varepsilon^2 - 1)\nabla\varphi. \tag{4.32}$$

Applying Schwarz's inequality to the second term on the right we find that

$$\|\nabla\varphi(dv_\delta - dv)\nabla\varphi\|_1 \leq \|\nabla\varphi\|_2\|(dv_\delta - dv)\nabla\varphi\|_2. \tag{4.33}$$

Since $(dv_\delta - dv) \;\to 0$ uniformly on compact sets with $\|dv_\delta - dv\|_\infty$ uniformly bounded, by familiar arguments we conclude that for any $\eta > 0$ and δ sufficiently small,

$$\|(dv_\delta - dv)\nabla\varphi\|_2 \leq \eta(\|\nabla\varphi\|_2 + c) \tag{4.34}$$

so that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \|\nabla\varphi(dv_\delta - dv)\nabla\varphi\|_1 \;\; \le \;\; \eta\left(\mathcal{M}(\|\nabla\varphi\|_2^2 + c\mathcal{M}(\|\nabla\varphi\|_2)\right) \tag{4.35}$$

and hence

$$\lim_{\delta\to 0}\;\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}\nabla\varphi(dv_\delta - dv)\nabla\varphi \;\; = \;\; 0. \tag{4.36}$$

The third term on the right hand side of equation (4.32) is treated analogously by noting that $\|dv_\delta\|_\infty$ is uniformly bounded and then using the Fourier transform to show that $\|(R_\varepsilon^2 - 1)\nabla\varphi\|_2 \le \eta(\|\nabla\varphi\|_2 + 1)\forall t$ for any $\eta > 0$ and ε sufficiently small (cf. (4.17)). Thus,

$$\lim_{\varepsilon,\delta\to 0}\;\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}\nabla\varphi\, dv\, \nabla R_\varepsilon^2\nabla\varphi \;\; = \;\; \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}\nabla\varphi\, dv\, \nabla\varphi. \tag{4.37}$$

To treat the nonlinear term we begin by writing

$$\begin{aligned}\int_{\mathrm{R}^N} f(\varphi)\psi_{\varepsilon,\delta} \;\; &= \;\; -\int_{\mathrm{R}^N}\nabla\cdot v\, F(\varphi) + \int_{\mathrm{R}^N}(\nabla\cdot v_\delta - \nabla\cdot v)F(\varphi) \\ &\quad + \int_{\mathrm{R}^N} f(\varphi)v_\delta\cdot(R_\varepsilon^2 - 1)\nabla\varphi\end{aligned} \tag{4.38}$$

Since $F(\varphi)\in\mathcal{AP}(\mathrm{R}, L^1(\mathrm{R}^N))$ (Lemma 4.9) with $(\nabla\cdot v_\delta - \nabla v)\to 0$ uniformly on compact sets we conclude, by arguments made before, that

$$\lim_{\delta\to 0}\;\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}(\nabla\cdot v_\delta - \nabla v)F(\varphi) \;\; = \;\; 0. \tag{4.39}$$

Using the Fourier transform as before we can show that

$$\|(R_\varepsilon^2 - 1)\nabla\varphi\|_2 \;\; \le \;\; \eta(\|\nabla\varphi\|_2 + 1) \quad \forall\, t \tag{4.40}$$

for any $\eta > 0$ and ε sufficiently small. Now apply Schwarz's inequality to the third term on the right hand side of equation (4.38) to conclude that

$$\left|\int_{\mathrm{R}^N} f(\varphi)v_\delta\cdot(R_\varepsilon^2 - 1)\nabla\varphi\right| \;\; \le \;\; \eta\|v_\delta\|_\infty\|f(\varphi)\|_2(\|\nabla\varphi\|_2 + 1). \tag{4.41}$$

Both $\|f(\varphi)\|_2$ and $(\|\nabla\varphi\|_2 + 1)$ are numerical almost periodic functions (Lemma 4.9), i.e., are members of $\mathcal{AP}(\mathrm{R},\mathrm{R})$ and hence their product is almost periodic ([LZ] pp 6). Then, for any $\eta > 0$ and ε sufficiently small,

$$\Big|\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N} f(\varphi)v_\delta\cdot(R_\varepsilon^2-1)\nabla\varphi\Big| \;\leq\; \eta\|v_\delta\|_\infty\mathcal{M}\Big(\|f(\varphi)\|_2(\|\nabla\varphi\|_2+1)\Big). \tag{4.42}$$

