5 Directed Graphs

What is a directed graph?

Directed Graph: A *directed graph*, or *digraph*, D, consists of a set of *vertices* $V(D)$, a set of *edges* $E(D)$, and a function which assigns each edge e an ordered pair of vertices (u, v). We call u the *tail* of e, v the *head* of e, and u, v the *ends* of e. If there is an edge with tail u and head v, then we let (u, v) denote such an edge, and we say that this edge is directed *from u to v*.

Loops, Parallel Edges, and Simple Digraphs: An edge $e = (u, v)$ in a digraph D is a *loop* if $u = v$. Two edges e, f are *parallel* if they have the same tails and the same heads. If D has no loops or parallel edges, then we say that D is *simple*.

Drawing: As with undirected graphs, it is helpful to represent them with drawings so that each vertex corresponds to a distinct point, and each edge from u to v is represented by a curve directed from the point corresponding to u to the point corresponding to v (usually we indicate this direction with an arrowhead).

Orientations: If D is a directed graph, then there is an ordinary (undirected) graph G with the same vertex and edge sets as D which is obtained from D by associating each edge (u, v) with the ends u, v (in other words, we just ignore the directions of the edges). We call G the *underlying (undirected)* graph, and we call D an *orientation* of G.

Standard Diraphs

Null digraph	the (unique) digraph with no vertices or edges.
Directed Path	a graph whose vertex set may be numbered $\{v_1, \ldots, v_n\}$ and edges may be numbered $\{e_1, \ldots, e_{n-1}\}$ so that $e_i = (v_i, v_{i+1})$ for every $1 \leq i \leq n-1$.
Directed Cycle	a graph whose vertex set may be numbered $\{v_1, \ldots, v_n\}$ and edges may be numbered $\{e_1, \ldots, e_n\}$ so that $e_i = (v_i, v_{i+1})$ (*modulo* n) for every $1 \leq i \leq n$
Tournament	A digraph whose underlying graph is a complete graph.

Subgraphs and Isomorphism: These concepts are precisely analogous to those for undirected graphs.

Degrees: The *outdegree* of a vertex v, denoted $deg^+(v)$ is the number of edges with tail v, and the *indegree* of v, denoted $deg^-(v)$ is the number of edges with head v.

Theorem 5.1 *For every digraph D*

$$\sum_{v \in V(D)} deg^+(v) = |E(D)| = \sum_{v \in V(D)} deg^-(v)$$

Proof: Each edge contributes exactly 1 to the terms on the left and right. $\square$

Connectivity

Directed Walks & Paths: A *directed walk* in a digraph D is a sequence $v_0, e_1, v_1, \ldots, e_n v_n$ so that $v_i \in V(D)$ for every $0 \leq i \leq n$, and so that e_i is an edge from v_{i-1} to v_i for every $1 \leq i \leq n$. We say that this is a walk from v_0 to v_n. If $v_0 = v_n$ we say the walk is *closed* and if $v_0, v_1, \ldots, v_n$ are distinct we call it a *directed path.*

Proposition 5.2 *If there is a directed walk from u to v, then there is a directed path from u to v.*

Proof: Every directed walk from u to v of minimum length is a directed path. $\square$

δ^+ and δ^-: If $X \subseteq V(D)$, we let $\delta^+(X)$ denote the set of edges with tail in X and head in $V(G) \setminus X$, and we let $\delta^-(X) = \delta^+(V(G) \setminus X)$.

Proposition 5.3 *Let D be a digraph and let $u, v \in V(D)$. Then exactly one of the following holds.*

(i) *There is a directed walk from u to v.*

(ii) *There exists $X \subseteq V(D)$ with $u \in X$ and $v \notin X$ so that $\delta^+(X) = \emptyset$.*

Proof: It is immediate that (i) and (ii) are mutually exclusive, so it suffices to show that at least one holds. Let $X = \{w \in V(D) : \text{there is a directed walk from } u \text{ to } w\}$. If $v \in X$ then (i) holds. Otherwise, $\delta^+(X) = \emptyset$, so (ii) holds. $\square$

Strongly Connected: We say that a digraph D is *strongly connected* if for every $u, v \in V(D)$ there is a directed walk from u to v.

