1 Graph Basics

What is a graph?

Graph: a *graph* G consists of a set of vertices, denoted $V(G)$, a set of edges, denoted $E(G)$, and a relation called *incidence* so that each edge is incident with either one or two vertices - its *ends*. We assume that $V(G)$ and $E(G)$ are finite.

Adjacent: Two distinct vertices u, v are *adjacent* if there is an edge with ends u, v. In this case we let uv denote such an edge.

Loops, Parallel Edges, and Simple Graphs: An edge with only one endpoint is called a *loop*. Two (or more) distinct edges with the same ends are called *parallel*. A graph is *simple* if it has no loops or parallel edges.

Drawing: It is helpful to represent graphs by drawing them so that each vertex corresponds to a distinct point, and each edge with ends u, v is realized as a curve which has endpoints corresponding to u and v (a loop with end u is realized as a curve with both endpoints corresponding to u). Below is a drawing of a famous graph.

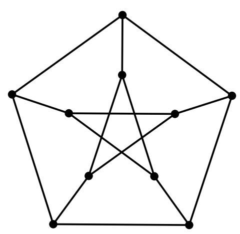

Figure 1: the Petersen Graph

Standard Graphs

Null graph	the (unique) graph with no vertices or edges.
Complete graph K_n	a simple graph on n vertices every two of which are adjacent.
Path P_n	a graph whose vertex set may be numbered $\{v_1, \ldots, v_n\}$ and edges may be numbered $\{e_1, \ldots, e_{n-1}\}$ so that every e_i has endpoints v_i and v_{i+1}. The *ends* of the path are v_1, v_n.
Cycle C_n	a graph whose vertex set may be numbered $\{v_1, \ldots, v_n\}$ and edges may be numbered $\{e_1, \ldots, e_n\}$ so that every e_i has endpoints v_i and v_{i+1} (*modulo* n)

Subgraph: If G is a graph and H is another graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ with the same incidence relation on these sets, we say that H is a *subgraph* of G and we write $H \subseteq G$. If $H_1, H_2 \subseteq G$ we let $H_1 \cup H_2$ denote the subgraph of G with vertices $V(H_1) \cup V(H_2)$ and edges $E(H_1) \cup E(H_2)$. We define $H_1 \cap H_2$ analogously. A *path* (*cycle*) of G is a subgraph of G which is a path (cycle).

Degree: The *degree* of a vertex v, denoted $deg(v)$ is the number of edges incident with v where loops are counted twice.

Theorem 1.1 *For every graph G*

$$\sum_{v \in V(G)} deg(v) = 2|E(G)|$$

Proof: Each edge contributes exactly 2 to the sum of the degrees. □

Corollary 1.2 *Every graph has an even number of vertices of odd degree.*

Isomorphic: We say that two graphs G and H are *isomorphic* if there exist bijections between $V(G)$ and $V(H)$ and between $E(G)$ and $E(H)$ which preserve the incidence relation. Informally, we may think of G and H as isomorphic if one can be turned into the other by renaming vertices and edges.

Connectivity

Walk: A *walk* W in a graph G is a sequence $v_0, e_1, v_1, \ldots, e_n v_n$ so that $v_0, \ldots, v_n \in V(G)$, $e_1, \ldots, e_n \in E(G)$, and every e_i has ends v_{i-1} and v_i. We say that W is a walk *from v_0 to v_n*

with *length* n. If $v_0 = v_n$ we say the W is *closed*, and if $v_0, \ldots, v_n$ are distinct, then (abusing notation) we call this walk a *path*.

Connected: A graph G is *connected* if for every $u, v \in V(G)$ there is a walk from u to v.