Since $\|v_\delta\|_\infty\mathcal{M}\Big(\|f(\varphi)\|_2(\|\nabla\varphi\|_2+1)\Big)$ is bounded uniformly in δ, we have that

$$\lim_{\varepsilon\to 0}\;\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N} f(\varphi)v_\delta\cdot(R_\varepsilon^2-1)\nabla\varphi \;=\; 0 \tag{4.43}$$

uniformly in δ. Thus,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N} f(\varphi)\psi_{\varepsilon,\delta} \;=\; \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N} -tr\,dv\,F(\varphi) \quad + \quad E(\varepsilon,\delta) \tag{4.44}$$

where $E(\varepsilon,\delta)$ can be made arbitrarily small first by choosing δ and then ε sufficiently small. This, along with equations (4.9), (4.20), (4.24) and (4.37), completes the proof of Theorem 4.10 □

5 Nonexistence of almost periodic solutions

We use the virial relation equation (4.7) to prove a nonexistence theorem for almost periodic solutions. A special case of this result ($N=1$ and $c=\frac{1}{2}$) was proven in [SV].

Theorem 5.1 *(Nonexistence of almost periodic solutions of NLW) Let $\varphi \in \mathcal{AP}_q$ for some $q\in[2,\infty]$ be a weak almost periodic solution of NLW (cf. Definition 4.3) with f satisfying (H) with this same q and, for some $c\in[\frac{N-2}{2N},\frac{1}{2}]$, the inequality*

$$F(z)-czf(z)\leq 0 \quad \textit{for all } z \textit{ such that } \mid z\mid\leq\|\varphi\|_{L^\infty(\mathrm{R}^{N+1})}.$$

Then φ is independent of time.

Proof:

As in Proposition 3.3, we use the vector field associated to dilations on R^N along with the Gauge transformation to derive the following identity valid for any $c\in\mathrm{R}$ (we use Lemma 4.9, part (v) for the Gauge transformation);

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}\left\{(\frac{1}{2}-c)(\partial_t\varphi)^2+(c+\frac{2-N}{2N})\mid\nabla\varphi\mid^2\right\}=\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}\{F(\varphi)-c\varphi f(\varphi)\}. \tag{5.1}$$

By hypothesis both sides of this equation must be zero. In the case $c\in(\frac{N-2}{2N},\frac{1}{2}]$ this implies that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}\mid\nabla\varphi\mid^2 \;=\; 0 \;=\; \lim_{T\to\infty}\frac{1}{T}\int_0^T\|\nabla\varphi\|_2^2. \tag{5.2}$$

Parseval's relation for Hilbert space valued almost periodic functions ([LZ] pp 31) states that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\|\nabla\varphi\|_2^2=\sum_{k=1}^{\infty}\|a_k\|_2^2 \tag{5.3}$$

where $\sum_{k=1}^{\infty}a_ke^{i\omega_kt}$ is the Fourier series associated to $\nabla\varphi$. By the uniqueness of these series it follows from equations (5.2) and (5.3) that $\nabla\varphi(t)=0$ (in $L^2(\mathrm{R}^N)$) for all t which implies that, since $\varphi(t)\in L^2(\mathrm{R}^N)$, $\varphi(t)=0$ for all t.

If $c=\frac{N-2}{2N}$ then a similar argument applied to $\partial_t\varphi$ leads to the conclusion that $\partial_t\varphi(t)=0$ (in $L^2(\mathrm{R}^N)$) for all t. That is, $\varphi(t)=\psi$ for some $\psi\in L^2(\mathrm{R}^N)$ and for all t $\square$

Remark: Example 1 in Section 3 and Corollary 3.6 hold also for almost periodic solutions, and by the same proofs as presented there.

6 Appendix

In this appendix we illustrate how virial relations for NLW can be derived and understood in a natural way by formulating NLW as a variational problem. From this aspect the vector fields characterizing the virial relations derived in this paper define infinitesimal generators of transformation groups on the space $H^1(\mathrm{R}^N)$ on which the action functional is defined. Our purpose here is to describe virial relations and conservation laws from a common point of view; in terms of the behavior of the action under such transformation groups. All transformation groups generate virial relations. If the transformation group happens to leave the action invariant, i.e., is a symmetry group of the action, then the associated virial relation is "trivial" in the sense that it is merely the consequence of the fact that the integrand is a divergence. At the same time from this one can infer a conservation law for the associated Euler-Lagrange equation (Noether's Theorem). For general transformation groups the integrand is not necessarily a divergence. How far from a divergence it is produces the "nontrivial" virial relations derived in this paper.