Proposition 5.4 *Let D be a digraph and let $H_1, H_2 \subseteq D$ be strongly connected. If $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is strongly connected.*

Proof: If $v \in V(H_1) \cap V(H_2)$, then every vertex has a directed walk both to v and from v, so it follows that $H_1 \cup H_2$ is strongly connected. $\square$

Strong Component: A *strong component* of a digraph D is a maximal strongly connected subgraph of D.

Theorem 5.5 *Every vertex is in a unique strong component of D.*

Proof: This follows immediately from the previous proposition, and the observation that a one-vertex digraph is strongly connected.

Observation 5.6 *Let D be a digraph in which every vertex has outdegree ≥ 1. Then D contains a directed cycle.*

Proof: Construct a walk greedily by starting at an arbitrary vertex v_0, and at each step continue from the vertex v_i along an arbitrary edge with tail v_i (possible since each vertex has outdegree ≥ 1) until a vertex is repeated. At this point, we have a directed cycle. $\square$

Acyclic: A digraph D is *acyclic* if it has no directed cycle.

Proposition 5.7 *The digraph D is acyclic if and only if there is an ordering $v_1, v_2, \ldots, v_n$ of $V(D)$ so that every edge (v_i, v_j) satisfies $i < j$.*

Proof: The "if" direction is immediate. We prove the "only if" direction by induction on $|V(D)|$. As a base, observe that this is trivial when $|V(D)| = 1$. For the inductive step, we may assume that D is acyclic, $|V(D)| = n \geq 2$, and that the proposition holds for all digraphs with fewer vertices. Now, apply the Observation 5.6 to choose a vertex v_n with $deg^+(v_n) = 0$. The digraph $D - v_n$ is acyclic, so by induction we may choose an ordering $v_1, v_2, \ldots, v_{n-1}$ of $V(D - v_n)$ so that every edge (v_i, v_j) satisfies $i < j$. But then $v_1, \ldots, v_n$ is such an ordering of $V(D)$. $\square$

Proposition 5.8 *Let D be a digraph, and let D' be the digraph obtained from D by taking each strong component $H \subseteq D$, identifying $V(H)$ to a single new vertex, and then deleting any loops. Then D' is acyclic.*

Proof: If D' had a directed cycle, then there would exist a directed cycle in D not contained in any strong component, but this contradicts Theorem 5.5. $\square$

Theorem 5.9 *If G is a 2-connected graph, then there is an orientation D of G so that D is strongly connected.*

Proof: Let $C, P_1, \ldots, P_k$ be an ear decomposition of G. Now, orient the edges of C to form a directed cycle, and orient the edges of each path P_i to form a directed path. It now follows from the obvious inductive argument (on k) that the resulting digraph D is strongly connected. $\square$

Eulerian, Hamiltonian, & path partitions

Proposition 5.10 *Let D be a digraph and assume that $deg^+(v) = deg^-(v)$ for every vertex v. Then there exists a list of directed cycles $C_1, C_2, \ldots, C_k$ so that every edge appears in exactly one.*

Proof: Choose a maximal list of cycles $C_1, C_2, \ldots, C_k$ so that every edge appears in at most one. Suppose (for a contradiction) that there is an edge not included in any cycle C_i and let H be a component of $D \setminus \cup_{i=1}^{k} E(C_i)$ which contains an edge. Now, every vertex $v \in V(H)$ satisfies $deg_H^+(v) = deg_H^-(v) \neq 0$, so by Observation 17.5 there is a directed cycle $C \subseteq H$. But then C may be appended to the list of cycles $C_1, \ldots, C_k$. This contradiction completes the proof. $\square$

Eulerian: A closed directed walk in a digraph D is called *Eulerian* if it uses every edge exactly once. We say that D is *Eulerian* if it has such a walk.

