# 10 Hamiltonian Cycles

In this section, we consider only simple graphs.

## Finding Hamiltonian Cycles

**Hamiltonian:** A cycle $C$ of a graph $G$ is *Hamiltonian* if $V(C) = V(G)$. A graph is *Hamiltonian* if it has a Hamiltonian cycle.

**Closure:** The *(Hamiltonian) closure* of a graph $G$, denoted $Cl(G)$, is the simple graph obtained from $G$ by repeatedly adding edges joining pairs of nonadjacent vertices with degree sum at least $|V(G)|$ until no such pair remains.

**Lemma 10.1** *A graph $G$ is Hamiltonian if and only if its closure is Hamiltonian.*

*Proof:* Suppose (for a contradiction) that the lemma is false. Then we may choose a graph $G$ with $|V(G)| = n$ and a pair of non-adjacent vertices $u, v \in V(G)$ with $deg(u) + deg(v) \geq n$ so that $G$ is not Hamiltonian, but adding a new edge $uv$ to $G$ results in a Hamiltonian graph. Every Hamiltonian cycle in this new graph contains the new edge $uv$, so in the original graph $G$ there is a path from $u$ to $v$ containing every vertex. Let $v = v_1, v_2, \ldots, v_n = u$ be the vertex sequence of this path. Set

$$\begin{aligned} P &= \{v_i : i \geq 2 \text{ and } v_i \text{ is adjacent to } v_1\} \\ Q &= \{v_i : i \geq 2 \text{ and } v_{i-1} \text{ is adjacent to } v_n\} \end{aligned}$$

Then $|P| + |Q| = deg(v) + deg(u) \geq n$ and since $P \cup Q \subseteq \{v_2, \ldots, v_n\}$, it follows that there exists $2 \leq i \leq n$ with $v_i \in P \cap Q$, so there is an edge $e$ with ends $v_1$ and $v_i$ and an edge $e'$ with ends $v_n$ and $v_{i-1}$. Using these two edges, we may form a Hamiltonian cycle in $G$ as desired. □

**Theorem 10.2 (Dirac)** *If $G$ is a graph with $n = |V(G)| \geq 3$ and $\delta(G) \geq \frac{n}{2}$, then $G$ is Hamiltonian.*

*Proof:* The graph $Cl(G)$ is complete, so this follows from the above lemma. □

**Theorem 10.3 (Chvátal)** *Let $G$ be a graph with $n = |V(G)| \geq 3$ and vertex degrees $d_1 \leq d_2 \leq \ldots \leq d_n$. If either $d_i > i$ or $d_{n-i} \geq n - i$ for every $1 \leq i < \frac{n}{2}$, then $G$ is Hamiltonian.*

*Proof:* It suffices to prove that $Cl(G)$ is complete for any graph satisfying the above assumption. Suppose (for a contradiction) that $G$ is a graph with $E(G)$ maximal which satisfies the above assumption but has $Cl(G)$ not complete. It follows from our maximality assumption that $G = Cl(G)$. Now, choose nonadjacent vertices $u, v$ with $deg(u) + deg(v)$ maximum, and assume that $deg(u) \leq deg(v)$. Set $i = deg(u)$ and note that by assumption $deg(u) + deg(v) \leq n - 1$ so $i < \frac{n}{2}$ and $deg(v) \leq n - i - 1$.

Since $deg(v) \leq n - i - 1$ there are at least $i$ vertices nonadjacent to $v$, and by our assumption each of these has degree $\leq deg(u) = i$. Thus, $G$ has at least $i$ vertices with degree $\leq i$ and we have $d_i \leq i$.

Similarly, $deg(u) = i$ so there are exactly $n - i - 1$ vertices nonadjacent to $u$, and by our assumption, each has degree $\leq deg(v) \leq n - i - 1$. Since $u$ also has degree $i = deg(u) \leq deg(v) \leq n - i - 1$, this gives us a total of at least $n - i$ vertices with degree $< n - i$ so $d_{n-i} < n - i$. This contradicts our assumption and completes the proof. □

**Lemma 10.4** *If $G$ is a graph with $\delta(G) \geq 2$, then $G$ has a cycle of length $\geq \delta(G) + 1$.*

*Proof:* Set $\delta = \delta(G)$. Let $P$ be a maximal path in $G$ and let $v$ be an end of $P$. By assumption, $v$ has at least $\delta$ neighbors all of which must lie on $P$. If $u$ is the neighbor of $v$ which is furthest from $v$ on $P$, then the subpath of $P$ from $v$ to $u$ together with $uv$ is a cycle of length $\geq \delta + 1$. □

**Theorem 10.5 (Chvátal-Erdös)** *If $G$ is a $k$-connected graph with $|V(G)| \geq 3$ and $\alpha(G) \leq k$, then $G$ is Hamiltonian.*