Variational Calculus and Transformation Groups

Let X be a Banach space of functions defined on R^m and S an action functional $X \to \mathrm{R}$ of the form

$$S[\varphi] = \int_\Omega \mathcal{S}(\varphi, \nabla_m \varphi) \tag{A.1}$$

where $\mathcal{S} : \mathrm{R} \times \mathrm{R}^m \to \mathrm{R}$, $(u,p) \to \mathcal{S}(u,p)$ is the Lagrangian, ∇_m denotes the gradient with respect to all the variables, and $\Omega \subset \mathrm{R}^m$. We consider S as being associated to a differential equation $K(\varphi) = 0$ through the relation $S'[\varphi] = 0 \Leftrightarrow \varphi$ is a (weak) solution of $K(\varphi) = 0$, where $S'[\varphi]$ denotes the Fréchet derivative. In this case $K(\varphi) = 0$ is the Euler-Lagrange equation associated to S;

$$K(\varphi) = \mathcal{S}_u - \nabla_m \cdot \mathcal{S}_p \;=\; \frac{\partial \mathcal{S}}{\partial \varphi} - \frac{\partial}{\partial x_i}\frac{\partial \mathcal{S}}{\partial \varphi_i} = 0, \tag{A.2}$$

where $\varphi_i \equiv \frac{\partial \varphi}{\partial x_i}$ and summation over $i = 1, \ldots, m$ is implied.

For example, if we are interested in l-quasiperiodic solutions of NLW with frequency ω, then we formulate a variational problem on a space of (generating) functions defined on $\Omega_{N,l}$ by defining the Lagrangian,

$$\mathcal{S} = \mathcal{S}(\gamma, \nabla\gamma, \mathcal{D}\gamma) = -\frac{1}{2}(\mathcal{D}_\omega\gamma)^2 + \frac{1}{2} \mid \nabla\gamma \mid^2 + F(\gamma). \tag{A.3}$$

From this formula we see that a natural choice for X is $H^1(\Omega_{N,l})$. Then,

$$S[\gamma] \;=\; \int_{\Omega_{N,l}} \Big\{ -\frac{1}{2}(\mathcal{D}_\omega\gamma)^2 + \frac{1}{2}|\nabla\gamma|^2 + F(\gamma) \Big\} \tag{A.4}$$

and

$$S'[\gamma](\beta) = \int_{\Omega_{N,l}} \Big\{ -\mathcal{D}_\omega\gamma\mathcal{D}_\omega\beta + \nabla\gamma\cdot\nabla\beta + f(\gamma)\beta \Big\}. \tag{A.5}$$

(cf. equation (2.2)). To formulate NLW as a variational problem for almost periodic solutions $\varphi \in \mathcal{AP}$ we define the action,

$$S[\varphi] \;=\; \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathrm{R}^N} \Big\{ -\frac{1}{2}(\partial_t\varphi)^2 + \frac{1}{2} \mid \nabla\varphi \mid^2 + F(\varphi) \Big\} \tag{A.6}$$

so that

$$S'[\varphi](\psi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathrm{R}^N} \Big\{ -\partial_t\varphi\partial_t\psi + \nabla\varphi\cdot\nabla\psi - f(\varphi)\psi \Big\} \tag{A.7}$$

(cf. equation (4.5)).

We can show that the actions (A.4) and (A.6) are positive at critical points;

$$S'[\gamma] \;=\; 0 \implies S[\gamma] \;>\; 0 \quad \text{(cf. equation (A.4))} \tag{A.8}$$

and

$$S'[\varphi] \;=\; 0 \implies S[\varphi] \;>\; 0 \quad \text{(cf. equation (A.6))}\ . \tag{A.9}$$

This follows from the virial relation associated to the dilations in x (equations (3.4) and (5.1) with $c = 0$) from which we infer that

$$\int_{\Omega_{N,l}} F(\gamma) \;=\; \int_{\Omega_{N,l}} \Big\{ \frac{1}{2}(\mathcal{D}_\omega\gamma)^2 + \frac{2-N}{2N}|\nabla\gamma|^2 \Big\} \tag{A.10}$$

and

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N} F(\varphi) \;=\; \lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^N}\Big\{\frac{1}{2}(\partial_t\varphi)^2+\frac{2-N}{2N}|\nabla\varphi|^2\Big\}, \tag{A.11}$$

for solutions γ of NLW on $\Omega_{N,l}$ and almost periodic solutions φ of NLW on R^{N+1} respectively. Substituting these into (A.4) and (A.6) justifies the claims (A.8) and (A.9).

