4 Connectivity

2-connectivity

Separation: A *separation* of G of order k is a pair of subgraphs (H, K) with $H \cup K = G$ and $E(H \cap K) = \emptyset$ and $|V(H) \cap V(K)| = k$. Such a separation is *proper* if $V(H) \setminus V(K)$ and $V(K) \setminus V(H)$ are nonempty.

Observation 4.1 *G has a proper separation of order 0 if and only if G is disconnected.*

Cut-vertex: A vertex v is a *cut-vertex* if $comp(G - v) > comp(G)$.

Observation 4.2 *If G is connected, then v is a cut-vertex of G if and only if there exists a proper 1-separation (H, K) of G with $V(H) \cap V(K) = \{v\}$.*

Proposition 4.3 *Let e, f be distinct non-loop edges of the graph G. Then exactly one of the following holds:*

(i) *There exists a cycle C with $e, f \in E(C)$*

(ii) *There is a separation (H, K) of order ≤ 1 with $e \in E(H)$ and $f \in E(K)$.*

Proof: It is clear that (i) and (ii) are mutually exclusive, so it suffices to show that (i) or (ii) holds. For this, we may assume that G is connected, and set k to be the length of the shortest walk containing e, f. We proceed by induction k. For the base case, if $k = 2$, then we may assume $e = uv$ and $f = vw$. If u, w are in the same component of the graph $G - v$, then (i) holds. Otherwise, v is a cut-vertex and (ii) holds.

For the inductive step, we may assume $k \geq 3$. Let $f = uv$ and choose an edge $f' = vw$ so that there is a walk containing e, f' of length $k - 1$. First suppose that there is a cycle C containing e, f'. If $C - v$ and u are in distinct components of $G - v$, then v is a cut-vertex and (ii) holds. Otherwise, we may choose a path $P \subseteq G - v$ from u to a vertex of $V(C) \setminus \{v\}$. Now $P \cup C + f$ has a cycle which contains e, f, so (i) holds. If there is no cycle containing e, f', then it follows from our inductive hypothesis that there is a 1-separation (H, K) with $e \in E(H)$ and $f' \in E(K)$. Suppose (for a contradiction) that $f \in E(H)$. Then $V(H) \cap V(K) = \{v\}$, the shortest walk containing e, f has length k and the shortest walk containing e, f' has length $< k$ which is contradictory. Thus, $f \in E(K)$ and (H, K) satisfy (ii). This completes the proof. $\square$

2-connected: A graph G is 2-connected if it is connected, $|V(G)| \geq 3$, and G has no cut-vertex.

Theorem 4.4 *Let G be a graph with at least three vertices. Then the following are equivalent:*

(i) *G is 2-connected*

(ii) *For all $x, y \in V(G)$ there exists a cycle C with $x, y \in V(C)$.*

(iii) *G has no vertex of degree 0, and for all $e, f \in E(G)$ there exists a cycle C with $e, f \in E(C)$.*

Proof: It is easy to see that (iii) implies (ii): to find a cycle C with $x, y \in V(C)$ just choose an edge e incident with x and an edge f incident with y and apply (iii) to e, f. Trivially, (ii) implies (i). So, to complete the argument, we need only show that (i) implies (iii), but this is an immediate consequence of the previous proposition. $\square$

Block: A *block* of G is a maximal connected subgraph $H \subseteq G$ so that H does not have a cut-vertex. Note that if H is a block, then either H is 2-connected, or $|V(H)| \leq 2$.

Proposition 4.5 *If H_1, H_2 are distinct blocks in G, then $|V(H_1) \cap V(H_2)| \leq 1$.*

Proof: Suppose (for a contradiction) that $|V(H_1) \cap V(H_2)| \geq 2$. Let $H' = H_1 \cup H_2$, let $x \in H'$ and consider $H' - x$. By assumption, $H_1 - x$ is connected, and $H_2 - x$ is connected. Since these graphs share a vertex, $H' - x = (H_1 - x) \cup (H_2 - x)$ is connected. Thus, H' has no cut-vertex. This contradicts the maximality of H_1, thus completing the proof. $\square$

Block-Cutpoint graph: If G is a graph, the *block-cutpoint* graph of G, denoted $BC(G)$ is the simple bipartite graph with bipartition (A, B) where A is the set of cut-vertices of G, and B is the set of blocks of G, and $a \in A$ and $b \in B$ adjacent if the block b contains the cut-vertex a.