Proposition 1.3 *If there is a walk from u to v, then there is a path from u to v.*

Proof: Choose a walk W from u to v, say $u = v_0, e_1, v_1, \ldots, e_n, v_n = v$ of minimum length. Then $v_0, v_1, \ldots, v_n$ are distinct, so W is a path. $\square$

Proposition 1.4 *If G is not connected, there is a partition $\{X, Y\}$ of $V(G)$ so that no edge has an end in X and an end in Y.*

Proof: Choose u and v so that there is no walk from u to v. Now, let $X \subseteq V(G)$ be the set of all vertices w so that there exists a walk from u to w and let $Y = V(G) \setminus X$. Then $u \in X$ and $v \in Y$ so $X, Y \neq \emptyset$. Furthermore, there cannot be an edge with an end in X and an end in Y (why?). $\square$

Proposition 1.5 *If $H_1, H_2 \subseteq G$ are connected and $H_1 \cap H_2 \neq \emptyset$, then $H_1 \cup H_2$ is connected.*

Proof: Let $u, v \in V(H_1 \cup H_2)$ and choose $p \in V(H_1) \cap V(H_2)$. Now, there exists a walk W_1 from u to p in $H_1 \cup H_2$ (in H_1 if $u \in V(H_1)$ and otherwise in H_2). Similarly, there is a walk W_2 from p to v. By concatenating W_1 and W_2, we obtain a walk from u to v. Since u, v were arbitrary vertices, it follows that G is connected. $\square$

Component: A *component* of G is a maximal nonempty connected subgraph of G. We let *comp*(G) denote the number of components of G.

Theorem 1.6 *Every vertex is in a unique component of G.*

Proof: Every vertex v is in a connected subgraph (consisting of only that vertex with no edges), so v must be contained in at least one component. However, by the previous proposition, no two components can share a vertex. It follows that every vertex is in exactly one, as required. $\square$

Cut-edge: An edge e is called a *cut-edge* if there is no cycle containing e.

Theorem 1.7 *Let G be a graph, let $e \in E(G)$ be an edge with ends u, v and let G' be the graph obtained from G by deleting e. Then one of the following holds:*

1. *e is a cut-edge of G and u, v are in different components of G'.*
2. *e is not a cut-edge of G and u, v are in the same component of G'.*

Further, $comp(G') = comp(G) + 1$ in the first case and $comp(G') = comp(G)$ in the second.

Proof: To see that 1 or 2 holds, observe that e is in a cycle of $G \Leftrightarrow$ there is a path in G' from u to $v \Leftrightarrow u$ and v are in the same component of G'. To see how $comp(G)$ and $comp(G')$ are related, let $H_1, \ldots, H_m$ be the components of G' and assume that $u \in V(H_i)$ and $v \in V(H_j)$. If $i \neq j$, then $H_i \cup H_j$ together with the edge e is a component of G and $comp(G') = comp(G) + 1$. If $i = j$, the vertex set of every component of G is also the vertex set of a component of G', so we have $comp(G') = comp(G)$. □

Bipartite, Eulerian, and Hamiltonian

Bipartite: A *bipartition* of a graph G is a pair (A, B) of disjoint subsets of $V(G)$ with $A \cup B = V(G)$ so that every edge has one end in A and one end in B. We say that G is *bipartite* if it has a bipartition.

Complete Bipartite: The *complete bipartite graph* $K_{m,n}$ is a simple bipartite graph with bipartition (A, B) where $|A| = m$, $|B| = n$, and every vertex in A is adjacent to every vertex in B.

Theorem 1.8 *For every graph G, the following are equivalent (the length of a cycle is its number of edges).*

(i) *G is bipartite*

(ii) *G has no cycle of odd length*

(iii) *G has no closed walk of odd length*

Proof: First we show that (ii) and (iii) are equivalent. If G has a cycle of odd size, then it has a closed walk of odd length, so (ii) implies (iii) On the other hand, if G has a closed walk of odd length, then choose such a walk $v_0, e_1, \ldots, e_n, v_n$ of minimum length. We claim that $\{v_0, \ldots, v_n\}$ and $\{e_1, \ldots, e_n\}$ form a cycle of odd length - or equivalently $v_1, \ldots, v_n$ are distinct. Suppose (for a contradiction) that $v_i = v_j$ for some $1 \leq i < j \leq n$. If $j - i$ is odd, then $v_i, e_i, \ldots, v_j$ is a shorter closed walk of odd length - a contradiction. If $j - i$ is even, then $v_0, e_1, v_1, \ldots, v_i, v_{j+1}, \ldots, v_n$ is a shorter closed walk of odd length - again contradicting our assumption. Thus (iii) implies (ii).