Returning to the abstract set-up, variations of S are defined through transformation groups acting on X. Let $T_\lambda : X \to X;\ \varphi \mapsto \varphi_\lambda \equiv T_\lambda\varphi$, with $T_0 = \mathbf{1}$, be a strongly continuous 1-parameter group of transformations with infinitesimal generator A, $A : D(A) \subset X \to X$, $D(A)$ denoting the domain of A. Here A is defined by $A\varphi = \frac{d}{d\lambda}\varphi_\lambda\big|_{\lambda=0}$. If $\varphi \in D(A)$ is a critical point of S, then applying the chain rule to the function $S[\varphi_\lambda] : \mathrm{R} \to \mathrm{R}$ we find that

$$\frac{d}{d\lambda}S[\varphi_\lambda]\Big|_{\lambda=0} \;=\; S'[\varphi](A\varphi) \;=\; 0. \tag{A.12}$$

If A is a differential operator then this equation is an integral formula involving the solution φ and its derivatives. It is the virial relation associated to the transformation group T_λ and corresponds to equation (2.4), as we will describe in more detail below.

A particular class of transformations on X arise from diffeomorphisms of R^m. Suppose $v : \mathrm{R}^m \to \mathrm{R}^m$ is a vector field on R^m that generates a global flow $\Phi_\lambda : \mathrm{R}\times\mathrm{R}^m \to \mathrm{R}^m$, ($\Phi_0 = \mathbf{1}$). Then the map $\varphi \mapsto \varphi_\lambda \equiv \varphi\circ\Phi_\lambda \equiv T_\lambda\varphi$ defines a 1-parameter group of transformations T_λ on X with infinitesimal generator $v\cdot\nabla_m$. For the remainder of the appendix we will always assume that T_λ is of this form.

Formally,

$$\frac{d}{d\lambda}S[\varphi_\lambda]\Big|_{\lambda=0} \;=\; S'[\varphi](v\cdot\nabla_m\varphi) \;=\; \int_\Omega\Big\{\mathcal{S}_u(v\cdot\nabla_m\varphi) \;+\; \mathcal{S}_p\cdot\nabla_m(v\cdot\nabla_m\varphi)\Big\}. \tag{A.13}$$

We use the word formally here because $v\cdot\nabla_m\varphi$ may not lie in X (that is, φ may not lie in $D(v\cdot\nabla_m)$) in which case a regularization procedure is required, as was done in the proof of Theorem 2.7. If φ is a critical point of S, then we have the virial relation

$$0 \;=\; \int_\Omega\Big\{\mathcal{S}_u(v\cdot\nabla_m\varphi) \;+\; \mathcal{S}_p\cdot\nabla_m(v\cdot\nabla_m\varphi)\Big\}. \tag{A.14}$$

We will see that this equation is precisely the virial relation (2.4) of Theorem 2.7 corresponding to the vector field v. The hypothesis of Theorem 2.7 concerning vector fields on R^N (cf. (2.3)) guarantees that these vector fields generate global flows on R^N, which in turn defines transformation groups on $H^1(\Omega_{N,l})$. We now consider how such transformation groups affect the action S and in this way we will distinguish between transformation groups that preserve S or not and the consequences thereof for the associated virial relations.

Symmetries, Conservation Laws, and Virial Relations

Definition A.1 *$T_\lambda : X \to X$ is a* symmetry group of S *if $S[\varphi_\lambda] = S[\varphi]$ for all $\varphi \in X$ and for all $\lambda \in \mathrm{R}$, where $\varphi_\lambda \equiv T_\lambda \varphi \equiv \varphi \circ \Phi_\lambda$.*

Lemma A.2 (Noether) *If T_λ is a symmetry group of S with infinitesimal generator $v \cdot \nabla_m$, then for any $\varphi \in X$ the expression*

$$\mathcal{S}_u(v \cdot \nabla_m \varphi) \ + \ \mathcal{S}_p \cdot \nabla_m (v \cdot \nabla_m \varphi)$$

is a divergence (cf. (A.13)).

In fact ([GF] Thm 2 §37),

$$\mathcal{S}_u(v \cdot \nabla_m \varphi) \ + \ \mathcal{S}_p \cdot \nabla_m (v \cdot \nabla_m \varphi) \ = \ \nabla_m \cdot \mathcal{S} v. \tag{A.15}$$

Therefore, if T_λ is a symmetry group of S and φ is a critical point of S, then using the Euler-Lagrange equation (A.2) we obtain from (A.15)

$$\begin{aligned} 0 \ &= \ \mathcal{S}_u(v \cdot \nabla_m \varphi) \ + \ \mathcal{S}_p \cdot \nabla_m (v \cdot \nabla_m \varphi) \ - \ \nabla_m \cdot \mathcal{S} v \\ &= \ (\mathcal{S}_u - \nabla_m \cdot \mathcal{S}_p) v \cdot \nabla_m \varphi \ + \ \nabla_m \cdot (\mathcal{S}_p v \cdot \nabla_m \varphi - \mathcal{S} v) \\ &= \ \nabla_m \cdot (\mathcal{S}_p v \cdot \nabla_m \varphi - \mathcal{S} v). \end{aligned} \tag{A.16}$$

We describe how this formula leads to a conservation law.