Observation 4.6 *If G is connected, then $BC(G)$ is a tree.*

Proof: Let (A, B) be the bipartition of $BC(G)$ as above. It follows from the connectivity of G that $BC(G)$ is connected. If there is a cycle $C \subseteq BC(G)$, then set H to be the union of all blocks in $B \cap V(C)$. It follows that H is a 2-connected subgraph of G (as in the proof of the previous proposition). This contradicts the maximality of the blocks in $B \cap V(C)$. □

Ears: An *ear* of a graph G is a path $P \subseteq G$ which is maximal subject to the constraint that all interior vertices of P have degree 2 in G. An *ear decomposition* of G is a decomposition of G into $C, P_1, \ldots, P_k$ so that C is a cycle of length ≥ 3, and for every $1 \leq i \leq k$, the subgraph P_i is an ear of $C \cup P_1 \cup \ldots P_i$.

Theorem 4.7 *A graph G is 2-connected if and only if it has an ear decomposition.*

Proof: For the "if" direction, let $C, P_1, \ldots, P_k$ be an ear decomposition of G. We shall prove that G is 2-connected by induction on k. As a base, if $k = 0$, then $G = C$ is 2-connected. For the inductive step, we may assume that $k \geq 1$ and that $C \cup P_1 \cup \ldots P_{k-1}$ is 2-connected. It then follows easily that $G = C \cup P_1 \cup \ldots P_k$ is also 2-connected.

We prove the "only if" direction by a simple process. First, choose a cycle $C \subseteq G$ of length ≥ 3 (this is possible by Theorem 4.4). Next we choose a sequence of paths $P_1, \ldots, P_k$ as follows. If $G' = C \cup P_1 \cup \ldots P_{i-1} \neq G$, then choose an edge $e \in E(G')$ and $f \in E(G) \setminus E(G')$, and then choose a cycle $D \subseteq G$ containing e, f (again using 4.4). Finally, let P_i be the maximal path in D which contains the edge f but does not contain any edge in $E(G')$. Then P_i is an ear of $C \cup P_1 \cup \ldots P_i$, and when this process terminates, we have an ear decomposition. □

Menger's Theorem

Theorem 4.8 *Let G be a graph, let $A, B \subseteq V(G)$ and let $k \geq 0$ be an integer. Then exactly one of the following holds:*

(i) *There exist k pairwise (vertex) disjoint paths $P_1, \ldots, P_k$ from A to B.*

(ii) *There is a separation (H, K) of G of order $< k$ with $A \subseteq V(H)$ and $B \subseteq V(K)$.*

Proof: It is clear that (i) and (ii) are mutually exclusive, so it suffices to show that (i) or (ii) holds. We prove this by induction on $|E(G)|$. As a base, observe that the theorem holds trivially when $|E(G)| \leq 1$. For the inductive step, we may then assume $|E(G)| \geq 2$. Choose an edge e and consider the graph $G' = G - e$. If G' contains k disjoint paths from A to B, then so does G, and (i) holds. Otherwise, by induction, there is a separation (H, K) of G' of order $< k$ with $A \subseteq V(H)$ and $B \subseteq V(K)$.

Now consider the separations $(H + e, K)$ and $(H, K + e)$ of G. If one of these separations has order $< k$, then (ii) holds. Thus, we may assume that e has one end in $V(H) \setminus V(K)$, the other end in $V(K) \setminus V(H)$, and both $(H + e, K)$ and $(H, K + e)$ have order k. Choose (H', K') to be one of these two separations with $E(H'), E(K') \neq \emptyset$ (this is possible since $|E(G)| \geq 2$) and set $X = V(H') \cap V(K')$ (note that $|X| = k$). Now, we apply the theorem inductively to the graph H' for the sets A, X and to K' for the sets X, B. If there are k disjoint paths from A to X in H' and k disjoint paths from X to B in K', then (i) holds. Otherwise, by induction there is a separation of H' or K' in accordance with (ii), and it follows that (ii) is satsified. $\square$

Note: The above theorem implies Theorem 11.2 (König Egerváry). Simply apply the above theorem to the bipartite graph G with bipartition (A, B). Then (i) holds if and only if $\alpha'(G) \geq k$, and (ii) holds if and only if $\beta(G) < k$ (here $V(H) \cap V(K)$ is a vertex cover).

Internally Disjoint: The paths $P_1, \ldots, P_k$ are *internally disjoint* if they are pairwise vertex disjoint except for their ends.