It is immediate that (i) implies (iii), so to complete the proof, it suffices to show that (iii) implies (i) To do this, let G be a graph which satisfies (iii), and assume (without loss) that G is connected. Now choose a vertex $u \in V(G)$, let $A \subseteq V(G)$ ($B \subseteq V(G)$) be the set of all vertices v so that there is a path of odd length (even length) from u to v. Suppose (for a contradiction) that there exists a vertex $w \in A \cap B$. Then there is a walk W_1 from u to w of odd length, and a walk W_2 from u to w of even length. By concatenating W_1 with the reverse of W_2 we obtain a closed walk of odd length - a contradiction. It follows that $A \cap B = \emptyset$. Since G is connected, we have $A \cup B = V(G)$. Now, (A, B) is a bipartition of G, so G satisfies (i) - as required. □

Deletion: If G is a graph and $S \subseteq E(G)$, we let $G - S$ denote the graph obtained from G by deleting every edge in S. Similarly, if $X \subseteq V(G)$, we let $G - X$ denote the graph obtained from G by deleting every vertex in X and any edge incident with such a vertex.

Induced Subgraph: A subgraph $H \subseteq G$ is *induced* if $E(H)$ contains every edge with both ends in $V(H)$. Equivalently, H is an induced subgraph of G if and only if $H = G - (V(G) \setminus V(H))$.

Theorem 1.9 *If G is a simple graph, then G is bipartite if and only if every induced cycle of G has even length.*

Proof: In light of the above result, to prove this theorem, we need only show that every graph with a cycle of odd length has an induced cycle of odd length. To see this G be a simple graph which is not bipartite, and choose an odd cycle $C \subseteq G$ of shortest length. If C is not induced, say $e \in E(G) \setminus E(C)$ has both ends in $V(C)$, then there is a smaller odd cycle using edges in $E(C) \cup \{e\}$ - a contradiction. Thus, C is an induced cycle of odd length - as required. □

Theorem 1.10 *Every loopless graph G has a bipartite subgraph H with $|E(H)| \geq \frac{1}{2}|E(G)|$.*

Proof: Choose disjoint sets $A, B \subseteq V(G)$ with $A \cup B = V(G)$ so that the number of edges of G with one end in A and one end in B is maximum. Let H be the subgraph of G with vertex set $V(G)$ and all edges with one end in A and one end in B. Let H' be the subgraph of G with vertex set $V(G)$ and all edges with either both ends in A or both ends in B. Let $v \in A$ and assume that v has s edges with other endpoint in A and t edges with other endpoint in B. If $s > t$, then moving v to B increases the number of edges with one end in A and one in B - a contradiction. Thus, the degree of v in H is at least the degree of v in H'. Since this holds for every vertex, it follows from Theorem 1.1 that $|E(H)| \geq |E(H')|$ and this implies $|E(H)| \geq \frac{1}{2}|E(G)|$ as required. $\square$

Proposition 1.11 *If G is a connected graph in which every vertex has degree ≥ 2, then G contains a cycle.*

Proof: If G has a loop edge, then this edge with its endpoint forms a cycle. Thus, we may assume (without loss) that G has no loop. Now, construct a walk $v_0, e_1, v_2, \ldots$ in a greedy manner by beginning at a vertex v_0, following an edge e_1 to a new vertex v_1, then following a new edge $e_2 \neq e_1$ to a vertex v_2, and so on (maintaining the property that $e_{i+1} \neq e_i$), stopping when we first revisit a vertex. Since each vertex has degree ≥ 2 (and no loops) it is always possible to choose e_{i+1} with $e_{i+1} \neq e_i$. Since $V(G)$ is finite, at some point we have $v_i = v_j$ for some $i < j$ and our procedure terminates. Now $v_i, e_{i+1}, \ldots, v_j$ forms a cycle. $\square$

Proposition 1.12 *If G is a graph in which every vertex has even degree, then there exists a list of cycles $C_1, \ldots, C_m$ of G so that every edge appears in exactly one of these cycles.*