Consider NLW on $\Omega_{N,l}$, where the action S is defined by (A.4). Here $\nabla_m = (\nabla, \mathcal{D})$. Writing $v = (v_a, v_b) \in \mathrm{R}^{N+l}$ where v_a is a vector field on R^N and v_b is a vector field on T^l, the last line in (A.16) reads

$$0 \;=\; \nabla\cdot\Big(\frac{\partial\mathcal{S}}{\partial\nabla\gamma}v\cdot\nabla_m\gamma-\mathcal{S}v_a\Big)\;+\;\mathcal{D}\cdot\Big(\frac{\partial\mathcal{S}}{\partial\mathcal{D}\gamma}v\cdot\nabla_m\gamma-\mathcal{S}v_b\Big). \tag{A.17}$$

Here $\partial\mathcal{S}/\partial\nabla\gamma$ and $\partial\mathcal{S}/\partial\mathcal{D}\gamma$ denote the vectors $(\partial\mathcal{S}/\partial\gamma_1,\dots,\partial\mathcal{S}/\partial\gamma_N)$ and $(\partial\mathcal{S}/\partial\gamma_{N+1},\dots,\partial\mathcal{S}/\partial\gamma_{N+l})$ respectively, where $\gamma_i=\partial\gamma/\partial x_i,\ \ i=1,\dots,N$, and $\gamma_{N+j}=\partial\gamma/\partial\theta_j,\ j=1,\dots,l$. If we set

$$\mathbf{E}_v(\gamma)\;\equiv\;\int_{\mathrm{R}^N}\Big(\frac{\partial\mathcal{S}}{\partial\mathcal{D}\gamma}v\cdot\nabla_m\gamma-\mathcal{S}v_b\Big), \tag{A.18}$$

and if $(\partial\mathcal{S}/\partial\nabla\gamma)v\cdot\nabla_m\gamma-\mathcal{S}v_a$ vanishes sufficiently rapidly as $|x|\to\infty$, then from (A.17) and the divergence theorem we then have the conservation law

$$\mathcal{D}\cdot\mathbf{E}_v(\gamma)=0. \tag{A.19}$$

As an example, let v_T be the infinitesimal generator of translation on T^l along Γ_ω (cf. Definition 2.1); $v_T=(0,\,\omega)$. Then,

$$\mathbf{E}_{v_T}(\gamma)\;=\;-\omega\int_{\mathrm{R}^N}\Big\{\frac{1}{2}(\mathcal{D}_\omega\gamma)^2+\frac{1}{2}|\nabla\gamma|^2+F(\gamma)\Big\}\;\equiv\;-\omega E(\gamma) \tag{A.20}$$

where $E(\gamma)$ is the energy of γ. In this case $\mathcal{D}_\omega E(\gamma)=\mathcal{D}\cdot\mathbf{E}_{v_T}(\gamma)=0$. Therefore, $E(\gamma)$ is constant along Γ_ω ($\mathcal{D}_\omega$ is the directional derivative along Γ_ω). If $E(\gamma)$ is continuous, then because Γ_ω is dense in T^l, $E(\gamma)$ is constant on T^l. Considering quasiperiodic solutions φ of NLW; $\varphi(x,t)=\gamma(x,\omega t)$, the argument just given shows that the energy $E(\varphi)$ of φ is conserved;

$$\frac{d}{dt}E(\varphi)\;=\;0\quad\text{where}\quad E(\varphi)\;\equiv\;\int_{\mathrm{R}^N}\Big\{\frac{1}{2}(\partial_t\varphi)^2+\frac{1}{2}|\nabla\varphi|^2+F(\varphi)\Big\}. \tag{A.21}$$

The energy of a solution γ, like any functional of γ that is constant on T^l, can be used to derive an integral identity simply by integrating it over T^l;

$$(2\pi)^l E(\gamma)\;=\;\int_{\mathrm{T}^l}E(\gamma)\;=\;\int_{\Omega_{N,l}}\Big\{\frac{1}{2}(\mathcal{D}_\omega\gamma)^2+\frac{1}{2}|\nabla\gamma|^2+F(\gamma)\Big\}. \tag{A.22}$$

Using (A.10) we derive the inequality

$$E(\gamma) \;=\; (2\pi)^{-l}\int_{\Omega_{N,l}} \left\{(\mathcal{D}_\omega\gamma)^2 + \frac{1}{N}|\nabla\gamma|^2\right\} \;\geq\; 0, \tag{A.23}$$

which shows that the energy of a quasiperiodic solution is positive, and zero if and only if the solution is zero. By performing the same arguments with the action defined by (A.6) we can see that the energy of almost periodic solutions is positive also.