Theorem 5.11 *Let D be a digraph D whose underlying graph is connected. Then D is Eulerian if and only if $deg^+(v) = deg^-(v)$ for every $v \in V(D)$.*

Proof: The "only if" direction is immediate. For the "if" direction, choose a closed walk $v_0, e_1, \ldots, v_n$ which uses each edge at most once and is maximum in length (subject to this constraint). Suppose (for a contradiction) that this walk is not Eulerian. Then, as in the undirected case, it follows from the fact that the underlying graph is connected that there exists an edge $e \in E(D)$ which does not appear in the walk so that e is incident with some vertex in the walk, say v_i. Let $H = D - \{e_1, e_2, \ldots, e_n\}$. Then every vertex of H has indegree equal to its outdegree, so by the previous proposition, there is a list of directed cycles in H containing every edge exactly once. In particular, there is a directed cycle $C \subseteq H$ with $e \in C$. But then, the walk obtained by following $v_0, e_1, \ldots, v_i$, then following the directed cycle C from v_i back to itself, and then following $e_{i+1}, v_i, \ldots, v_n$ is a longer closed walk which contradicts our choice. This completes the proof. $\square$

Hamiltonian: Let D be a directed graph. A cycle $C \subseteq D$ is *Hamiltonian* if $V(C) = V(D)$. Similarly, a path $P \subseteq D$ is *Hamiltonian* if $V(P) = V(D)$.

In & Out Neighbors: If $X \subseteq V(D)$, we define

$$\begin{aligned} N^+(X) &= \{y \in V(D) \setminus X : (x, y) \in E(D) \text{ for some } x \in X\} \\ N^-(X) &= \{y \in V(D) \setminus X : (y, x) \in E(D) \text{ for some } x \in X\} \end{aligned}$$

We call $N^+(X)$ the *out-neighbors* of X and $N^-(X)$ the *in-neighbors* of X. If $x \in X$ we let $N^+(x) = N^+(\{x\}$ and $N^-(x) = N^-(\{x\})$.

Theorem 5.12 (Rédei) *Every tournament has a Hamiltonian path.*

Proof: Let T be a tournament. We prove the result by induction on $|V(T)|$. As a base, if $|V(T)| = 1$, then the one vertex path suffices. For the inductive step, we may assume that $|V(T)| \geq 2$. Choose a vertex $v \in V(T)$ and let T^- (resp. T^+) be the subgraph of T consisting of all vertices in $N^-(v)$ (resp. $N^+(v)$) and all edges with both ends in this set. If both T^- and T^+ are not null, then each has a Hamiltonian path, say P^- and P^+ and we may form a Hamiltonian path in T by following P^- then going to the vertex v, then following P^+. A similar argument works if either T^- or T^+ is null. $\square$

Theorem 5.13 (Camion) *Every strongly connected tournament has a Hamiltonian cycle.*

Proof: Let T be a strongly connected tournament, and choose a cycle $C \subseteq T$ with $|V(C)|$ maximum. Suppose (for a contradiction) that $V(C) \neq V(T)$. If there is a vertex $v \in V(T) \setminus V(C)$ so that $N^+(v) \cap V(C) \neq \emptyset$ and $N^-(v) \cap V(C) \neq \emptyset$, then there must exist an edge $(w, x) \in E(C)$ so that $(w, v), (v, x) \in E(T)$. However, then we may use these edges to find a longer cycle. It follows that the vertices in $V(T) \setminus V(C)$ may be partitioned into $\{A, B\}$ so that every $x \in A$ has $V(C) \subseteq N^+(v)$ and every $y \in B$ has $V(C) \subseteq N^-(y)$. It follows from the strong connectivity of T that $A, B \neq \emptyset$ and that there exists an edge (y, z) with $y \in B$ and $z \in A$. However, then we may replace an edge $(w, x) \in E(C)$ with the path containing the edges $(w, y), (y, z), (z, x)$ to get a longer cycle. This contradiction completes the proof. $\square$

Path Partition: A *path partition* of a digraph D is a collection $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ so that P_i is a directed path for $1 \leq i \leq k$ and $\{V(P_1), V(P_2), \ldots, V(P_k)\}$ is a partition of $V(D)$. We let $heads(\mathcal{P})$ ($tails(\mathcal{P})$) denote the set of vertices which are the initial (terminal) vertex in some P_i.