*Proof:* Let $C$ be a cycle of $G$ of maximum length, and suppose (for a contradiction) that $C$ is not Hamiltonian. Since $G$ has minimum degree $\geq k$, it follows from Lemma 10.4 that $C$ has length $\geq k + 1$. Let $H$ be a component of $G - V(C)$ and let $S$ be the set of vertices in $V(C)$ which have a neighbor in $V(H)$. Since $G$ is $k$-connected and $|V(G)| \geq k + 1$ we must have $|S| \geq k$. Also, observe that no two vertices in $S$ can be consecutive on $C$ since this would yield a cycle longer than $C$ (contradicting our assumption). Let $T$ be the set of all vertices

$v \in V(C)$ so that $v$ is the clockwise neighbor of a point in $S$ (on the cycle $C$). Note that $S \cap T = \emptyset$ by our earlier observation. If there exist $t_1, t_2 \in T$ which are adjacent, then let $s_1, s_2 \in S$ be the counterclockwise neighbors of $t_1, t_2$ (respectively) and choose a path $P \subseteq G$ from $s_1$ to $s_2$ with all internal vertice in $V(H)$. Now the graph $C - s_1t_1 - s_2t_2 + P + t_1t_2$ is a cycle longer than $C$ contradicting our assumption. Thus $T$ is an independent set of size $|T| = |S| \geq k$, and we may add to $T$ any vertex in $V(H)$ to obtain an independent set of size $\geq k+1$. However, this contradicts the assumption $\alpha(G) \leq k$, thus completing the proof. □

## Structure

**Observation 10.6** *Let $G$ be a graph and let $X \subseteq V(G)$. If $|X| < comp(G - X)$, then $G$ is not Hamiltonian.*

*Proof:* We prove the contrapositive. If $C \subseteq G$ is a Hamiltonian cycle, then

$$|X| \geq comp(C - X) \geq comp(G - X). \qquad \square$$

**Theorem 10.7 (Smith)** *If $G$ is a d-regular graph where d is odd and $e \in E(G)$, then there are an even number of Hamiltonian cycles in $G$ which pass through the edge $e$.*

*Proof:* Choose an end $v$ of $e$, and construct a simple graph $H$ as follows. Define $V(H)$ to be the set of all Hamiltonian paths in $G$ which have $v$ as an end and contain $e$. If $P$ is such a path with ends $v, u$, then for every $uw \in E(G)$ with $w \neq v$, add an edge in the graph $H$ from $P$ to other Hamiltonian path contained in $P + uw$. Now, a vertex of $H$ has odd degree if and only if this Hamiltonian path may be extended to a Hamiltonian cycle. Further, for every Hamiltonian cycle containing $e$, the Hamiltonian path obtained by removing the other edge incident with $v$ appears as a vertex of $H$ with odd degree. Thus, the number of Hamiltonian cycles containing $e$ is exactly equal to the number of vertices of odd degree in $H$, and this is necessarily even. □

**Observation 10.8** *Every 3-regular graph which is Hamiltonian is 3-edge-colourable.*

*Proof:* Let $G$ be 3-regular and Hamiltonian. Then $|V(G)|$ is even (since all degrees are odd), so if $C$ is a Hamiltonian cycle, we can colour the edges of $C$ alternately *red* and *blue* and colour all other edges *green*. □

**Theorem 10.9 (Grinberg)** *If $G$ is a plane graph with a Hamiltonian cycle $C$, and $G$ has $f'_i$ faces of length $i$ inside $C$ and $f''_i$ faces of length $i$ outside $C$ for every $i$, then $\sum_i (i - 2)(f'_i - f''_i) = 0$*

*Proof:* We shall prove that $\sum_i (i-2)f'_i = |V(G)| - 2 = \sum_i (i-2)f''_i$ by induction on $|E(G) \setminus E(C)|$. As a base case, observe that the formula holds trivially whenever $|E(G) \setminus E(C)| = 0$. For the inductive step, let $G$ be a plane graph with Hamiltonian cycle $C$ and $|E(G) \setminus E(C)| > 0$, and assume that the theorem holds for every such graph and cycle with $|E(G) \setminus E(C)|$ of smaller value. Let $e \in E(G) \setminus E(C)$. We shall assume that $e$ lies inside the cycle $C$, the other case is similar. Let $S$ and $T$ be the faces on either side of $e$ and assume that $S$ has size $s$ and $T$ has size $t$. By induction, the formula holds for $G - e$, and since the outside of $C$ is the same in $G - e$ as in $G$ we have $\sum_i (i-2)f''_i = |V(G)| - 2$. For the inner faces, we see that $G - e$ has lost a contribution of $s - 2$ from $S$ and $t - 2$ from $T$, but has gained a contribution of $(s - t - 2) - 2$ from the new face formed from $S$ and $T$. Thus, the formula holds for $G$. $\square$