We return to general transformations on X (i.e., not necessarily symmetry groups of S). For an arbitrary transformation group T_λ the formula (A.15) may not be true. In this regard, to each T_λ we define a function g on X, $g = g(\varphi, \nabla_m\varphi, v)$ where $v \cdot \nabla_m$ is the infinitesimal generator of T_λ, by the following formula;

$$g(\varphi) \;\equiv\; \mathcal{S}_u(v\cdot\nabla_m\varphi) \;+\; \mathcal{S}_p\cdot\nabla_m(v\cdot\nabla_m\varphi) - \nabla_m\cdot\mathcal{S}v. \tag{A.24}$$

Then, when T_λ is a symmetry group of S, $g \equiv 0$ by Lemma A.2. Otherwise g may not be zero. This motivates us to think of the function g as measuring of how far T_λ is from being a symmetry of S.

Let us now consider the variational problem associated to NLW on $\Omega_{N,l}$ and let v be the vector field on R^N that generates the flow Φ_λ through which T_λ is defined. Here the independent variables are space-time variables $(x,\theta) \in \Omega_{N,l}$ so that $\nabla_m = \nabla_{x,\theta} \equiv (\nabla, \mathcal{D})$. With g defined by (A.24), and noting that a divergence term vanishes when integrated over $\Omega_{N,l}$ (viz. the term $\nabla_m\cdot\mathcal{S}v$), if γ is a critical point of S the virial relation associated to the vector field v (equation (A.13)) can be written as

$$0 \;=\; S'[\gamma](v\cdot\nabla\gamma) \;=\; \int_{\Omega_{N,l}} g. \tag{A.25}$$

If in addition γ solves the Euler-Lagrange equation, then rewriting (A.24) we have

$$\begin{aligned} g &= (\mathcal{S}_u - \nabla_{x,\theta}\cdot\mathcal{S}_p)v\cdot\nabla\varphi + \nabla_{x,\theta}\cdot(\mathcal{S}_p v\cdot\nabla\varphi - \mathcal{S}v) \\ &= \nabla_{x,\theta}\cdot(\mathcal{S}_p v\cdot\nabla\varphi - \mathcal{S}v) \end{aligned}$$

$$= \quad \mathcal{D}\cdot\mathbf{e} \;+\; \nabla\cdot\mathbf{p} \tag{A.26}$$

where $\mathbf{e} = (\partial\mathcal{S}/\partial\mathcal{D}\gamma)v\cdot\nabla\gamma$ and $\mathbf{p} = (\partial\mathcal{S}/\partial\nabla\gamma)v\cdot\nabla\gamma - \mathcal{S}v$ (cf. (A.17)). With the definitions $\mathbf{E}_v(\gamma) \equiv \int_{\mathrm{R}^N}\mathbf{e}$ and $G \equiv \int_{\mathrm{R}^N} g$, we see from (A.26) that G acts as source of $\mathbf{E}_v(\gamma)$;

$$\mathcal{D}\cdot\mathbf{E}_v(\gamma) \;=\; G. \tag{A.27}$$

This corroborates the statement made above about g measuring how far T_λ is from being a symmetry group of S: if T_λ is a symmetry group of S then $\mathbf{E}_v(\gamma)$ as defined through (A.26) is divergence free; if T_λ is not a symmetry group of S then the divergence of $\mathbf{E}_v(\gamma)$ is determined by g.

For the Lagrangian (A.3) associated to NLW on $\Omega_{N,l}$, equation (A.24) is

$$g = tr\,dv\,\left(\frac{1}{2}\,(\mathcal{D}_\omega\gamma)^2 - F(\gamma)\right) + \nabla\gamma\cdot[dv - \frac{1}{2}tr\,dv\,\mathbf{1}]\nabla\gamma \tag{A.28}$$

which was derived in Theorem 2.7 (cf. (A.25)). By Lemma A.2 this function vanishes if T_λ is a symmetry group of NLW. For example, rotations and translations of R^N are symmetries of NLW. In the former case $dv \in so(N)$ and in the latter $v = constant$. In both cases it is apparent from (A.28) that $g \equiv 0$.