Lemma 5.14 (Bondy) *Let $\mathcal{P}$ be a path partition of the digraph D, and assume $|\mathcal{P}| > \alpha(D)$. Then there is a path partition $\mathcal{P}'$ of D so that $|\mathcal{P}'| = |\mathcal{P}| - 1$, and $tails(\mathcal{P}') \subseteq tails(\mathcal{P})$.*

Proof: We proceed by induction on $|V(D)|$. As a base, observe that the result is trivial when $|V(D)| = 1$. For the inductive step, note that since $\alpha(D) < |tails(\mathcal{P})|$ there must exist an edge (x, y) with $x, y \in tails(\mathcal{P})$. Choose i so that $y \in V(P_i)$. If $|V(P_i)| = 1$, then we may remove P_i from $\mathcal{P}$ and then append the edge (x, y) to the path containing x to get a suitable path partition. Thus, we may assume that $|V(P_i)| > 1$, and choose $w \in V(D)$ so that $(w, y) \in E(P_i)$. Now, $\mathcal{P}' = \{P_1, \ldots, P_{i-1}, P_i - y, P_{i+1}, \ldots, P_k\}$ is a path partition of $D - y$ and $\alpha(D - y) \leq \alpha(D) < |\mathcal{P}'|$, so by induction, there is a path partition $\mathcal{P}''$ of $D - y$ with $|\mathcal{P}''| = |\mathcal{P}'| - 1$ and $tails(\mathcal{P}'') \subseteq tails(\mathcal{P}')$. Since $x, w \in tails(\mathcal{P}')$, at least one of x, w is in $x, w \in tails(\mathcal{P}'')$. Since $(x, y), (w, y) \in E(D)$, we may extend $\mathcal{P}''$ to a suitable path partition of D by using one of these edges. $\square$

Theorem 5.15 (Gallai-Milgram) *Every digraph D has a path partition $\mathcal{P}$ with $|\mathcal{P}| = \alpha(D)$.*

Proof: This follows immediately from the observation that every digraph has a path partition (for instance, take each vertex as a one vertex path), and (repeated applications of) the above lemma. $\square$

Note: This is a generalization of Theorem 5.12.

Partially Ordered Set: A *partially ordered set* (or *poset*) consists of a set X and a binary relation $\prec$ which is reflexive ($x \prec x$ for every $x \in X$), antisymmetric ($x \prec y$ and $y \prec x$ imply

$x = y$), and transitive ($x \prec y$ and $y \prec z$ imply $x \prec z$). We say that two points $x, y \in X$ are *comparable* if either $x \prec y$ or $y \prec x$.

Chains and Antichains: In a poset, a *chain* is a subset $A \subseteq X$ so that any two points in A are comparable. An *antichain* is a subset $B \subseteq X$ so that no two points in B are comparable.

Theorem 5.16 (Dilworth) *Let $(X, \prec)$ be a poset and let k be the size of the largest antichain. Then there is a partition of X into k chains.*

Proof: Form a digraph D with vertex set X by adding an edge from x to y whenever $x \neq y$ and $x \prec y$. Now $\alpha(D) = k$, so the Gallai-Milgram Theorem gives us a path partition of D of size k. However, the vertex set of a directed path is a chain in the poset, so this yields a partition of X into k chains. $\square$

The Ford-Fulkerson Theorem

Flows: If D is a digraph and $s, t \in V(D)$, then an (s,t)-*flow* is a map $\phi : E(D) \to \mathbb{R}$ with the property that for every $v \in V(D) \setminus \{s,t\}$ the following holds.

$$\sum_{e \in \delta^+(v)} \phi(e) = \sum_{e \in \delta^-(v)} \phi(e).$$

The *value* of ϕ is $\sum_{e \in \delta^+(s)} \phi(e) - \sum_{e \in \delta^-(s)} \phi(e)$.

Proposition 5.17 *If ϕ is an (s,t)-flow of value q, then every $X \subseteq V(D)$ with $s \in X$ and $t \notin X$ satisfies*

$$\sum_{e \in \delta^+(X)} \phi(e) - \sum_{e \in \delta^-(X)} \phi(e) = q.$$

Proof:

$$\begin{aligned} q &= \sum_{e \in \delta^+(s)} \phi(e) - \sum_{e \in \delta^-(s)} \phi(e) \\ &= \sum_{x \in X} \left(\sum_{e \in \delta^+(x)} \phi(e) - \sum_{e \in \delta^-(x)} \phi(e) \right) \\ &= \sum_{e \in \delta^+(X)} \phi(e) - \sum_{e \in \delta^-(X)} \phi(e) \end{aligned}$$

□

Capacities: We shall call a weight function $c : E(D) \to \mathbb{R}^+ \cup \{\infty\}$ a *capacity function*. If $X \subseteq V(D)$, we say that $\delta^+(X)$ has *capacity* $\sum_{e \in \delta^+(X)} \phi(e)$.

