12 Extremal Graph Theory II

In this section, graphs are assumed to have no loops or parallel edges.

Average Degree: The *average degree* of a graph G is $\frac{2|E(G)|}{|V(G)|} = \frac{1}{|V(G)|} \sum_{v \in V(G)} deg(v)$.

Observation 12.1 *For every $r \in \mathbb{N}$, every graph of average degree $\geq 2r$ contains a subgraph of minimum degree $\geq r+1$.*

Proof: We prove the observation by induction on $|V(G)|$. If G has minimum degree $\geq r+1$, then we are done. Otherwise, let $v \in V(G)$ satisfy $deg(v) \leq r$. Then we have

$$\frac{2|E(G-v)| + 2|\delta(v)|}{|V(G-v)| + 1} = \frac{2|E(G)|}{|V(G)|} \geq 2r \geq \frac{2|\delta(v)|}{1}.$$

It follows from this (and the observation $\frac{a+b}{c+d} \geq \frac{b}{d} \Rightarrow \frac{a}{c} \geq \frac{a+b}{c+d}$) that $\frac{2|E(G-v)|}{|V(G-v)|} \geq \frac{2|E(G)|}{|V(G)|} \geq 2r$. (So, deleting a vertex of degree $\leq r$ from a graph with average degree $\geq 2r$ can only increase the average degree). Now, by induction, $G - v$ has a subgraph of minimum degree $\geq r+1$ and this completes the proof. $\square$

Theorem 12.2 (Mader) *For every positive integer r, every graph with average degree $\geq 2^{\binom{r}{2}}$ contains a subdivision of K_r.*

Proof: We will prove by induction that for $m = r-1, r, r+1, \ldots, \binom{r}{2}$, every graph of average degree $\geq 2^m$ contains a subgraph which is a subdivision of a (simple) graph on r vertices with m edges. As a base, when $m = r-1$, we may choose a vertex v with $deg(v) \geq 2^{r-1} \geq r-1$. Now v together with $r-1$ of its neighbors induce a graph with r vertices and $r-1$ edges.

For the inductive step, we let $r \leq m \leq \binom{r}{2}$ and let G be a graph with average degree $\geq 2^m$. We may assume (without loss) that G is connected. Choose a maximal set $U \subseteq V(G)$ so that the graph induced on U is connected and so that the graph G' obtained from G by identifying U to a new vertex u and then deleting all loops and parallel edges has average degree $\geq 2^m$. Note that such a set U exists since $U = \{v\}$ works for every $v \in V(G)$.

Now, let H be the subgraph of G' induced by the neighbors of u. If there exists a vertex $v \in V(H)$ with $deg_H(v) < 2^{m-1}$, then replacing U by $U \cup \{v\}$ reduces the number of vertices in G' by 1 and the number of edges by $< 2^{m-1}$ so this gives a G' with average degree $\geq 2^m$, contradicting our choice of U. It follows that H has average degree $\geq 2^{m-1}$. By induction, we may choose a subgraph K of H which is a subdivision of an r vertex graph with $m-1$ edges. Since every point in $V(H)$ is adjacent to a point in U and the graph induced on U is connected, we may extend K to a subgraph K' of G which is a subdivision of an r vertex graph with m edges, as desired. $\square$

Ball: If $x \in V(G)$ and $n \in \mathbb{N}$, the *ball of radius n around x* is

$$B_n(x) = \{v \in V(G) : dist(x, v) \leq n\}.$$

Theorem 12.3 *Every graph G with $\delta(G) \geq 3$ and no cycle of length $\leq 8\binom{r}{2} + 2$ contains K_r as a minor.*

Proof: Set $k = \binom{r}{2}$ and choose a maximal subset of vertices $X \subseteq V(G)$ with the property that $dist(x, y) \geq 2k + 1$ for every distinct $x, y \in X$. Now let $X = \{x_1, \ldots, x_m\}$ and define a function $f : V(G) \to X$ by the rule that for every $v \in V(G)$, the vertex $f(v) = x_i$ is a point in X with minimum distance to v, and subject to this has i as small as possible. Now for $1 \leq i \leq m$ let $V_i = \{v \in V(G) : f(v) = x_i\}$. The following inclusion follows immediately from our definitions (it holds for every $1 \leq i \leq m$).

$$B_k(x_i) \subseteq V_i \subseteq B_{2k}(x_i)$$

Claim: If $v \in V_i$ and P is a shortest path from v to x_i then $V(P) \subseteq V_i$.