As another application of the variational calculus, we point out that the formula (A.28) may be derived directly from (A.13) (cf. (A.25)) as follows. First note that

$$S[\gamma_\lambda] \;=\; \int_{\Omega_{N,l}} \Big\{ -\frac{1}{2}(\mathcal{D}_\omega\gamma_\lambda)^2 + \frac{1}{2}|\nabla\gamma_\lambda|^2 + F(\gamma_\lambda)\Big\}\,dxd\theta.$$

By making the change of variables $y = \Phi_\lambda(x)$, this becomes

$$S[\gamma_\lambda] \;=\; \int_{\Omega_{N,l}} \Big\{ -\frac{1}{2}(\mathcal{D}_\omega\gamma)^2 + \frac{1}{2}|\nabla\gamma_\lambda\circ\Phi_{-\lambda}|^2 + F(\gamma)\Big\}\,det\,(J_\lambda\circ\Phi_{-\lambda})^{-1}\,dyd\theta$$

where $[J_\lambda(x)]_{i,j} = \partial\Phi^i_\lambda/\partial x_j$ is the Jacobian matrix associated to the transformation $x \mapsto y$

and $(J_\lambda \circ \Phi_{-\lambda})^{-1}$ is the Jacobian matrix associated to the inverse mapping $y \mapsto x$. Note that

$$\frac{\partial}{\partial \lambda} det\,(J_\lambda \circ \Phi_{-\lambda})^{-1}\Big|_{\lambda=0} \;=\; tr\,dv.$$

In addition,

$$|\nabla \gamma_\lambda(x)|^2 \;=\; \nabla \gamma_\lambda(x) \cdot \nabla \gamma_\lambda(x) \;=\; A \nabla \gamma(y) \cdot \nabla \gamma(y), \quad \text{where } [A]_{i,j} = \sum_{k=1}^{N} \frac{\partial \Phi_\lambda^i}{\partial x_k} \frac{\partial \Phi_\lambda^j}{\partial x_k},$$

and

$$\frac{\partial}{\partial \lambda}\Big[A \circ \Phi_\lambda\Big]\Big|_{\lambda=0} \;=\; dv + dv^T,$$

which together imply that

$$\frac{\partial}{\partial \lambda} |\nabla \gamma_\lambda \circ \Phi_{-\lambda}|^2\,\Big|_{\lambda=0} \;=\; 2\,\nabla \gamma \cdot\, dv\,\nabla \gamma.$$

Therefore,

$$\begin{aligned}\frac{d}{d\lambda} S[\gamma_\lambda]\Big|_{\lambda=0} &= \int_{\Omega_{N,l}} \frac{\partial}{\partial \lambda}\Big\{-\frac{1}{2}(\mathcal{D}_\omega \gamma)^2 + \frac{1}{2}|\nabla \gamma_\lambda \circ \Phi_{-\lambda}|^2 + F(\gamma)\Big\} det\,(J_\lambda \circ \Phi_{-\lambda})^{-1}\Big|_{\lambda=0} dy d\theta \\ &= \int_{\Omega_{N,l}} g\end{aligned}$$

with g as in (A.28).

References

[A] Adams, R.A. *Sobolev Spaces.* Academic Press, New York (1975).

[B] Berger, M.S. *On the Existence and Structure of Stationary States for a Nonlinear Klein-Gordon Equation.* J. Fun. Ana., **9**, 249-261 (1972).

[BCN] Brezis, H., Coron, J.-M., Nirenberg, L. *Free Vibrations for a Nonlinear Wave Equation and a Theorem of P. Rabinowitz.* Comm. Pure Appl. Math. **33**, 667-689 (1980).

[BL] Berestzcki, H., Lions, P.-L. *Nonlinear Scalar Field Equations I, Existence of a Ground State.* Arch. Rat. Mech. Ana. **84**, 313-345 (1983).

[BrLi] Brezis, H., Lieb, E. *Minimum Action Solutions of Some Vector Field Equations.* Commun. Math. Phys. **96**, 97-113 (1984).

[BMW] Birnir, B., McKean, H.P., Weinstein, A. *The Rigidity of Sine-Gordon Breathers.* Comm. Pure Appl. Math., **47**, 1043-1051 (1994).

[C] Corduneanu, C. *Almost Periodic Functions.* John Wiley and Sons, New York (1968)

[Co] Coron, J.-M. *Minimal Period for a Vibrating String of Infinite Length.* C.R. Acad. Sc. Paris, **294**, 127-129 (1982).

[CW] Craig, W., Wayne, C.E. *Newton's Method and Periodic Solutions of Nonlinear Wave Equations.* Commun. Pure. Appl. Math., **46**, 1409-1498 (1993).

[D] Denzler, J. *Nonpersistence of Breathers for the Perturbed Sine Gordon Equation.* Commun. Math. Phys. **158**, #2, 397-430 (1993).

[DS] Dunford, N. and Schwartz, J.T. *Linear Operators.* John Wiley and Sons, New York (1957).

[GF] Gelfand, I.M., and Fomin, S.V. *Calculus of Variations.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1963).