Proof: Choose a maximal list of cycles $C_1, \ldots, C_m$ which are pairwise edge-disjoint (i.e. they have disjoint edge sets). Suppose (for a contradiction) that some component H of $G - (\cup_{i=1}^m E(C_i))$ has at least one edge. Since every vertex of G has even degree, it follows that every vertex in H has even degree. Since H is connected with $E(H) \neq \emptyset$, it then follows that every vertex of H has degree ≥ 2. But then, the previous proposition shows that H contains a cycle, contradicting our choice of $C_1, \ldots, C_m$. Thus $\cup_{i=1}^m E(C_i) = E(G)$, and every edge occurs exactly once in $C_1, \ldots, C_m$ as required. $\square$

Eulerian: A closed walk of a graph G is called *Eulerian* if it uses every edge exactly once. A graph is *Eulerian* if it has an Eulerian walk.

Theorem 1.13 *A connected graph G is Eulerian if and only if every vertex of G has even degree.*

Proof: The "only if" direction is immediate. For the "if" direction, assume that every vertex of G has even degree, and choose a closed walk $v_0, e_1, \ldots, v_n$ of maximum length subject to the constraint that each edge is used at most once. Let $S = \{e_1, \ldots, e_n\}$, and let $X = \{v_0, \ldots, v_n\}$. Suppose (for a contradiction) that $S \neq E(G)$. We claim that there must exist an edge $e \in E(G) \setminus S$ with at least one end in X. This is certainly true if $X = V(G)$. If $X \subset V(G)$, then since G is connected, there must be an edge e with one end in X and one in $V(G) \setminus X$, so $e \in E(G) \setminus S$ has one end in X. Now, let $v_i \in X$ be an end of e and consider the component H of $G - S$ which contains e and v_i. It follows from our assumptions that every vertex of H has even degree, so by the previous proposition, we may choose a cycle C of H which contains the edge e (and thus the vertex v_i). Now we may extend the walk W by taking the initial part of this walk from v_0 to v_i, then traversing the cycle C once from v_i back to itself, and then taking the final part of W from v_i to v_n. This new walk contradicts our original choice. Thus $S = E(G)$, and W is an Eulerian walk. □

Hamiltonian: A cycle C of a graph G is *Hamiltonian* if $V(C) = V(G)$. A graph is *Hamiltonian* if it has a Hamiltonian cycle.

Observation 1.14 *Let G be a graph and $X \subseteq V(G)$. If $|X| < comp(G - X)$, then G is not Hamiltonian.*

Proof: Suppose (for a contradiction) that G has a Hamiltonian cycle $C \subseteq G$. Then for every $X \subseteq V(G)$ we must have $|X| \geq comp(C - X) \geq comp(G - X)$, a contradiction. □

Theorem 1.15 *Let G be a simple graph with $n \geq 3$ vertices. If $deg(u) + deg(v) \geq n$ for every two non-adjacent vertices u, v, then G is Hamiltonian.*

Proof: We proceed by induction on $t = \binom{n}{2} - |E(G)|$. If $t = 0$, then G is complete, so it has a Hamiltonian cycle. Thus we may assume that $t > 0$ and we may choose two distinct non-adjacent vertices u, v. Now, add a new edge with e with ends u, v to form the graph

G'. By induction, G' has a Hamiltonian cycle. If this cycle does not use e, then it is also a Hamiltonian cycle of G, so we are done. Thus, we may assume that this cycle uses e. Therefore, we may number $V(G)$ as $v = v_1, v_2, \ldots, v_n = u$ so that v_i is adjacent to v_{i+1} for $1 \leq i \leq n-1$. Set

$$\begin{aligned} P &= \{v_i : i \geq 2 \text{ and } v_i \text{ is adjacent to } v_1\} \\ Q &= \{v_i : i \geq 2 \text{ and } v_{i-1} \text{ is adjacent to } v_n\} \end{aligned}$$

Then $|P| + |Q| = deg(v) + deg(u) \geq n$ and since $P \cup Q \subseteq \{v_2, \ldots, v_n\}$, it follows that there exists $2 \leq i \leq n$ with $v_i \in P \cap Q$, so there is an edge e with ends v_1 and v_i and an edge e' with ends v_n and v_{i-1}. Using these two edges, we may form a Hamiltonian cycle in G as desired. $\square$