13 Ramsey Theory

In this section, graphs are assumed to have no loops or parallel edges.

Ramsey Numbers: If s, t are positive integers, the *Ramsey Number* $R(s,t)$ is the smallest integer n with the property that however the edges of K_n are assigned the colours *red* and *blue*, there always must exist either a complete subgraph on s vertices with all edges *red*, or a complete subgraph on t vertices with all edges *blue*. More generally, if H_1, H_2 are graphs, then $R(H_1, H_2)$ is the smallest integer n with the property that however the edges of K_n are assigned *red* and *blue* there must always exist either a subgraph isomorphic to H_1 with all edges *red* or a subgraph isomorphic to H_2 with all edges *blue*.

Theorem 13.1 (Ramsey) *$R(s,t) \leq \binom{s+t-2}{s-1}$ for every pair of positive integers s, t.*

Proof: We proceed by induction on $s + t$. As a base, observe that the result holds trivially whenever $s = 1$ or $t = 1$. For the inductive step, we let s, t be positive integers with $s > 1$ and $t > 1$. Set $n = \binom{s+t-2}{s-1}$ and consider an arbitrary *red/blue*-colouring of the edges of K_n. Choose a vertex $v \in V(K_n)$ and let S be the set of vertices joined to v by a *red* edge and T be the set of vertices joined to v by a *blue* edge. Since $|S| + |T| + 1 = \binom{s+t-2}{s-1} = \binom{s+t-3}{s-1} + \binom{s+t-3}{s-2}$ we must have either $|S| \geq \binom{s+t-3}{s-2}$ or $|T| \geq \binom{s+t-3}{s-1}$. In the former case, the theorem follows by applying induction to the graph induced by S (with the parameters $s - 1$ and t). In the latter case, the theorem follows by applying induction to the graph induced by T (with the parameters s and $t - 1$). $\square$

Hypergraph: A *hypergraph* H consists of a set of vertices, denoted $V(H)$, a set of edges (sometimes called hyperedges), denoted $E(H)$, and an incidence relation on $V(H) \times E(H)$. If $e \in E(H)$ is an edge, we think of e as containing those vertices it is incident with, so we call the number of vertices contained in e the *size* of e. Note that a graph is a special case of a hypergraph where all edges have size two.

Complete Hypergraphs: We let K_n^q denote the hypergraph on n vertices with exactly one size q edge containing each q element subset of our n vertices (and no other edges). So K_n^2 is the complete graph on n vertices.

Hypergraph Ramsey Numbers: If s, t are positive integers, $R^q(s,t)$ is the smallest integer n with the property that however the edges of K_n^q are coloured *red* and *blue*, there either exists a subgraph isomorphic to K_s^q with all edges *red* or a subgraph isomorphic to K_t^q with all edges *blue*

Theorem 13.2 (Ramsey) *If s, t, q are positive integers, then $R^q(s,t)$ exists (is finite).*

Proof: We proceed by induction on q and for fixed q by induction on $s+t$. As a base for the first induction, note that the result is trivial when $q = 1$. For the inductive step, we may then assume that $q > 1$. If $s = 1$ or $t = 1$, then the result is again trivial, so we may further assume that $s > 1$ and $t > 1$. Now, let $n = R^{q-1}(R^q(s-1,t), R^q(s,t-1)) + 1$ (note that by induction these numbers are finite) and consider an arbitrary *red/blue* colouring of the edges of K_n^q. Choose a vertex x and consider the edge-coloured hypergraph H on $V(K_n^q) \setminus x$ obtained by taking each edge e which contains x, and simply removing x from this edge (keeping its colour the same). Our hypergraph H is a 2-edge-colouring of K_{n-1}^{q-1}, so by our assumptions, it must have either a subset R of vertices of size $R^q(s-1,t)$ with all edges *red* or a subset B of size $R^q(s,t-1)$ with all edges *blue*. In the former case, consider the subgraph of our original graph induced on R. By assumption, this graph must either have a t element subset with all edges *blue* (in which case we are done) or an $s-1$ element subset with all edges *red* which can be extended to an s element subset with this property by adding x. A similar argument resolves the latter case using the subset B. Thus, $n \leq R^q(s,t)$ and this value is finite. □

Theorem 13.3 *If $R^4(5,t)$ points are placed in the plane, with no three on a line, then there exist t points in convex position.*

Proof: Construct a hypergraph on our $n = R^4(5,t)$ points by adding an edge of colour *red* containing every set of 4 points which do not lie in convex position, and adding an edge of colour *blue* containing every set of 4 points which do lie in convex position. This hypergraph is a 2-edge-colouring of K_n^4 so by construction, it must contain either a 5 point set with all edges *red* or a t point set with all edges *blue*. The former case is impossible (in any 5 point set at least 4 lie in convex position), and in the latter case the t points in our set lie in convex position. □