11 Extremal Graph Theory

In this section, graphs are assumed to have no loops or parallel edges.

Complete t-Partite If $m_1, m_2, \ldots, m_t$ are nonnegative integers, the *complete t-partite* graph $K_{m_1,m_2,\ldots,m_t}$ is a simple graph with vertex partition $\{I_1, I_2, \ldots, I_t\}$ where $|I_j| = m_j$ for every $1 \leq j \leq t$ and adjacency is determined by the rule that vertices x, y are adjacent if and only if they lie in different members of this partition.

Turán Graph Let n, t be positive integers with $n \geq t$, and choose ℓ and $0 \leq j < t$ so that $n = t\ell + j$. Then the *Turán Graph* $T_{n,t}$ is defined as follows.

$$T_{n,t} = K_{\underbrace{\ell,\ldots,\ell}_{t-j},\underbrace{\ell+1,\ldots,\ell+1}_{j}}$$

Observation 11.1 *The Turán graph $T_{n,t}$ has the maximum number of edges over all complete t-partite graphs on n vertices.*

Proof: Let $m_1, m_2, \ldots, m_t$ be positive integers with $\sum_{i=1}^{t} m_i = n$ and consider the graph $K_{m_1,m_2,\ldots,m_t}$. If there exist distinct $i, j \in \{1, 2, \ldots, t\}$ with $m_i \leq m_j + 2$, then replacing m_i by $m_i + 1$ and m_j by $m_j - 1$ increases the number of edges by $(m_i + 1)(m_j - 1) - m_i m_j = m_j - m_i - 1 > 0$. We may repeat this operation until $|m_i - m_j| \leq 1$ for every $i, j \in \{1, 2, \ldots, t\}$ (since the number of edges increases each time, it can only be repeated finitely many times) at which point we have the desired graph. $\square$

Theorem 11.2 (Turán) *The Turán graph $T_{n,t}$ has the maximum number of edges over all n vertex graphs which do not contain a clique of order $t + 1$.*

Proof: We proceed by induction on t. As a base, observe that the result holds trivially when $t = 1$. For the inductive step, let G be an n-vertex graph with no clique of order $t + 1$. Our approach will be to construct a complete t-partite graph G' on n vertices so that $|E(G')| \geq |E(G)|$. To do this, let $\Delta = \Delta(G)$, choose a vertex $v \in V(G)$ with $deg(v) = \Delta$, and let H be the subgraph of G induced by $N(v)$. Now, H does not have a clique of order t, so by induction there is a complete $(t-1)$-partite graph H' on Δ vertices with $|E(H')| \geq |E(H)|$. Now, extend H' to a new graph G' by adding an independent set X of size $n - \Delta$ and joining every vertex in X to every vertex in $V(H')$. Now G' is a complete t-partite graph on n vertices and

$$\begin{aligned} |E(G)| &= |E(H)| + |E(G) \setminus E(H)| \\ &\leq |E(H)| + \Delta|V(G) \setminus V(H)| \\ &\leq |E(H')| + \Delta(n - \Delta) \\ &= |E(G')|. \end{aligned}$$

It now follows from the previous observation that $|E(G)| \leq |E(G')| \leq |E(T_{n,t}|$, thus completing the proof. $\square$