Theorem 4.9 (Menger's Theorem) *Let u, v be distinct non-adjacent vertices of G, and let $k \geq 0$ be an integer. Then exactly one of the following holds:*

(i) *There exist k internally disjoint paths $P_1, \ldots, P_k$ from u to v.*

(ii) *There is a separation (H, K) of G of order $< k$ with $u \in V(H) \setminus V(K)$ and $v \in V(K) \setminus V(H)$.*

Proof: Let $A = N(u)$ and $B = N(v)$ and apply the above theorem to $G - \{u, v\}$. $\square$

***k*-Connected:** A graph G is *k-connected* if $|V(G)| \geq k + 1$ and $G - X$ is connected for every $X \subseteq V(G)$ with $|X| < k$. Note that this generalizes the notion of 2-connected from Section 13. Also note that 1-connected is equivalent to connected.

Corollary 4.10 *A simple graph G with $|V(G)| \geq k+1$ is k-connected if and only if for every $u, v \in V(G)$ there exist k internally disjoint paths from u to v.*

Line Graph: If G is a graph, the *line graph* of G, denoted $L(G)$, is the simple graph with vertex set $E(G)$, and two vertices $e, f \in E(G)$ adjacent if e, f share an endpoint in G.

Edge cut: If $X \subseteq V(G)$, we let $\delta(X) = \{xy \in E(G) : x \in X \text{ and } y \notin X\}$, and we call any set of this form an *edge cut*. If $v \in V(G)$ we let $\delta(v) = \delta(\{v\})$.

Theorem 4.11 (Menger's Theorem - edge version) *Let u, v be distinct vertices of G and let $k \geq 0$ be an integer. Then exactly one of the following holds:*

(i) *There exist k edge disjoint paths $P_1, \ldots, P_k$ from u to v.*

(ii) *There exists $X \subseteq V(G)$ with $u \in X$ and $v \notin X$ so that $|\delta(X)| < k$.*

Proof: Apply Theorem 4.8 to the graph $L(G)$ for $\delta(u)$ and $\delta(v)$ and k. □

k-edge-connected: A graph G is k-edge-connected if $G - S$ is connected for every $S \subseteq E(G)$ with $|S| < k$.

Corollary 4.12 *A graph G is k-edge-connected if and only if for every $u, v \in V(G)$ there exist k pairwise edge disjoint paths from u to v.*

Fans and Cycles

Subdivision: If $e = uv$ is an edge of the graph G, then we *subdivide* e by removing the edge e, adding a new vertex w, and two new edges uw and wv.

Observation 4.13

1. *Subdividing an edge of a 2-connected graph yields a 2-connected graph.*
2. *Adding an edge to a k-connected graph results in a k-connected graph.*
3. *If G is k-connected and $A \subseteq V(G)$ satisfies $|A| \geq k$, then adding a new vertex to G and an edge from this vertex to each point in A results in a k-connected graph.*

Fan: Let $v \in V(G)$ and let $A \subseteq V(G) \setminus \{v\}$. A (v, A)-*fan* of *size* k is a collection of k paths $\{P_1, \ldots, P_k\}$ so that each P_i is a path from v to a point in A, and any two such paths intersect only in the vertex v.

Lemma 4.14 *If G is k-connected, $v \in V(G)$ and $A \subseteq V(G) \setminus \{v\}$ satisfies $|A| \geq k$, then G contains a (v, A)-fan of size k.*

Proof: Construct a new graph G' from G by adding a new vertex u and then adding a new edge between u and each point of A. By the above observation, G' is k-connected, and $u, v \in V(G')$ are nonadjacent, so by Menger's theorem there exist k internally disjoint paths from u to v. Removing the vertex u from each of these paths yields a (v, A)-fan of size k in G. □

Theorem 4.15 *Let G be a k-connected graph with $k \geq 2$ and let $X \subseteq V(G)$ satisfy $|X| = k$. Then there exists a cycle $C \subseteq G$ with $X \subseteq V(C)$.*

Proof: Choose a cycle $C \subseteq G$ so that $|V(C) \cap X|$ is maximum, and suppose (for a contradiction) that $X \not\subseteq V(C)$. Choose a vertex $v \in X \setminus V(C)$ and set $k' = \min\{k, |V(C)|\}$. It follows from the above lemma that G has a $(v, V(C))$-fan of size k', say $\{P_1, \ldots, P_{k'}\}$. Since $|X \cap V(C)| < k$, it follows that there exists a cycle $C' \subseteq C \cup P_1 \cup \ldots \cup P_{k'}$ so that $\{v\} \cup (X \cap V(C)) \subseteq V(C')$. This contradiction completes the proof. □