Proof of Claim: Suppose (for a contradiction) that $u \in V(P)$ satisfies $u \in V_j$ for $j \neq i$. If $dist(u, x_j) < dist(u, x_i)$, then we find $dist(v, x_j) < dist(u, x_i)$, which is contradictory. It follows from this (and $u \in V_j$) that $dist(u, x_j) = dist(u, x_i)$. From this we deduce $dist(v, x_j) = dist(v, x_i)$. However, now $v \in V_i$ implies $i < j$ and $u \in V_j$ implies $j < i$ which is a contradiction.

With this claim, we now deduce the following properties of V_i.

- The graph induced by V_i is connected.
- $dist(v, x_i) \leq 2k$ for every $v \in V_i$.
- If $x, y \in V_i$ then $dist(x, y) \leq 4k$.

For every $1 \le i \le m$ let T_i be the graph induced on V_i. Suppose (for a contradiction) that T_i has a cycle and choose such a cycle $C \subseteq T_i$ of minimum length. By assumption, $|E(C)| \ge 8k+2$, but then we may choose two points $u, v \in V(C)$ which are distance $\ge 4k+1$ on this cycle. Now there is a path $P \subseteq T_i$ from u to v of length $\le 4k$ and now $P \cup C$ contains a shorter cycle than C, giving us a contradiction. Thus, we find that T_i is a tree.

Now by degree sum arguments, we have

$$\begin{aligned} |\delta(V_i)| &= \sum_{v \in V_i} deg(v) - 2|E(T_i)| \\ &\ge 3|V_i| - 2|V_i| \\ &= |V_i| \\ &\ge |B_k(x_i)| \\ &\ge 2^k \end{aligned}$$

Now, construct a graph G' from G by contracting every edge in T_i for every $1 \le i \le m$. It follows from the assumption that any two points in T_i are joined by a path of length $\le 4k$ in T_i and the assumption that all cycles in G have length $\ge 4k+3$ that G' has no loops or parallel edges. So, G' is a minor of G which is simple with minimum degree $\ge 2^k = 2^{\binom{r}{2}}$, and by Theorem 12.2 we find that G' contains a subdivision of K_r. This gives us a K_r minor in G, as required. $\square$

Linking: A graph G is *k-linked* if $|V(G)| \ge 2k$ and for every $\{s_1, \ldots, s_k, t_1, \ldots, t_k\} \subseteq V(G)$, there exist vertex disjoint paths $P_1, \ldots, P_k$ so that P_i has ends s_i and t_i for $1 \le i \le k$.

Theorem 12.4 *For every positive integer k, every $2^{\binom{3k}{2}}$-connected graph is k-linked.*

Proof: Let $\{s_1, \ldots, s_k, t_1, \ldots, t_k\} \subseteq V(G)$ and apply Theorem 12.2 to choose a subdivision of K_{3k} in G. Call this subdivision H and let $U \subseteq V(G)$ be the set of vertices with degree $3k$ in H. By Menger's Theorem, we may choose a collection of paths $Q_1 \ldots, Q_{2k} \subseteq G$ so that each of these paths has a distinct starting point in $\{s_1 \ldots, s_k, t_1, \ldots, t_k\}$ and a distinct endpoint in U. Subject to this, choose $Q_1, \ldots, Q_{2k}$ so that each Q_i has a minimum number of edges in $E(G) \setminus E(H)$.

Let $U = \{s'_1, \ldots, s'_k, t'_1, \ldots, t'_k, u_1, \ldots, u_k\}$ and assume that for $1 \le i \le k$ the path Q_i has ends s_i and s'_i and that the path Q_{k+i} has ends t_i and t'_i. Let $1 \le i \le k$ and let $R_i, R'_i \subseteq H$

be the paths from u_i to s_i' and t_i' which correspond to (possibly subdivided) edges of our K_{3k}. Let v_i be the first vertex on the path R_i from u_i to s_i' which is contained in one of the paths $Q_1, \ldots, Q_{2k}$ and suppose that $v_i \in Q_j$. Now, consider rerouting Q_j along R_i to the vertex u_i. The resulting paths $Q_1, \ldots, Q_{2k}$ would be vertex disjoint, so it follows from our choice that Q_j uses no edges in $E(G) \setminus E(H)$ after v_i. It follows from this that $j = i$ and after the vertex v_i, the path Q_i follows R_i to s_i'. By a similar argument, we find that the first vertex v_i' on R_i' which lies on one of $Q_1, \ldots, Q_{2k}$ is on the path Q_{k+i}. Now, define the path P_i to be the path from s_i to t_i obtained by following Q_i from s_i to v_i, then following R_i to u_i, then following R_i' to v_i' and then following Q_{k+i} to t_i. It is a consequence of our construction that $P_1, \ldots, P_k$ are vertex disjoint, thus completing the proof. $\square$