[GV] Ginibre, J., Velo, G. *The Global Cauchy Problem for the Nonlinear Klein-Gordon Equation.* Math. Z. **189**, 487-505 (1985).

[H] Hale, J.K. *Periodic Solutions of a Class of Hyperbolic Equations Containing a Small Parameter.* Arch. Rat. Mech. Anal., **23**, 380-398 (1967).

[HR] Hewitt, E., Ross, K.A. *Abstract Harmonic Analysis.* 2 vols. Springer Verlag, Berlin. (1963-1970).

[Ki] Kichenassamy, S. *Breather Solutions of the Nonlinear Wave Equation.* Comm. Pure

Appl. Math. **44**, 789-818 (1991).

[Kr] Krasnosel'skii, M.A. *Topological Methods in the Theory of Nonlinear Integral Equations.* Pergamon Press, New York (1964).

[LL] Landau, L.D., Lifshitz, E.M. *Mechanics.* Pergamon Press, New York (1976).

[LZ] Levitan, B.Z., and Zhikov, V.V. *Almost Periodic Functions and Differential Equations.* Cambridge University Press, New York (1982).

[M] Milgrom, M. *The Virial Theorem for Action-Governed Theories.* Physical Letters A, **190**, 17-21 (1994).

[MS] McLaughlin, D.W., Scott, A.C. *Perturbation Analysis of Fluxon Dynamics.* Physical Review A, **18**, # 4, 1652-1680 (1978).

[NMPZ] Novikoff, S., Manakov, S.V., Pitaevskii, L.P., Zakharov, V.E. *Theory of Solitons: The Inverse Scattering Method.* Consultants Bureau, New York and London (1984).

[Pe] Petersen, K., *Ergodic Theory.* Cambridge University Press, New York (1983).

[Py] Pyke, R. *A Characterization of Nonlinear Bound States.* In preparation.

[Po] Pohozaev, S.I. *Eigenfunctions of the Equation* $\Delta u + \lambda f(u) = 0$. Sov. Math. Doklady, **5**, 1408-1411 (1965).

[PS] Pyke, R.M., Sigal, I.M. *Nonlinear Wave Equations: Constraints on Periods and Exponential Bounds.* Submitted to Duke Math.J.

[R] Ruelle, D. *A Remark on Bound States in Potential Scattering Theory.* Nuovo Cimento **61A**, 655-662 (1969).

[Sc] Scarpellini, B. *Decay Properties of Periodic, Radially Symmetric Solutions of Sine-Gordon-type Equations.* Math. Meth. Appl. Sci., **16**, 359-378 (1993).

[Si] Sigal, I.M., *Nonlinear Wave and Schrodinger Equations I. Instability of Periodic and Quasiperiodic Solutions.* Comm. Math. Phys., **153**, 297-320 (1993).

[Sm] Smiley, M.W., *Complex-Valued Time-Periodic Solutions and Breathers of Nonlinear Wave Equations.* Diff. and Int. Eq. **4**#4, 851-859 (1991).

[Str1] Strauss, W. *Existence of Solitary Waves in Higher Dimensions.* Commun. Math. Phys., **55**, 149-162 (1977).

[Str2] Strauss, W. *Nonlinear Wave Equations.* Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, no. 73, American Mathematical Society

(1989).

[SK] Segur,H., Kruskal, M. *Nonexistence of Small Amplitude Breather Solutions in* ϕ^4 *Theory.* Phys. Rev. Lett. **58**, 747-750 (1987).

[SV] Scarpellini, B., and Vuillermot, P.-A. *Smooth Manifolds for Semilinear Wave Equations on* R^2*: On the Existence of Almost-Periodic Breathers.* J. Diff. Eq., **77**, 123-166 (1989).

[V1] Vuillermot, P.-A. *Nonexistence of Spacially Localized Vibrations for a Class of Nonlinear Wave Equations.* Comment. Helvetici **62**, 573-586 (1987).

[V2] Vuillermot, P.-A. *Quasiperiodic Soliton Solutions to Nonlinear Klein-Gordon Equations on* R^2. Math. Z., **203**, 235-253 (1990).

[V3] Vuillermot, P.-A. *Periodic Soliton Solutions to Nonlinear Klein-Gordon Equations on* R^2. Diff. Integral Eq., **3**, 541-570 (1990).

[Wa] Wayne, C.E. *Periodic and Quasiperiodic Solutions of Nonlinear Wave Equations via KAM Theory.* Comm. Math. Phys. **127**, 479-528 (1990).

[We] Weinstein, A. *Periodic Nonlinear Waves on a Half Line.* Comm. Math. Phys. **99**, 385-388 (1985). Erratum: Comm. Math. Phys., **107**, 177 (1986).