Admissible Flows: An (s,t)-flow ϕ is *admissible* if $0 \le \phi(e) \le c(e)$ for every edge e.

Augmenting Paths: Let c be a capacity function and $\phi : E(D) \to \mathbb{R}$ an admissible (s,t)-flow. A path P from u to v is called *augmenting* if for every edge $e \in E(P)$, either e is traversed in the forward direction and $\phi(e) < c(e)$ or e is traversed in the backward direction and $\phi(e) > 0$.

Theorem 5.18 (Ford-Fulkerson) *Let D be a digraph, let $s,t \in V(D)$, and let c be a capacity function. Then the maximum value of an (s,t)-flow is equal to the minimum capacity of a cut $\delta^+(X)$ with $s \in X$ and $t \notin X$. Furthermore, if c is integer valued, then there exists a flow of maximum value ϕ which is also integer valued.*

Proof: It follows immediately from Proposition 19.1 that every admissible (s,t)-flow has value less than or equal to the capacity of any cut $\delta^+(X)$ with $s \in X$ and $t \notin X$.

We shall prove the other direction of this result only for capacity functions $c : E(D) \to \mathbb{Q}^+$ (although it holds in general). For every edge e, let $\frac{p_e}{q_e}$ be a reduced fraction equal to $c(e)$, and let n be the least common multiple of $\{q_e : e \in E(D)\}$. We shall prove that there exists a flow $\phi : E(D) \to \mathbb{Q}^+$ so that $\phi(e)$ can be expressed as a fraction with denominator n for every edge e. To do this, choose a flow ϕ with this property of maximum value. Define the set X as follows.

$$X = \{v \in V(D) : \text{there is an augmenting path from } s \text{ to } v\}$$

If $t \in X$, then there exists an augmenting path P from s to t. However, then we may modify the flow ϕ to produce a new admissible flow of greater value by increasing the flow by $\frac{1}{n}$ on every forward edge of P and decreasing the flow by $\frac{1}{n}$ on every backward edge of P. Since this new flow would contradict the choice of ϕ, it follows that $t \notin X$.

It follows from the definition of X that every edge $e \in \delta^+(X)$ satisfies $\phi(e) = c(e)$ and every edge $f \in \delta^-(X)$ satisfies $\phi(f) = 0$. Thus, our flow ϕ has value equal to the capacity of the cut $\delta^+(X)$ and the proof is complete. □

Note: The above proof for rational valued flows combined with a simple convergence argument yields the proof in general. However, the algorithm inherent in the above proof does not yield a finite algorithm for finding a flow of maximum value for arbitrary capacity functions.

Corollary 5.19 (edge-digraph version of Menger) *Let D be a digraph and let $s, t \in V(D)$. Then exactly one of the following holds:*

(i) *There exist k pairwise edge disjoint directed paths $P_1, \ldots, P_k$ from s to t.*

(ii) *There exists $X \subseteq V(D)$ with $s \in X$ and $t \notin X$ so that $|\delta^+(X)| < k$*

Proof: It is immediate that (i) and (ii) are mutually exclusive, so it suffices to show that at least one holds. Define a capacity function $c : E(D) \to \mathbb{R}$ by the rule that $c(e) = 1$ for every edge e. Apply the Ford-Fulkerson Theorem to choose an admissible integer valued (s,t)-flow $\phi : E(D) \to \mathbb{Z}$ and a cut $\delta^+(X)$ with $s \in X$ and $t \notin X$ so that the value of ϕ and the capacity of $\delta^+(X)$ are both equal to the integer q. Now, let $H = D - \{e \in E(D) : \phi(e) = 0\}$. Then H is a digraph with the property that $\delta^+_H(s) - \delta^-_H(s) = q = \delta^-_H(t) - \delta^+_H(t)$ and $\delta^+_H(v) = \delta^-_H(v)$ for every $v \in V(H) \setminus \{s, t\}$. By Problem 3 of Homework 10, we find that H contains q edge disjoint directed paths from s to t. So, if $q \leq k$, then (i) holds, and if $q > k$ (ii) holds. $\square$